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Linear Algebra and its Applications 373 (2003) 67–74
LINEAR ALGEBRA AND ITS APPLICATIONS
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A note on k-primitive directed graphs

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Received 19 April 2002; accepted 3 September 2002

Submitted by B. Shader

Abstract

We consider the problem of which primitive directed graphs can be k-colored to yield a k-primitive directed graph. If such a k-coloring exists, then certainly such a graph must have at least k cycles. We prove that any primitive directed graph admits a 2-coloring that is 2-primitive. By contrast, for each $k \geqslant 4$, we construct examples of primitive directed graphs having k cycles for which no k-coloring is k-primitive. We also give some partial results for the case that $k = 3$.

Keywords: Directed graph; k-primitive; k-coloring

1. Introduction and preliminaries

Suppose that G is a simple directed graph on at least two vertices, possibly with loops. G is *primitive* if there is some $r \in \mathbb{N}$ such that for any pair of vertices u, v of G, there is a walk from u to v of length r. It is well-known (see [1], for example) that G is primitive if and only if it is strongly connected and the greatest common divisor of its cycle lengths is 1. Observe that the directed graph consisting of a loop at a single vertex can also be thought of as 'primitive'. However, in this paper it will be terminologically convenient to exclude that graph from consideration as primitive; that convention will help to avoid some trivialities in our discussion.

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1 NSERC Grant no. OGP0138251.

0024-3795/$ - see front matter
doi:10.1016/S0024-3795(02)00574-8

A *k-coloring* of G is a k-tuple of spanning subgraphs $(G_1, \dots, G_k)$ such that the subgraphs $G_1, \dots, G_k$ partition the arcs of G into k (nonempty) subsets. We think of each G_i as containing the arcs given color i in the directed graph G. In parallel with the notion of primitivity for G, we say that the k-coloring $(G_1, G_2, \dots, G_k)$ is *k-primitive* if there is a k-tuple of positive integers $(r_1, r_2, \dots, r_k)$ such that between any pair of vertices u, v of G, there is a walk from u to v in G having exactly r_i arcs in G_i for each i, $i = 1, \dots, k$ (equivalently, that walk has exactly r_i arcs of color i for each $i = 1, \dots, k$).

Evidently the case $k = 1$ corresponds to the definition of primitivity given above. Further, it is clear that if there is a k-coloring of G that is k-primitive, then G itself must be a primitive directed graph. In this paper we deal with the following natural question: Given a primitive directed graph G, under what circumstances is there a k-coloring of G that is k-primitive?

Fortunately, there is a matrix theoretic technique that enables one to determine whether a specific k-coloring is k-primitive. Suppose that we have a directed graph G with k-coloring $(G_1, \dots, G_k)$. Label the cycles of G by $C_1, \dots, C_l$, and construct the $k \times l$ matrix $M = [m_{i,j}]$, where $m_{i,j}$ denotes the number of arcs on cycle j having color i. The following result, found in [2], shows how M can be used to determine whether or not the coloring is k-primitive.

Theorem 1.1 [2]. *Let G be a directed graph, and suppose that $(G_1, \dots, G_k)$ is a k-coloring of G. Then G is k-primitive if and only if the $k \times k$ minors of M are relatively prime.*

From Theorem 1.1 we see that if G admits a k-primitive k-coloring, then necessarily G must have at least k cycles. In this paper, we show any primitive directed graph admits a 2-primitive 2-coloring. Further, for each $k \geqslant 4$, we provide an example of a primitive directed graph having exactly k cycles, but which cannot be k-colored to be k-primitive. Finally, we present a few partial results dealing with the case $k = 3$.

