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Compactness and finite dimension in asymmetric normed linear spaces ☆

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Abstract

We describe the compact sets of any asymmetric normed linear space. After that, we focus our attention in finite dimensional asymmetric normed linear spaces. In this case we establish the equivalence between T_1 separation axiom and normable spaces. It is proved an asymmetric version of the Riesz Theorem about the compactness of the unit ball. We also prove that the Heine–Borel Theorem characterizes finite dimensional asymmetric normed linear spaces that satisfies the T_2 separation axiom. Finally we focus our attention on the T_0 separation axiom and results that are related to the dual p-complexity spaces.

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1. Introduction

Let E be a linear space on the set of real numbers $\mathbb{R}$. We say that a function $q : E \to \mathbb{R}^+$ (where $\mathbb{R}^+$ is the set of nonnegative real numbers) is an *asymmetric norm* on E if for all $x, y \in E$ and $a \in \mathbb{R}^+$:

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(i) $q(x) = q(-x) = 0$ if and only if $x = 0$.
(ii) $q(ax) = aq(x)$.
(iii) $q(x + y) \leqslant q(x) + q(y)$.

Asymmetric norms are called quasi-norms in [4,1,9].

The function $q^{-1}: E \to \mathbb{R}^+$ defined by $q^{-1}(x) := q(-x)$ is also an asymmetric norm. The function $q^s: E \to \mathbb{R}^+$ given by the formula $q^s(x) := \max\{q(x), q^{-1}(x)\}$ is a norm on E.

An asymmetric norm q induces a quasi-metric d_q by mean of the formula

$$d_q(x, y) = q(y - x), \quad x, y \in E.$$

If d_q is a bicomplete quasi-metric on E, then (E, q) is called a biBanach space [10]. Each quasi-metric d on E generates a T_0 topology τ_{d_q} on E, for which the basic open sets are the balls $B_d(x, r) = \{y \in E\colon d(x, y) < r\}$. Hence if q is an asymmetric norm on E, the sets

$$B_\varepsilon(0) := \{x \in E\colon q(x) < \varepsilon\}, \quad \varepsilon > 0,$$

form a fundamental system of neighborhoods of zero for the topology generated by d_q. In the same way the translations of these sets $B_\varepsilon(y) := y + B_\varepsilon(0)$, define a fundamental system of neighborhoods of y for all $y \in E$. Note from the definition that $B_\varepsilon(y) = B_{d_q}(y, \varepsilon)$.

We denote by $\overline{B_\varepsilon}(0)$ the sets

$$\overline{B_\varepsilon}(0) := \{x \in E\colon q(x) \leqslant \varepsilon\}, \quad \varepsilon > 0.$$

When necessary, we will indicate the space as a subscript, together with the radius separated by one coma, and the used asymmetric norm as a superscript.

The pair (E, q) is called an *asymmetric normed linear space*.

The complexity (quasi-metric) space was introduced by Schellekens [12] in order to develop a topological foundation for the complexity analysis of programs and algorithms, based on the notion of a "complexity distance", i.e., a quasi-metric which intuitively measures relative improvements in the complexity of programs and algorithms. The complexity space accepts, among others, many important kinds of exponential time algorithms. In particular, some applications of this theory to the complexity analysis of Divide and Conquer algorithms were given in [12].

Later on, it was introduced in [11] the so-called dual complexity (quasi-metric) space, to discuss in a more handy context several quasi-metric properties of the complexity space which are interesting from a computational point of view. In fact, while the complexity space cannot be modelled as a quasi-normed cone, the dual space admits a structure of quasi-normed (asymmetric normed, in our terminology) semilinear space [10] (in our context, a semilinear space is a cone in the sense of Keimel and Roth [8]) and, by other hand, it can be directly used for the complexity analysis of certain algorithms, where the running time of computing is the complexity measure (compare [12, Section 4] and [11, p. 313]).

