Linear Groups Definable in $o$-Minimal Structures

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We study subgroups $G$ of $GL(n,R)$ definable in $o$-minimal expansions $M = (R,+,\cdot,\ldots)$ of a real closed field $R$. We prove several results such as: (a) $G$ can be defined using just the field structure on $R$ together with, if necessary, power functions, or an exponential function definable in $M$. (b) If $G$ has no infinite, normal, definable abelian subgroup, then $G$ is semialgebraic. We also characterize the definably simple groups definable in $o$-minimal structures as those groups elementarily equivalent to simple Lie groups, and we give a proof of the Kneser–Tits conjecture for real closed fields.

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1. INTRODUCTION

In most of this paper, $R$ will be a real closed field and $M$ will be an $o$-minimal expansion of $(R,<,+,\cdot)$. This means that the structure $M$ has universe $R$ and basic relations and functions $<,+,\cdot$ as well as

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possibly more, but subject to the requirement that the structure $M$ be $o$-minimal; that is, every subset of $R$ itself, definable with parameters in $M$, should be a finite union of points and intervals $(a, b)$, $a, b \in R$. A basic reference for $o$-minimality is [3]. The point is that the assumption of $o$-minimality brings with it a host of “good” geometric properties of definable sets. First order definability is of course a key notion. Given a structure such as $M$ above, the “category of definable sets in $M$” can be defined inductively: Each basic relation as well as the graph of any basic function so a subset of some $R^n$. A basic reference for $o$-minimality is 3. The point is that the assumption of $o$-minimality brings with it a host of “good” geometric properties of definable sets. First order definability is of course a key notion. Given a structure such as $M$ above, the “category of definable sets in $M$” can be defined inductively: Each basic relation as well as the graph of any basic function so a subset of some $R^n$. A basic reference for $o$-minimality is 3. The point is that the assumption of $o$-minimality brings with it a host of “good” geometric properties of definable sets. First order definability is of course a key notion. Given a structure such as $M$ above, the “category of definable sets in $M$” can be defined inductively: Each basic relation as well as the graph of any basic function so a subset of some $R^n$. A basic reference for $o$-minimality is 3. The point is that the assumption of $o$-minimality brings with it a host of “good” geometric properties of definable sets. First order definability is of course a key notion. Given a structure such as $M$ above, the “category of definable sets in $M$” can be defined inductively: Each basic relation as well as the graph of any basic function so a subset of some $R^n$.

The present paper is a natural continuation of [6, 7] but can be also seen as a contribution to the problem of describing/classifying $o$-minimal expansions of real closed fields. The paper [6] was concerned with definably simple groups $G$ which are definable in arbitrary $o$-minimal structures. It was shown that any such $G$ is a semialgebraic linear group over a real closed field. The paper [7] explored the model-theoretic relation between such a group $G$ (viewed as a structure $(G, \cdot)$ in its own right) and the real closed field $(R, <, +, \cdot)$ where it lives. We showed that $(G, \cdot)$ is bi-interpretable either with this real closed field $R$ or with its algebraic closure $R(i)$. In the present paper we begin with an arbitrary linear group $G < GL(n, R)$ definable in the $o$-minimal expansion $M$ of $R$. We are concerned with several kinds of questions. Examples of the first kind are as follows. Is $G$ already semialgebraic? If not, can we isolate key additional definable relations and functions needed to define $G$? Is $G$ definably isomorphic to a semialgebraic group? These questions are clearly related to the issue of classifying $o$-minimal expansions of a real closed field $R$ by a predicate for a linear group. The next kind of question concerns whether the structure $(G, \cdot)$ is elementarily equivalent to a Lie group. The third kind of question asks when we can deduce the simplicity of $G$ as an abstract group from the definable simplicity of $G$ in the structure $M$. These are all natural model-theoretic questions. It turns out that the first kinds of questions are very much related to generalizing classical results about real Lie groups. For example it is known that both simple linear real Lie groups and compact linear real Lie groups are (semi) algebraic. The third kind of question turns out to be related to the Kneser–Tits conjecture specialized to real closed fields. This Kneser–Tits conjecture for a field $K$ asks whether the group of $K$-rational points of certain algebraic
groups defined over $K$ is abstractly simple. In any case we obtain fairly complete answers to all our questions, which will be described below.