2. The case $k = 2$

Lemma 2.1. *Let G be a directed graph with two cycles labeled C_1 and C_2 whose lengths are l_1 and l_2, respectively. Suppose that $l_2 \geqslant l_1 + 1$, and let a and b be the smallest positive integers such that $al_2 - bl_1 = \pm \gcd(l_1, l_2)$. There is a coloring of C_1 and C_2 which colors a arcs of C_1 color 1, b arcs of C_2 color 1, and the remaining arcs of these two cycles color 2.*

Proof. Let $q_1 = l_1/\gcd(l_1, l_2)$ and $q_2 = l_2/\gcd(l_1, l_2)$, and note that $aq_2 - bq_1 = \pm 1$. We claim that $a \leqslant q_1$. To see the claim, suppose to the contrary that $a \geqslant q_1 + 1$. Then we have $bq_1 = aq_2 \mp 1 \geqslant aq_2 - 1 \geqslant q_1q_2 + q_2 - 1 \geqslant q_1q_2 + 1$, the last inequality since $q_2 \geqslant 2$. Thus we find that $b \geqslant q_2 + 1/q_1 > q_2$. Since b and q_2 are integers, we see that $b \geqslant q_2 + 1$. But then letting $a' = a - q_1$ and $b' = b - q_2$, we have $a'q_2 - b'q_1 = \pm 1$, so that $a'l_2 - b'l_1 = \pm\gcd(l_1, l_2)$, contradicting the fact a and b were chosen as the smallest positive integers such that $al_2 - bl_1 = \pm\gcd(l_1, l_2)$. Consequently we have $a \leqslant q_1$, as claimed.

Next we claim that $a \leqslant b$. To see the claim, suppose to the contrary that $a \geqslant b + 1$. Then we have $aq_2 - bq_1 \geqslant (b + 1)(q_1 + 1) - bq_1 = b + q_1 + 1 > 1$, which contradicts the fact that $aq_2 - bq_1 = \pm 1$. We thus conclude that $a \leqslant b$, as claimed.

Let r denote the number of arcs common to C_1 and C_2 (here we admit the possibility that $r = 0$). If $r < a$, then we can obtain the desired coloring as follows: color the r shared arcs color 1, the $a - r$ unshared arcs of C_1 color 1, the $b - r$ unshared arcs of C_2 color 1, and all other arcs color 2.

Suppose now that $r \geqslant a$. We claim that $l_2 - l_1 + 1 \geqslant b - a$. To see the claim, note that $aq_2 - bq_1 \geqslant -1$, so that $a(q_2 - q_1) \geqslant (b - a)q_1 - 1$, which yields $q_2 - q_1 \geqslant (b - a)(q_1/a) - (1/a) \geqslant b - a - 1$, the last inequality following from the fact that $1 \leqslant a \leqslant q_1$. If we have $q_2 - q_1 \geqslant b - a$, then certainly $l_2 - l_1 + 1 > q_2 - q_1 \geqslant b - a$. On the other hand, if $q_2 - q_1 = b - a - 1$, then note that $\pm 1 = aq_2 - bq_1 = a(q_1 + b - a - 1) - bq_1 = (a - b)(q_1 - a) - a$. Since $a \leqslant q_1$ and $a \leqslant b$, we thus have $aq_2 - bq_1 = \pm 1 = (a - b)(q_1 - a) - a \leqslant -a$. Evidently this is possible only if $a = 1$ and $aq_2 - bq_1 = -1$. Solving that last equation for b yields $b = (q_2 + 1)/q_1$. But then we have $l_2 - l_1 + 1 \geqslant q_2 - q_1 + 1 \geqslant (q_2 - q_1 + 1)/q_1 = b - a$. In either case, we see that $l_2 - l_1 + 1 \geqslant b - a$, as claimed.

From the claim, we see that since $r \leqslant l_1 - 1$, we have $b - a \leqslant l_2 - r$. We can now obtain the desired coloring as follows: color a of the shared arcs color 1, $b - a$ of the unshared arcs of C_2 color 1 (this is possible to do since $b - a \leqslant l_2 - r$), and all other arcs color 2.

Thus we see that in either case, it is possible to produce a coloring of C_1 and C_2 with the desired properties. □

Remark 2.1. The coloring produced in Lemma 2.1 does indeed use both colors. To see this fact, first note that since both a and b are positive, at least one arc in the graph is given color 1. Further, there are $l_1 - a$ arcs of C_1 given color 2, and $l_2 - b$ arcs of C_2 given color 2. If $l_1 - a$ is positive, then at least one arc is given color 2, and so both colors are used.