Motivated by the fact that, in this dual context, the complexity analysis of algorithms with running time $\mathcal{O}(2^n/n^r)$, $0 < r \leqslant 1$, cannot be performed via the dual complexity space, the authors have recently introduced [6] the so-called dual p-complexity space ($p \geqslant 1$), which provides, for $p > 1$, an appropriate framework to discuss complexity functions generating this kind of algorithms. In particular, it was shown that the dual p-complexity space is an asymmetric normed semilinear space which is isometrically isomorphic to the positive cone of $(l_p, \|\cdot\|_{+p})$ (see Section 2 in [6] for definitions and details). In order to obtain a general theory of dual complexity it was introduced in [6] the following class of spaces. For each $p \in [1, \infty)$ set $\mathcal{B}_p^* := \{f \in \mathbb{R}^\omega\colon \sum_{n=0}^{\infty}(2^{-n}|f(n)|)^p < \infty\}$, where ω are the nonnegative integers numbers. If for any $f, g \in \mathcal{B}_p^*$ and $a \in \mathbb{R}$ we define $f + g$ and $a \cdot f$ in the usual pointwise way, then it easily follows that $(\mathcal{B}_p^*, +, \cdot)$ is a linear space.

Now denote by q_p the nonnegative real valued function defined on $\mathcal{B}_p^*$ by $q_p(f) = (\sum_{n=0}^{\infty}(2^{-n} f(n)^+)^p)^{1/p}$, where $f(n)^+$ are the nonnegative values $(f(n) \vee 0)$. Then $(\mathcal{B}_p^*, q_p)$ is a biBanach space. In Corollary 4 of [6] it is shown that $(\mathcal{B}_p^*, q_p)$ and $(l_p, \|\cdot\|_{+p})$, with $\|\mathbf{x}\|_{+p} = (\sum_{n=0}^{\infty}(x_n^+)^p)^{1/p}$, are isometrically isomorphic. For each $p \in [1, \infty)$ let $\mathcal{C}_p^* = \{f \in \mathcal{B}_p^*\colon\ f(n) \geqslant 0 \text{ for all } n \in \omega\}$. Following [6], the asymmetric normed semilinear space $(\mathcal{C}_p^*, q_p)$ will be called the dual p-complexity space, where the restriction of q_p to $\mathcal{C}_p^*$ is also denoted by q_p. If we denote by l_p^+ the positive cone of l_p, in [6] is proved that $(\mathcal{C}_p^*, q_p)$ and $(l_p^+, \|\cdot\|_{+p})$ are isometrically isomorphic.

Observe that the quasi-metric d_{q_p} induced on $\mathcal{C}_p^*$ by q_p is given by

$$d_{q_p}(f, g) = \left(\sum_{n=0}^{\infty} 2^{-pn}\big(\big(g(n) - f(n)\big) \vee 0\big)^p\right)^{1/p}.$$

In particular $(\mathcal{C}_1^*, d_{q_1})$ is exactly the dual complexity space as defined in [11].

The computational interpretation [11] of the complexity distance d_{q_1} remains valid for each quasi-metric d_{q_p} [6]. Thus, the fact that $d_{q_p}(f, g) = 0$, with $f \neq g$, can be interpreted as g is more efficient than f. Furthermore $q_p(f)$ measures relative progress made in lowering complexity by replacing f by the "optimal" complexity function 0, assuming that the complexity measure is the running time of computing.

On the other hand, there is in the last years a renewed interest in automata of infinite objects due to their intimate relation with temporal and modal logics of programs. Thus, Emerson and Jutla [3] have successfully applied complexity of tree automata to obtain optimal deterministic exponential time algorithms in some important modal logics of programs, where by an exponential time algorithm we mean an algorithm with running time $\mathcal{O}(2^{P(n)})$, such that $P(n)$ is a polynomial with $P(n) > 0$ for all n. This running time corresponds to the function f given by $f(n) = 2^{P(n)}$ for all n, which does not belong to any dual p-complexity space whenever $P(n) \geqslant n$. The authors have introduced [7] the notion of the $\sup_{P(n)}$-complexity space in order to discuss complexity functions generating this kind of algorithms. In this case we work with the Banach space $(l_\infty, \|\cdot\|_\infty)$ where l_∞ is the bounded sequences space and we define the usual multiplicative operation between sequences from l_∞.