From now on we will assume a familiarity with model theory and $o$-minimality. As mentioned before [3] is a good reference. We will refer extensively to our earlier papers [6, 7]. We will also be using some notation and facts about algebraic groups and we will try to give accurate references for the non-expert.

We will shortly describe the main results of this paper. As above, $M$ will denote an $o$-minimal expansion of a real closed field $R$. “Definable” means “definable with parameters in $M$” unless we say otherwise. By an “exponential function” $e$ in $M$, we mean a definable additive isomorphism $e$ between $(R, +)$ and $(R_{>0}, \cdot)$. By a “power function” in $M$ we mean a definable automorphism of $(R_{>0}, \cdot)$. These have been studied in some detail in the case that $R$ is the real field. It is well known that the only definable endomorphisms of $(R, +)$ are the linear ones: multiplication by some element of $R$. Note that (a) if $e_1$ and $e_2$ are exponential functions in $M$ then there is some nonzero $c \in R$ such that $e_2(x) = e_1(cx)$ for all $x \in R$; (b) if some exponential function $e$ is definable in $M$, then every power function definable in $M$ is already definable in $(R, +, \cdot, e)$ (and is just the image under $e$ of some linear map). Not much appears to be known about the relation between “exponential functions” definable in $o$-minimal expansions of real closed fields, and the classical real exponential function $exp$. Is it the case that $(R, +, \cdot, e)$ is elementarily equivalent to $(R, +, \cdot, exp)$? In any case it should not be considered surprising that exponential and power functions crop up naturally when investigating linear groups definable in $M$. By a linear group definable in $M$ we mean a subgroup $G$ of $GL(n, R)$ (viewed naturally as a group of $n$ by $n$ matrices) which is definable in $M$. We usually assume $G$ to be definably connected, which is equivalent to $G$ having no proper definable subgroup of finite index, and it actually implies connectedness in the case that $R$ is the real field. A set $X$ definable in $M$ is said to be semialgebraic if $X$ is definable in the reduct $(R, +, \cdot)$ of $M$.

Let us first describe our main results for arbitrary $G$ (definably connected). The first two appear in Section 4.

**Theorem.** There are semialgebraic groups $G_1, G_2 < GL(n, R)$ such that $G_1 < G < G_2$, $G_1$ is normal in $G_2$ and $G_2/G_1$ is abelian. (So $G$ is an extension of a definable subgroup of an abelian semialgebraic group by a semialgebraic group).

**Theorem.** Either $G$ is already semialgebraic, or $G$ is definable in $(R, +, \cdot, f_1, \ldots, f_3)$ where the $f_i$ are power functions definable in $M$, or $G$ is definable in $(R, +, \cdot, e)$ where $e$ is some exponential function definable in $M$. 

In the case that $G$ is nilpotent we obtain, in Section 3:

**Theorem.** Suppose $G$ is nilpotent. Then $G$ is definably isomorphic to a semialgebraic group.

In certain cases we can show that $G$ is outright semialgebraic (proved in Section 4).

**Theorem.** Suppose that $G$ is either semisimple (no infinite normal definable abelian subgroup) or closed and bounded (as a subset of $R^n$). Then $G$ is semialgebraic.

Solvable groups present strong counterexamples to semialgebraicity. We will give two examples.

**Example.** Let $\alpha$ be a positive real number. Let $G$ be the semidirect product of $(R_{>0}, \cdot)$ with $(R, +) \times (R, +)$, where the action of the first group on the second is given by $(a, b)^t = (ta, t^\alpha b)$. $G$ can be realised as a subgroup of $GL(3, R)$ definable in the $o$-minimal structure $R_{\alpha} = (R, +, \cdot, (-)^\alpha)$:

$$G = \left\{ \begin{pmatrix} t & 0 & u \\ 0 & t^\alpha & v \\ 0 & 0 & 1 \end{pmatrix} : t > 0, u, v \in R \right\}.$$ 