If $l_1 - a$ is 0, then q_1 must be 1, since it is a factor of $aq_2 - bq_1$, so that $l_1 = \gcd(l_1, l_2)$. Thus the equation $aq_2 - bq_1 = \pm 1$ yields $b = bq_1 = aq_2 \mp 1 = l_1q_2 \mp 1 = \gcd(l_1, l_2)q_2 \mp 1 = l_2 \mp 1$. Since b is the smallest positive integer satisfying $al_2 - bl_1 = \mp\gcd(l_1, l_2)$, we conclude that $b = l_2 - 1$. But then $l_2 - b$ is positive, and again, both colors are used.

Theorem 2.1. *If G is a primitive directed graph, then there is a 2-coloring of G that is 2-primitive.*

Proof. Suppose that G has k cycles $C_1, \ldots, C_k$, say of lengths $l_1, \ldots, l_k$. Since G is primitive, we have $\gcd(l_1, \ldots, l_k) = 1$. In particular, there are at least two distinct cycle lengths. We assume that those cycles are C_1 and C_2, with lengths l_1 and l_2, respectively; without loss of generality, we take $l_2 \geqslant l_1 + 1$.

Select the smallest positive integers a and b such that $|al_2 - bl_1| = \gcd(l_1, l_2) \equiv g$. Applying Lemma 2.1 to C_1 and C_2, we find that there is a coloring of those two cycles such that a arcs of C_1 have color 1, b arcs of C_2 have color 1, and the remaining arcs on C_1 and C_2 have color 2. We complete the coloring of G with colors 1 and 2 by letting any remaining arcs be colored arbitrarily.

That coloring leads us to the following matrix:

$$M = \begin{bmatrix} a & b & c_3 & \cdots & c_k \\ l_1 - a & l_2 - b & l_3 - c_3 & \cdots & l_k - c_k \end{bmatrix},$$

where $c_3, \ldots, c_k$ denote the numbers of arcs of color 1 on the cycles $C_3, \ldots, C_k$, respectively.

Let $d_{i,j} = \det M[1, 2 \mid i, j]$ and let $z = \gcd\{d_{i,j} \mid 1 \leqslant i < j \leqslant k\}$. Then $z \mid d_{1,2}$ so that $z \mid l_1$ and $z \mid l_2$ since $d_{1,2} = |al_2 - bl_1| = g$. Also, for any $i \geqslant 3$, $z \mid d_{1,i}$ so that $z \mid al_i - c_i l_1$. It follows that $z \mid al_i$ for all i. Further, $z \mid d_{2,i}$ for all $i \geqslant 3$ so that $z \mid bl_i$ for all i. Suppose that $z \neq 1$, and let p be a prime divisor of z. Since the cycle lengths are relatively prime, there is some j such that $p \nmid l_j$. It follows that $p \mid a$ and $p \mid b$. But then, $p \mid (a(l_2/g) - b(l_1/g))$ and since $a(l_2/g) - b(l_1/g) = \pm 1$, we have $p = 1$. We conclude that z must be 1. Thus we see that the collection of 2×2 minors of M has greatest common divisor 1, and so by Theorem 1.1, we see that our 2-coloring of G is 2-primitive. □

Example 2.1. We illustrate the coloring technique of Theorem 2.1 on the graph constructed as follows: begin with a directed path on 15 vertices, say $i \to i + 1$, $i = 1, \ldots, 14$, then add in the arcs $15 \to 1$, $10 \to 1$ and $6 \to 1$. The resulting graph G has three cycles: a 6-cycle on vertices $1, \ldots, 6$, a 10-cycle on vertices $1, \ldots, 10$, and a Hamilton cycle.

Consider the 6-cycle and the 10-cycle. The greatest common divisor of their lengths is 2, and in the notation on Lemma 2.1, we have $a = 1$ and $b = 2$. Color the arcs $1 \to 2$ and $9 \to 10$ with color 1, and color the arcs $i \to i + 1$, $i = 2, \ldots, 8$, $6 \to 1$ and $10 \to 1$ all with color 2. Finally, of the remaining arcs $i \to i + 1$, $i = 10, \ldots, 14$ and $15 \to 1$, fix some j between 0 and 6, and color j of them with color 1 and the rest with color 2.