For each polynomial $P(n)$, with $P(n) > 0$ for all $n \in \omega$, we define $\mathcal{B}_{P(n),\infty}^* := \{f \in \mathbb{R}^\omega\colon \sup\{2^{-P(n)}|f(n)|\colon n \in \omega\} < \infty\}$. It easily follows that $\mathcal{B}_{P(n),\infty}^*$ is a linear space for the usual pointwise operations.

The function $q_{P(n),\infty}$ defines on $\mathcal{B}_{P(n),\infty}^*$ as $q_{P(n),\infty}(f) = \sup\{2^{-P(n)} f(n)^+\colon n \in \omega\}$ is an asymmetric norm.

For each polynomial $P(n)$, with $P(n) > 0$ for all $n \in \omega$, consider the biBanach space $(\mathcal{B}^*_{P(n),\infty}, q_{P(n),\infty})$ and let

$$\mathcal{C}^*_{P(n),\infty} := \{f \in \mathcal{B}^*_{P(n),\infty}\colon\ f(n) \geqslant 0 \text{ for all } n \in \omega\}.$$

Let $q_{P(n),\infty}$ the restriction of the asymmetric norm $q_{P(n),\infty}$ defined on $\mathcal{B}^*_{P(n),\infty}$ to $\mathcal{C}^*_{P(n),\infty}$. Then, $\mathcal{C}^*_{P(n),\infty}$ is an asymmetric normed semilinear space.

Observe that in any case, the unit ball $\overline{B}_1(0)$ of these spaces is not a bounded set for any norm, in particular, they are unbounded sets for the corresponding q_p or $q_{P(n)}$.

The aim of this paper is to extend the results about compact sets on finite dimensional normed spaces to the case of asymmetric normed linear spaces. In Section 2 we introduce the set theoretical arguments that allows to a general description of compact sets of an asymmetric normed linear space. In Section 3, we focus our attention in the finite dimensional case to reproduce the classical results of the normed spaces theory. In particular, we prove that a T_1 asymmetric normed linear space (E, q) is finite dimensional if and only if the unit ball $\overline{B}^q_1(0)$ is compact for the topology generated by the asymmetric norm (Theorem 13). This will be done via the compactness of $\overline{B}^{q^s}_1(0)$ in the supremum norm q^s. In fact, we will prove the equivalence between T_1 and normability of the spaces in the case of finite dimension and thus between T_1 and T_2 separation axioms. The T_2 separation axiom in the general case of asymmetric normed linear spaces has been studied in [5]. We also prove that the Heine–Borel Theorem characterizes finite dimensional asymmetric normed linear spaces that satisfies T_2 axiom (Theorem 15).

In this Section 3, we pay also attention to T_0 spaces and, in particular several properties in relation with the dual p-complexity spaces are discussed.

Basic references about quasi-metrics and asymmetric norms are [13,1,9,4,11].

Definitions and basic results on general topology can be found in [2].

2. Compact sets in asymmetric normed linear spaces

In this section we describe the compact sets of any asymmetric normed linear space. In particular, given a compact set in (E, q^s), we give a way to construct compact sets in (E, q) for the topology induced by q.

Definition 1. Let (E, q) be an asymmetric normed linear space and $x \in E$. We define the set $\theta(x)$ as:

$$\theta(x) = \{z \in E\colon\ d_q(x, z) = q(z - x) = 0\}.$$

In particular

$$\theta(0) = \{z \in E\colon\ d_q(0, z) = q(z) = 0\}.$$

Observe that $\theta(x)$ is the closure of $\{x\}$ in (E, q^{-1}).