We will sketch how $R_{\alpha}$ can be interpreted in the structure $(G, \cdot)$. Let $T$ be the group of diagonal matrices in $G$ (which we identify with $(R_{>0}, \cdot)$), and let $U$ be the group of unipotent matrices in $G$ (which we identify with $(R \times R)$). Both $U$ and $T$ are definable in $(G, \cdot)$ as the connected components of their centralizers. The action of $T$ on $U$ is by conjugation in $G$ and is precisely the action of $R_{>0}$ on $R \times R$ given above. By considering the orbits of $(1, 0)$ and $(0, 1)$ in $U$ under $T$, we can definably in $(G, \cdot)$ obtain $U$ as the direct sum of the two copies $U_1$ and $U_2$ of $(R, +)$. The action of $T$ on $U_i$ gives it definably the field structure $(R, +, \cdot)$ ($i = 1, 2$) (after choosing $(0, 1)$ and $(1, 0)$ as the multiplicative identities). The canonical isomorphism $i$, say, between these two copies $U_1$ and $U_2$ of $(R, +, \cdot)$ is definable in $(G, \cdot)$. The raising to the power $\alpha$ map can now be defined on the positive elements of $U_i$, using the action of $T$ and the isomorphism $i$. Thus $R_{\alpha}$ is interpretable in $(G, \cdot)$ (with parameters). (In fact it is not hard to see that these structures are bi-interpretable.) It follows that for $\alpha$ irrational, $(G, \cdot)$ is not even abstractly isomorphic to a real semialgebraic group: if it were then $R_{\alpha}$ would be interpretable in $(R, +, \cdot)$ which is known not to be the case.

**Example.** Let an action of $(R, \cdot)$ on $R \times R$ be given by $(a, b)^t = (ea + te'b, e'b)$, and let $G$ be the corresponding semidirect product. $G$ can be
realised as a subgroup of $GL(3, \mathbb{R})$ definable in the $o$-minimal structure $\mathbb{R}_{\text{exp}} = (\mathbb{R}, +, \cdot, \exp)$ where exp is the exponential function:

$$G = \left\{ \begin{pmatrix} e^t & te^v & u \\ 0 & e^t & v \\ 0 & 0 & 1 \end{pmatrix} : t, u, v \in \mathbb{R} \right\}.$$

One shows by similar methods to the previous example that $(G, \cdot)$ is bi-interpretable with $\mathbb{R}_{\text{exp}}$. So $G$ is not isomorphic to any real semialgebraic group. In fact $G$ is not even isomorphic to any group definable in any $\mathbb{R}_a$.

The results described so far are developed in Sections 2, 3, and 4 of this paper. As in [6], we exploit the availability of Lie algebras in the $o$-minimal context. The structure of commutative algebraic groups is also relevant.

Sections 5 and 6 of this paper contain some related results. In Section 5, we prove that the definably simple (nonabelian) groups $G$ definable in $o$-minimal structures are precisely the groups which are elementarily equivalent (in the group language) to simple Lie groups. (This is related to and generalises the bi-interpretability results in [7].) It is natural for a model-theorist to ask about the relation between definable simplicity and abstract simplicity of groups definable in certain categories. We show in Section 6 that if $G$ is a definably simple group in a saturated real closed field, then $G$ is abstractly simple if and only if $G$ is not definably compact. We point out that this reduces to the Kneser–Tits conjecture for real closed fields, which we also prove: If $R$ is a real closed field, and $G$ is a (noncommutative) almost $R$-simple (namely $G$ has no connected normal algebraic subgroup defined over $R$), simply connected, $R$-isotropic algebraic group defined over $R$, then the group $G(R)$ of $R$-rational points of $G$ is simple as an abstract group (modulo its finite centre). Platonov has informed us that he has an unpublished proof of this.

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**PRELIMINARIES—DEFINABLE LINEAR GROUPS AND THEIR LIE ALGEBRAS**

In this section, we set notation, review the definable group—Lie algebra correspondence from [6], and state some further results. We assume familiarity with the general machinery of $o$-minimality, as well as notions
of differentiability in o-minimal expansions of real closed fields. A reasonable reference for the latter is [3].