This coloring leads us to the matrix

$$M = \begin{bmatrix} 1 & 2 & j + 2 \\ 5 & 8 & 13 - j \end{bmatrix},$$

whose minors are -2, $3 - 6j$ and $10 - 10j$. Since the first two minors are mutually prime, we find that our coloring of G is indeed 2-primitive.

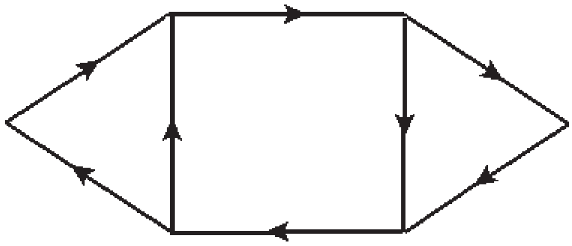

Fig. 1.

3. Examples and discussion for the case $k \geqslant 3$

By way of contrast with Theorem 2.1, in this section we give some examples to show that for each $k \geqslant 4$, there are primitive directed graphs having k cycles for which no k-coloring is k-primitive.

Our first example deals with the case $k = 4$.

Example 3.1. Consider the directed graph G shown in Fig. 1. Evidently G has exactly four cycles, of lengths 4, 5, 5, and 6. In particular G is primitive. Observe also that each arc of G is either contained in just two cycles, or in all four of the cycles. Suppose that we have a 4-coloring of G, say $(G_1, \ldots, G_4)$, and form the corresponding 4×4 matrix M of Theorem 1.1. From that result, the 4-coloring $(G_1, \ldots, G_4)$ is 4-primitive if and only if $\det(M) = \pm 1$. Since adding columns together does not affect the determinant, we see that if $\tilde{M}$ is the matrix formed from M by replacing its first column by the vector of row sums of M, then $\det(M) = \det(\tilde{M})$. For each arc a of G, let n_a denote the number of cycles in which a is contained. Letting r_i denote the ith row sum of M, we find that for each $i = 1, \ldots, 4$, $r_i = \sum_{a \in G_i} n_a$. In particular, since each n_a is even, we see that each entry in the first column of $\tilde{M}$ is divisible by 2. Consequently, $\det(M)$ is divisible by 2, from which we conclude that the 4-coloring of G is not 4-primitive.

Next, we use Example 3.1 to help in discussing the case $k \geqslant 5$.

Example 3.2. In this example, we use G of Fig. 1 to construct, for each $k \geqslant 5$, a primitive directed graph having k cycles for which no k-coloring is k-primitive. To do so, start with G, and at one of the vertices of degree 2, say v, attach $k - 4$ 2-cycles each having v as a vertex. The resulting graph H has k cycles, and $k + 2$ vertices. Label the cycles of H that are inherited from G by $C_1, \ldots, C_4$, and note that each arc of H which is inherited from G is contained in either two or four of the cycles $C_1, \ldots, C_4$.

Suppose now that we have a k-coloring of H, and construct the $k \times k$ matrix M of Theorem 1.1. Let $\tilde{M}$ be formed from M by replacing its first column by the sum of its first four columns (which correspond to the cycles $C_1, \ldots, C_4$). As in Example 3.1, we find that each entry in the first column of $\tilde{M}$ is divisible by 2, so that $\det(\tilde{M})$ is also divisible by 2. Since $\det(M) = \det(\tilde{M})$, we find from Theorem 1.1 that the k-coloring of H cannot be k-primitive.

Our last example provides a different kind of construction from that used above.

Example 3.3. Suppose that $n \geqslant 3$ is an odd integer and consider the directed graph D on n vertices labeled $1, \ldots, n$, with arcs $i \to i + 1 \pmod n$ and $i \to i - 1 \pmod n$, for $i = 1, \ldots, n$. Then we find that D has two cycles of length n and n cycles of length 2, for a total of $n + 2$ cycles. Note that D is primitive (since n and 2 are mutually prime) and that each arc of D is contained in exactly two cycles. It follows as in Example 3.1 that for any $n + 2$-coloring of D, the corresponding matrix M has each row sum divisible by 2. (Indeed the sum of row i is twice the number of arcs with color i.) We thus conclude that no $n + 2$-coloring of D is $n + 2$-primitive.