Lemma 2. *Given a set $A \subset E$ of an asymmetric normed linear space (E, q), we have that*

$$\bigcup_{x \in A} \theta(x) = A + \theta(0),$$

where

$$A + \theta(0) = \{z \in E\colon\ z = x + y,\ x \in A \text{ and } y \in \theta(0)\}.$$

Proof. Let $z \in \bigcup_{x\in A} \theta(x)$. Then there exists an $x \in A$ such that $q(z-x) = 0$. This implies that $z - x = y$, $y \in \theta(0)$ and then we can express z as $z = x + y$. Thus $\bigcup_{x\in A} \theta(x) \subset A + \theta(0)$.

Conversely, let $w \in A + \theta(0)$. There exists an $x \in A$ and an element $y \in \theta(0)$ such that $w = x + y$ and also $w - x = y$. Then $q(w - x) = q(y) = 0$ so $w \in \theta(x)$ and $w \in \bigcup_{x\in A} \theta(x)$. This implies that $A + \theta(0) \subset \bigcup_{x\in A} \theta(x)$. $\square$

Lemma 3. *Let (E, q) be an asymmetric normed linear space and $x \in E$. Then*

$$B_\varepsilon(x) = B_\varepsilon(x) + \theta(0).$$

Proof. $B_\varepsilon(x) \subset B_\varepsilon(x) + \theta(0)$ since $0 \in \theta(0)$ and every $x \in B_\varepsilon(x)$ can be written as $x = x + 0$.

Let $z \in B_\varepsilon(x) + \theta(0)$. Then there exists an $y \in B_\varepsilon(x)$ and $w \in \theta(0)$ such that $z = y + w$. Then

$$q(z - x) = q(y + w - x) \leqslant q(y - x) + q(w) < \varepsilon + 0 = \varepsilon.$$

As a consequence, $z \in B_\varepsilon(x)$ and $B_\varepsilon(x) + \theta(0) \subset B_\varepsilon(x)$. $\square$

Lemma 4. *Let (E, q) be an asymmetric normed linear space and $A \subset E$ an open set. Then*

$$A = A + \theta(0).$$

Proof. It is obvious that $A \subset A + \theta(0)$.

Let $z \in A + \theta(0)$. Then we can express z as $z = x + y$ where x is in A and y is an element of $\theta(0)$. But A is an open set. Consequently, there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A$. Taking into account that $B_\varepsilon(x) = B_\varepsilon(x) + \theta(0)$ by the preceding lemma, it can be concluded that z is in A. $\square$

Lemma 5. *Given a family $\{A_i\colon\ i \in I\}$ of sets in (E, q), then*

$$\bigcup_{i\in I} \big(A_i + \theta(0)\big) = \left(\bigcup_{i\in I} A_i\right) + \theta(0).$$

Proof. If $x \in \bigcup_{i\in I}(A_i + \theta(0))$ that means that there exists some i satisfying that $x \in A_i + \theta(0)$ such that x can be written as $x = x_i + z$ with $x_i \in A_i$ and $z \in \theta(0)$. Then $x_i \in \bigcup_{i\in I} A_i$ and $x = x_i + z$ is in $(\bigcup_{i\in I} A_i) + \theta(0)$.

If $x \in (\bigcup_{i\in I} A_i) + \theta(0)$ there exists an $x_i \in A_i$ and $z \in \theta(0)$ such that $x = x_i + z$ and then x is in $\bigcup_{i\in I}(A_i + \theta(0))$. $\square$

Proposition 6. *Let (E, q) be an asymmetric normed linear space and $K \subset E$. Then K is compact respect to the topology induced by the asymmetric norm τ_{d_q} if and only if $K + \theta(0)$ is compact for the same topology.*

Proof. (⇒) Let $\{A_i\colon i \in I\}$ an open cover of K. By Lemma 4 we have that

$$A_i = A_i + \theta(0).$$

Then by Lemma 5

$$K + \theta(0) \subset \bigcup_{i\in I} A_i + \theta(0).$$

But K is compact and there exists a finit subcover of K, $\{A_j\colon j \in J \subset I\}$ such that $K \subset \bigcup_{j\in J} A_j$. Then applying the same Lemma 5 we obtain that $K + \theta(0) \subset \bigcup_{j\in J}(A_j + \theta(0))$. This implies that $K + \theta(0)$ admits a finit subcover $\{A_j + \theta(0)\colon j \in J \subset I\}$ and $K + \theta(0)$ is a compact set.