$M$ will denote an o-minimal expansion $(R, <, +, \cdot, \ldots)$ of a real closed field $R$. “Definable” always means inside $M$. The focus of our attention in this paper is not so much arbitrary groups definable in $M$ but rather definable subgroups of $GL(n, R)$. Nevertheless, it is worth saying a word or two about “definable manifolds.” A definable $n$-dimensional $C^r$-manifold over $R$ is a topological space $X$ with a finite open cover $U_1, \ldots, U_r$, such that each $U_i$ is homeomorphic to an open definable subset $V_i$ of $R^n$, and the transition functions (between open subsets of the $V_i$) are definable and $C^r$. Note that $X$ is naturally a “definable object.” For our purposes we will identify, for $a \in X$, the tangent space $T_a(X)$ to $X$ at $a$ with $R^n$. (By means of one of the charts, identify $a$ with a point in $R^n$ and now the vectors in $R^n$ correspond to directions along which to differentiate $C^1$-functions on $X$ in a neighbourhood of $a$.) If $Y$ is another definable $C^r$-manifold, it makes sense to talk of a definable $C^1$ (or even $C^r$ for $i \leq r$) map $f$ from an open neighbourhood of $a \in X$ into $Y$ (by passing to the relevant charts $V_i$). Assuming $Y$ is $m$-dimensional, $f$ then has a differential $df_a$ at $a$, a linear map from $R^n$ to $R^m$. This map is independent of the charts in which $f$ is read. If the definable manifold is equipped with a definable $C^r$ group structure, we call it a definable $C^r$ group manifold, or with some ambiguity a “definable Lie group over $R$.”

The most elementary case of the above is when $X$ is already an open definable subset of $R^n$ (so $X = U = V$). An example is $GL(n, R)$, the group of $n \times n$ matrices over $R$ of determinant nonzero, an open definable subset of $R^N$ where $N = n^2$. As the group operation is given by polynomial maps, $GL(n, R)$ is clearly also a definable $C^r$ group manifold (for all $r$). We can identify $R^N$ canonically with $M(n, R)$, the set of all $n \times n$ matrices over $R$, and so we will identify the tangent space to $GL(n, R)$ at the identity $e$ with $M(n, R)$. Note $M(n, R)$ is among other things the vector space $\text{End}(R^n)$ of linear endomorphisms of $R^n$. If $a \in GL(n, R)$ the map $\text{Int}(a): GL(n, R) \to GL(n, R)$ defined by $\text{Int}(a)(x) = a^{-1}xa$ is an infinitely differentiable group automorphism of $GL(n, R)$. Its differential at $e$ is a vector space automorphism of $R^N$. The map taking $a$ to the differential of $\text{Int}(a)$ at $e$ is a definable differentiable homomorphism $Ad$ from $GL(n, R)$ into $GL(N, R)$ (often called the adjoint representation). (It turns out that $Ad(A)(X) = A^{-1}AX$.) Let $ad$ be $d(Ad)_e$. $ad$ is a linear map from $M(n, R)$ to $\text{End}(M(n, R))$. For $A, B \in M(n, R)$ let $[A, B] = ad(A)(B)$. Then $[A, B]$ turns out (by an easy computation) to be precisely $AB - BA$ (in the ring $M(n, R)$). $M(n, R)$ with the operation $[-, -]$ is a Lie algebra, $\text{gl}(n, R)$.

