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NOTES

Parity Patterns on Even Triangulated Polygons

Communicated by F. Harary

By a triangulated polygon (TP) is meant an elementary cycle in the plane, completely triangulated on one side, and with no points or lines on the other side. The triangulated side may or may not contain points in addition to points of the cycle. If all points not points of the cycle are of even degree, the TP will be called even. If there are no points other than points of the cycle, the TP is vacuously even. The graphs in Figure 1 are all examples of even triangulated polygons.

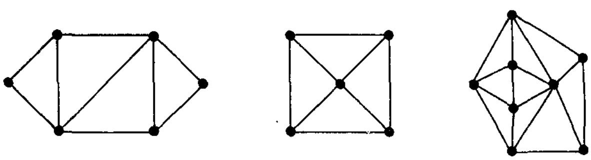

FIGURE 1

For any point x of a graph, $p(x)$ (parity of x) is defined as

$$p(x) = e, \quad \text{degree of } x \text{ even,}$$
$$p(x) = 0, \quad \text{degree of } x \text{ odd.}$$

For any cycle C, the corresponding cycle of $p(x)'s$, $x \in C$, will be called the parity-pattern $P(C)$ of the cycle.

THEOREM. *For any even* TP, *G, with bounding cycle C and parity-pattern $P(C)$, there exists an even* TP, *G', with bounding cycle C' of the same length as C with $P(C) = P(C')$ and G' contains at most one point not in C'.*

PROOF: By induction on the number of points n in C.

(1) The consequence holds for $n = 3$, since for the triangle there is only one possible P. Either all points in C are even or two are odd, since the number of odd points must be even. But it can be shown that, in a fully triangulated graph with all even nodes but two, these two cannot be adjacent.[1] Therefore the only P on C_3 is the case $p(x) = e$ for all points. This P is identical with that of a simple triangle.

(2) Assume the theorem holds for C_n. Consider a TP, G, with bounding cycle C_{n+1}.

CASE 1. $n+1$ is odd. In this case it is impossible that $p(x) = 0$ for all $x \in C$. Hence, there is at least one $x \in C$, $p(x) = e$. Let y, z be the two neighbors of x in C. If there is a line between y and z, then y and z are the only neighbors of x. Construct G' by eliminating x. If there is no line between y and z, construct G' with bounding cycle C' by adding the line (y, z) to G (see Fig. 2). Either construction produces a new even

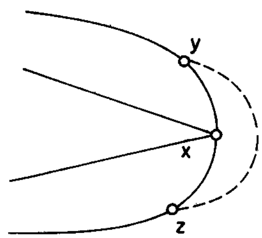

FIGURE 2

TP with a bounding cycle of n points. By hypothesis, there is a TP, G'', with a bounding cycle C'', $P(C'') = P(C')$ and G'' contains at most one point not in C''.

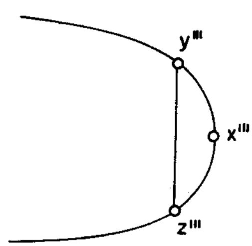

FIGURE 3

[1] This theorem appears in Grunbaum [1].

Now construct the graph G''' by adding the point x''' as in Figure 3 to the points corresponding to y and z. $p(x''') = p(x) = e$. In C'', the parities of y'' and z'' are reversed from the parities of the corresponding y and z in C, whence by adding the lines (y''', x''') (x''', z''') these parities are restored. Thus $P(C''') = P(C)$, and the theorem holds.

CASE 2. $n + 1$ even. If $p(x) = 0$ for every $x \in C$, G' consists of the simple cycle C' with the same number of points as C, with an additional point connected to each point of C'.

If $p(x) \neq 0$ for some x, proof as in Case 1.

REFERENCE

1. B. GRUNBAUM, *Convex Polytopes* (to be published).

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A Characterization of Planar Geodetic Graphs*

An undirected graph is *geodetic* if each pair of vertices is joined by a unique shortest arc (path), called a *geodesic*. The problem of characterizing the geodetic graphs has been posed by O. Ore [1]. The solution for planar graphs is announced.

Clearly a graph is geodetic if and only if each of its blocks is a geodetic subgraph. Let K_n denote the complete graph with n vertices. A *suspended arc* is an arc whose terminal vertices are of degree at least 3 while any intermediate vertices are of degree 2.

THEOREM. *A planar graph is geodetic if and only if it is connected and each of its blocks is* K_2, *an odd cycle, or a homeomorph of* K_4 *which satisfies the following three conditions*:

* The result in this note is contained in the author's dissertation, presented for the degree of Doctor of Philosophy in Yale University. This research was partially supported by National Science Foundation Grant No. 18895.