(⇐) If $K + \theta(0)$ is compact, given an open cover of the set K, $\{A_i\colon i \in I\}$, the family $\{A_i + \theta(0)\colon i \in I\}$ is an open cover of $K + \theta(0)$ and this set admits a finit subcover $\{A_j + \theta(0)\colon j \in J \subset I\}$. Then by Lemma 5 $K + \theta(0) \subset \bigcup_{j\in J} A_j + \theta(0)$ that implies $K \subset \bigcup_{j\in J} A_j$ and then $\{A_j\colon j \in J \subset I\}$ is a subcover of K obtained from the open cover $\{A_i\colon i \in I\}$. Thus, K is compact. □

Corollary 7. *Given a $K_0 \subset K + \theta(0)$, if $K + \theta(0)$ is a compact set and $K_0 + \theta(0) = K + \theta(0)$ then K_0 is also compact.*

Note that if K is a compact set of (E, q^s), then $K + \theta(0)$ is a compact set of (E, q).

3. Finite dimensional asymmetric normed linear spaces

Let (E, q) an asymmetric normed linear space endowed with the topology τ_{d_q} defined above. A set $M \subset E$ is said to be compact if it is compact considered as a subspace of E with the induced topology, that is, (M, q) is compact with respect to the topology $\tau_{d_q}|_M$. A set M of E is compact if every sequence in M has a convergent subsequence whose limit is in M.

Lemma 8. *Let $(e, \|\cdot\|)$ be a finite dimensional normed linear space, with base $\{e_1, e_2, \ldots, e_n\}$. Then, a sequence $(x_k)_{k\in\mathbb{N}}$ in E converges to $x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$ if and only if the i-co-ordinate sequence of $(x_k)_{k\in\mathbb{N}}$ converges to λ_i, with respect to the Euclidean norm, $i = 1, \ldots, n$.*

We generalize this classical result to asymmetric normed linear spaces as follows.

Theorem 9. *Let (E, q) be a finite dimensional T_1 asymmetric normed linear space, with base $\{e_1, e_2, \ldots, e_n\}$. Then, a sequence $(x_k)_{k\in\mathbb{N}}$ in E converges to $x = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$ with respect to q if and only if the i-co-ordinate sequence of $(x_k)_{k\in\mathbb{N}}$ converges to λ_i, with respect to the Euclidean norm, $i = 1, \ldots, n$.*

Proof. First suppose that the i-co-ordinate sequence of $(x_k)_{k\in\mathbb{N}}$ converges to λ_i, with respect to the Euclidean norm, $i = 1, \ldots, n$. Given a positive real number $M > 0$ and an $\frac{\varepsilon}{n}$ there is a k_0^i such that when $k \geqslant k_0^i$ then

$$\left|(x_k)_i - \lambda_i\right| < \frac{\varepsilon}{nM}.$$

Let be $k_0 = \max\{k_0^i,\ i = 1, \ldots, n\}$. Then, if $k \geqslant k_0$,

$$q(x_k - x) \leqslant \sum_{i=1}^{n} q\big((x_k)_i - \lambda_i\big) \leqslant \sum_{i=1}^{n} q^s\big((x_k)_i - \lambda_i\big) \leqslant \sum_{i=1}^{n} M\left|(x_k)_i - \lambda_i\right| \leqslant \varepsilon,$$

where we have used the fact that q^s is a norm and equivalent to the Euclidean norm with constant M.

Suppose now that $(x_k)_{k\in\mathbb{N}}$ is a sequence in E that converges to 0 with respect to q (if $(x_k)_{k\in\mathbb{N}}$ converges to x respect to q, the sequence $(x_k - x)_{k\in\mathbb{N}}$ converges to 0), but for some $n_0 \in 1, \ldots, n$ the co-ordinate sequence $((\lambda_k)_{n_0})_{k\in\mathbb{N}}$ is not convergent to 0 with respect to the Euclidean norm, where

$$x_k = (\lambda_k)_1 e_1 + (\lambda_k)_2 e_2 + \cdots + (\lambda_k)_n e_n$$

for each $k \in \mathbb{N}$.