Next, we give a few partial results on the problem of determining which primitive directed graphs having at least three cycles can be 3-colored to yield a 3-primitive graph.

Theorem 3.1. *Suppose that G is a primitive directed graph having at least three cycles. Suppose that there is a nonempty subset of arcs of G, say A, such that $G' \equiv G \setminus A$ can be written as the union of a primitive directed graph and a (possibly empty) collection of isolated vertices. Then G admits a 3-coloring that is 3-primitive.*

Proof. Denote the primitive component of G' by H (possibly they are equal). By Theorem 2.1, there is a 2-coloring of H, say using colors 1 and 2, that is 2-primitive. We now complete this to a 3-coloring of G by selecting a single arc $e \in A$ and giving it color 3, while coloring the arcs of $A \setminus e$ arbitrarily with colors 1 and 2.

We claim that this 3-coloring of G is 3-primitive. To see this, form the matrix M of Theorem 1.1 for the coloring of G. Let M_H denote the corresponding matrix arising from our 2-coloring of H. Observe that M contains a submatrix of the following form:

$$S = \left[\begin{array}{ccc|c} & M_H & & x \\ \hline 0 & \ldots & 0 & 1 \end{array}\right],$$

where the last column corresponds to a cycle containing the arc e. Observe that each 2×2 minor of M_H is equal to a 3×3 minor of S, and since those 2×2 minors are relatively prime, we conclude that the 3×3 minors of S (and hence of M) are relatively prime. Thus our coloring is 3-primitive. □

The following result is similar in spirit to that above.

Theorem 3.2. *Let G be a primitive directed graph with at least three cycles. Suppose that there are two cycles C_1, C_2 with lengths l_1, l_2, respectively, such that l_1*

and l_2 are relatively prime, and such that there is at least one arc of G not contained in $C_1 \cup C_2$. Then G admits a 3-coloring that is 3-primitive.

Proof. Let a and b be the smallest positive integers such that $al_2 - bl_1 = \pm 1$. By Lemma 2.1 there is a coloring of $C_1 \cup C_2$ that gives a arcs of C_1 and b arcs of C_2 color 1, and the rest color 2. Since there is an arc e of G (say on cycle C_3) that is not contained in $C_1 \cup C_2$, we can give e color 3, and color the remaining arcs of G arbitrarily with colors 1, 2 and 3. As in Theorem 3.1, we find that the matrix M corresponding to our coloring has a 3×3 submatrix of the following form:

$$S = \begin{bmatrix} a & b & c \\ l_1 - a & l_2 - b & l_3 - c - 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\det(S) = \pm 1$, we conclude that our 3-coloring is 3-primitive. $\square$

Example 3.4. Observe that neither Theorem 3.1 nor Theorem 3.2 applies to the graph G of Example 2.1, since that graph has no primitive subgraphs, and every pair of cycle lengths has a proper divisor. Nevertheless, we can produce a 3-coloring of G that is 3-primitive. For example, give the arc $9 \to 10$ color 1, give the arcs $6 \to 1$, $10 \to 1$, $14 \to 15$ and $15 \to 1$ color 2, and give all remaining arcs color 3. The resulting matrix M of Theorem 1.1 is

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 5 & 8 & 12 \end{bmatrix}.$$

Since $\det(M) = 1$, we see that the coloring is 3-primitive.

Informed by our work in Section 2, and by Example 3.4, we formulate the following.

Conjecture 3.1. *Suppose that G is a primitive directed graph with cycles C_1, C_2 and C_3 whose lengths are l_1, l_2 and l_3, respectively. Let the greatest common divisor of l_1, l_2 and l_3 be g. Then there is a 3-coloring of G with corresponding matrix M such that*

$$M[1, 2, 3 \mid 1, 2, 3] = \begin{bmatrix} a & b & c \\ x & y & z \\ l_1 - a - x & l_2 - b - y & l_3 - c - z \end{bmatrix}$$

has determinant $\pm g$.

Evidently the confirmation of this conjecture would be a key step in proving that every primitive directed graph with at least three cycles admits a 3-coloring that is 3-primitive.

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