We may assume that there is $r > 0$ such that $|(\lambda_k)_{n_0}| > r$ for all $k \in \mathbb{N}$.

For each $k \in \mathbb{N}$ put $M_k = \max\{|(\lambda_k)_i|,\ i = 1, \ldots, n\}$. Define a sequence $(y_k)_{k\in\mathbb{N}}$ by $y_k = x_k / M_k$ $\forall k \in \mathbb{N}$. Then

$$q(y_k) = \frac{q(x_k)}{M_k} < \frac{q(x_k)}{r},$$

for all $k \in \mathbb{N}$, so $(y_k)_{k\in\mathbb{N}}$ converges to 0 with respect to q.

Now observe that there exists a co-ordinate sequence of $(y_k)_{k\in\mathbb{N}}$ that has a co-ordinate subsequence which consist only of terms -1 or 1. Denote this subsequence by $((\lambda_{k_j})_m)_{j\in\mathbb{N}}$ where $m \in \{1, \ldots, n\}$. Consider the corresponding subsequence $(y_{k_j})_{j\in\mathbb{N}}$ of $(y_k)_{k\in\mathbb{N}}$ and its first co-ordinate sequence $((\lambda_{k_j})_1)_{j\in\mathbb{N}}$. Then $((\lambda_{k_j})_1)_{j\in\mathbb{N}}$ has a convergent subsequence. Continuing this process to the nth co-ordinate sequence, we obtain a subsequence $(y_{k_l})_{l\in\mathbb{N}}$ of $(y_k)_{k\in\mathbb{N}}$ which has each co-ordinate sequence convergent but the mth co-ordinate subsequence consist only of terms -1 or 1. So by the preceding lemma $(y_{k_l})_{l\in\mathbb{N}}$ converges to a point $y \neq 0$ with respect to the norm q^s. Since $q(y) \leqslant q(y - y_{k_l}) + q(y_{k_l})$ $\forall l \in \mathbb{N}$, it follows that $q(y) = 0$ so $y = 0$, a contradiction.

We conclude that each co-ordinate sequence $((\lambda_k)_i)_{k\in\mathbb{N}}$ converges for $i = 1, \ldots, n$.

Finally, if the sequence $(x_k)_{k\in\mathbb{N}}$ converges to x with respect to q, then the sequence $(x_k - x)_{k\in\mathbb{N}}$ converges to 0 with respect to q. So the i-co-ordinate sequence $((x_k)_i - (x)_i)_{k\in\mathbb{N}}$ converges to 0. Hence the i-co-ordinate sequence $((x_k)_i)_{k\in\mathbb{N}}$ converges to the i-co-ordinate $(x)_i$. This concludes the proof. □

Definition 10. An asymmetric normed linear space (E, q) is called normable if there is a norm $\|\cdot\|$ on the linear space E such that the topologies τ_{d_q} and $\tau_{d_{\|\cdot\|}}$ coincide on E.

Corollary 11. *Let (E, q) be a finite dimensional T_1 asymmetric normed linear space. Then (E, q) is normable by the norm q^s.*

Proof. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in E that converges to a point x with respect to a q. By Theorem 9 and Lemma 8, $(x_k)_{k\in\mathbb{N}}$ converges to x with respect to the norm q^s. □

In particular observe that, because of Corollary 11, T_1 separation axiom implies T_2 separation axiom in the finite dimensional case.

Lemma 12. *Let (E,q) be an asymmetric normed linear space and τ_{d_q} the topology generated by the quasi-metric d_q. Then τ_{d_q} is T_1 if and only if $q(y)\neq 0$, for each $y\in E\backslash\{0\}$.*

Theorem 13. *The unit ball $\overline{B}_1^q(0)$ of a T_1 asymmetric normed linear space (E,q) is compact if and only if it is finite dimensional.*

Proof. Suppose firstly that $\overline{B}_1^q(0)$ is a compact set of (E,q). Then $\overline{B}_1^q(0)$ is compact in (E,q^s) by the preceding corollary. Since $B_1^{q^s}(0)\subset\overline{B}_1^q(0)$ and $B_1^{q^s}(0)$ is closed in (E,q^s) it follows that $B_1^{q^s}(0)$ is compact in (E,q^s). Hence (E,q^s), and thus (E,q), are finite dimensional.

Conversely, let $\{e_1,e_2,\dots,e_n\}$ be a base of (E,q). For each $x\in E$ set

$$x=\lambda_1(x)e_1+\lambda_2(x)e_2+\cdots+\lambda_n(x)e_n.$$

Thus we have defined n functions $\lambda_i:E\to\mathbb{R}$, which are clearly linear functions on E.

By Theorem 9, each λ_i is continuous from (E,q) to $\mathbb{R}$ endowed with the Euclidean norm, so there exists n constants $M_i>0$, $M_i\in\mathbb{R}$, $i=1,\dots,n$, such that

$$\left|\lambda_i(x)\right|\leqslant M_i q(x),\quad i=1,\dots,n,\ \forall x\in E.$$

Now let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\overline{B}_1^q(0)$. Then $|\lambda_i(x_k)|\leqslant M_i$, $i=1,\dots,n$, $k\in\mathbb{N}$. Hence, the first co-ordinate sequence $(\lambda_1(x_k))_{k\in\mathbb{N}}$ has a convergent subsequence. The corresponding co-ordinate sequence $(\lambda_2(x_k))_{k\in\mathbb{N}}$ has also a convergent subsequence. Continuing this process, we obtain a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$, which has each co-ordinate sequence convergent. Therefore $(x_{k_j})_{j\in\mathbb{N}}$ converges to some $y\in E$ with respect to the norm q^s by Theorem 9. Since $q(x_{k_j})\leqslant 1$ and $q(y)-q(x_{k_j})\leqslant q(y-x_{k_j})$ $\forall j\in\mathbb{N}$, it follows that $q(y)\leqslant 1$. We conclude that $\overline{B}_1^q(0)$ is a compact set of the normed space (E,q^s) and by the preceding corollary it is a compact set of (E,q). □

Remark 14. The above proof is doing following the customary scheme but there is an straightforward argument to deduce the result from classical theorems. This comes from the observation that all asymmetric norms on a T_1 finite dimensional linear space are equivalent. Lemma 12 shows that an asymmetric normed linear space is T_1 if and only if $q(x)\neq 0$ for all $x\in E\backslash\{0\}$. Let (E,q) be a finite dimensional asymmetric normed linear space and q^s the supremum norm as usual. Then the restriction of q to $\{x\in E\colon q^s(x)=1\}$ does not attain zero because q is a continuous function in (E,q^s). Thus, it is bounded below, and so q and q^s are equivalent.

Theorem 15. *Let (E,q) a finite dimensional asymmetric normed linear space. Then (E,q) is normable if and only if each compact set is closed.*

Proof. Suppose that (E, q) is not normable. Then it is not Hausdorff by Corollary 11, so there exist a sequence $(x_n)_{n\in\mathbb{N}}$ in E and two points $x, y \in E$ with $x \neq y$ such that $x_n \to x$ and $x_n \to y$ with respect to the topology τ_{d_q}. Since $K = \{x\} \cup \{x_n \colon n \in \mathbb{N}\}$ is compact in (E, q) and $y \in \overline{K} - K$, K cannot be closed.

The converse is well known. □

Note that, in a finite dimensional linear space, every compact set is bounded and hence this theorem provides a version of the Heine–Borel Theorem for asymmetric normed linear spaces.

Definition 16. Let (X, q) be an asymmetric normed linear space. We say that $\overline{B}_1^q(0)$ is right-bounded if there exists a real constant $r > 0$, such that

$$r\overline{B}_1^q(0) \subset \overline{B}_1^{q^s}(0) + \theta(0).$$

Proposition 17. *Let (E, q) be a finite dimensional asymmetric normed linear space such that $\overline{B}_1^q(0)$ is right-bounded . Then $\overline{B}_1^q(0)$ is compact.*

Proof. $\overline{B}_1^{q^s}(0)$ is the unit ball of the normed space (E, q^s). Since E is finite dimensional, $\overline{B}_1^{q^s}(0)$ is compact. Let $\{A_i,\ i \in I\}$ be an open cover of $\overline{B}_1^q(0)$ in τ_{d_q}. Since $\overline{B}_1^{q^s}(0) \subset \overline{B}_1^{q^s}(0) + \theta(0) \subset \overline{B}_1^q(0)$, then $\{\overline{B}_1^{q^s}(0) \cap A_i,\ i \in I\}$ is an open cover of $\overline{B}_1^{q^s}(0)$ in $\tau_{d_{q^s}}|_{\overline{B}_1^{q^s}(0)}$. There exists a finite subcover $\{\overline{B}_1^{q^s}(0) \cap A_j,\ j = 1, \ldots, n\}$ of $\overline{B}_1^{q^s}(0)$ in $\tau_{d_{q^s}}|_{\overline{B}_1^{q^s}(0)}$. Then $\overline{B}_1^{q^s}(0) + \theta(0) \subset \bigcup_{j=1}^n (\overline{B}_1^{q^s}(0) \cap A_j) + \theta(0) \subset \bigcup_{j=1}^n A_j + \theta(0)$. But $\overline{B}_1^q(0)$ is right-bounded, so $r\overline{B}_1^q(0) \subset \bigcup_{j=1}^n A_j + \theta(0) \subset \bigcup_{j=1}^n A_j$ by Lemma 4. Then $r\overline{B}_1^q(0)$ is compact. Taking into account that the function $f(x) = rx$ is continuous for the topology τ_{d_q}, it is obvious that $\overline{B}_1^q(0)$ is compact. □

Lemma 18. $\overline{B}_1^{q^s}(0) + \theta(0) \subset \overline{B}_1^q(0)$.

Proof. Let $g \in \overline{B}_1^{q^s}(0) + \theta(0)$. Then we can write $g = f_1 + f_2$ such that $f_1 \in \overline{B}_1^{q^s}(0)$ and $f_2 \in \theta(0)$. Then $q(g) \leqslant q(f_1) + q(f_2) = q(f_1) \leqslant q^s(f_1) \leqslant 1$. Then $g \in \overline{B}_1^q(0)$. □

Lemma 19. *Consider the asymmetric normed linear spaces $(\mathcal{B}_p^*, q_p)$ for every $p \in [1, +\infty)$. Then $\overline{B}_{B_p^*,1}^{q_p}(0) \subset \overline{B}_{B_p^*,1}^{q_p^s}(0) + \theta(0)$.*

Proof. Let $f \in \overline{B}_{B_p^*,1}^{q_p}(0)$. Then we can split f as a $f = (f \vee 0) + (f \wedge 0)$. It is easy to see that $q_p^s(f \vee 0) = q_p(f) \leqslant 1$ and $q_p(f \wedge 0) = 0$ that implies that $(f \vee 0) \in \overline{B}_{B_p^*,1}^{q_p^s}(0)$ and $(f \wedge 0) \in \theta(0)$. Thus, $f \in \overline{B}_{B_p^*,1}^{q_p^s}(0) + \theta(0)$. □

Corollary 20. *$\overline{B}_{B_p^*}^{q_p}(0)$ is right-bounded and $\overline{B}_{B_p^*}^{q_p}(0) = \overline{B}_{B_p^*}^{q_p^s}(0) + \theta(0)$.*

Proof. The proof is a direct consequence of Lemmas 18 and 19. □

These results can be extended to the asymmetric normed linear spaces $(\mathcal{B}^*_{P(n),\infty}, q_{P(n),\infty})$ and $P(n) > 0$ for all n, taking into account that, in finite dimensional case, all asymmetric norms are equivalent.

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