Let us define the notion of a definable submanifold of $R^n$. A subset $X$ of $R^n$ will be called a definable $m$-dimensional $C^r$-submanifold of $R^n$ if $X$ is...
definable, and for each point \( a \in X \) there is an open definable subset \( U \) of \( \mathbb{R}^n \) containing \( a \) with the following property: after reordering coordinates, we can write \( U = V \times W \) for \( V \) an open definable subset of \( \mathbb{R}^m \) and \( W \) an open definable subset of \( \mathbb{R}^{n-m} \), and there is a definable \( C^r \) map \( f \) from \( V \) to \( W \) such that \( X \cap U = \{(x, f(x)) : x \in V \} \). It is not difficult to see that any such \( X \) has a natural structure of a definable \( m \)-dimensional \( C^r \) manifold over \( \mathbb{R} \). For example, with notation above, the open definable subset \( V \) of \( \mathbb{R}^m \) could be taken as a chart around \( a \). The only point is to see that finitely many such charts cover \( X \), which follows from the \( C^r \) cell decomposition in \( o \)-minimal expansions of real closed fields (see [3]). (Similarly we obtain the notion of a definable \( C^r \) submanifold of a definable \( C^r \) manifold (working in the charts).) In any case, with \( X \) as above (a definable submanifold of \( \mathbb{R}^n \)), and \( a, U, V, W, f \) also as above, let us write \( a = (a', a'') \) with \( a' \in V \) and \( a'' \in W \), and let us define the embedded tangent space of \( X \) at \( a \) to be \( \{(x, (df)_{x}(x)) : x \in \mathbb{R}^m \} \). This is a linear subspace of \( \mathbb{R}^n \). Note that choosing \( V \) to be a chart around \( a \), and defining \( g(x) = (x, f(x)) \) on \( V \), this embedded tangent space is precisely the image of \( \mathbb{R}^m \) under \( (dg)_a \). The reader should check that the embedded tangent space of \( X \) at \( a \) does not depend on the choice of \( V, W \), and \( f \) above. The reader should also check that

\[(\ast) \] if \( X \) is a definable \( C^r \)-submanifold of \( \mathbb{R}^n \) contained in an open definable subset \( U \) of \( \mathbb{R}^n \) and \( g \) is a smooth definable homeomorphism of \( U \) with an open subset \( U' \) of \( \mathbb{R}^n \), then \( g(X) \) is also a definable \( C^r \)-submanifold of \( \mathbb{R}^n \).

**Fact 2.1.** Let \( G \) be a definable subgroup of \( GL(n, \mathbb{R}) \). Suppose \( \dim(G) = m \). Then \( G \) is a definable \( m \)-dimensional \( C^r \) submanifold of \( \mathbb{R}^N \) \((N = n^2)\), for all \( r \). Moreover \( G \) is closed in \( GL(n, \mathbb{R}) \).

**Proof.** By [9] any definable subgroup of a group definable in an \( o \)-minimal structure is a closed subgroup. By \( o \)-minimality, there is some point \( a \in G \) and an open definable neighbourhood \( U \) of \( a \) in \( GL(n, \mathbb{R}) \) such that \( G \cap U \) is a definable \( m \)-dimensional \( C^r \)-submanifold of \( \mathbb{R}^N \). As translation by any \( g \in G \) is an infinitely differentiable definable homeomorphism of \( GL(n, \mathbb{R}) \) fixing \( G \) setwise, it follows by (\ast) above that for any \( b \in G \), there is an open neighbourhood \( U_b \) of \( b \) in \( GL(n, \mathbb{R}) \), such that \( G \cap U_b \) is an \( m \)-dimensional \( C^r \)-submanifold of \( \mathbb{R}^N \).

From now on given a definable submanifold \( X \) of \( \mathbb{R}^N \) and \( a \in X \) we identify \( T_a(X) \) with the embedded tangent space. The following is easily seen by chasing diagrams.

**Fact 2.2.** Let \( G \leq GL(n, \mathbb{R}) \) be definable. Then \( T_a(G) \) is not only a vector subspace but also a Lie subalgebra of \( gl(n, \mathbb{R}) \). Moreover, if we define
We write \( \text{Lie}(G) \) for \( T_0(G) \). We assume familiarity with basic notions around Lie algebras such as ideal, abelian, semisimple. The following is proved in [6]. More precisely (i) is Claim 2.20, (ii) is Claim 2.32(1), (iii) is Claim 2.32(2), and (iv) is Theorem 2.34.

**FACT 2.3.** Let \( G, H \) be definable, definably connected subgroups of \( GL(n, R) \).

(i) \( H < G \) if and only if \( \text{Lie}(H) < \text{Lie}(G) \).

(ii) \( G \) is abelian iff \( \text{Lie}(C) \) is abelian.

(iii) Assuming that \( G < H \), \( G \) is normal in \( H \) if and only if \( L(G) \) is an ideal of \( L(H) \).

(iv) \( G \) is semisimple if \( \text{Lie}(G) \) is semisimple.

There is a notion of the Lie algebra of an arbitrary algebraic group over an algebraically closed field of characteristic 0 (see [1]). It is important to note that this coheres with our notion. If \( K \) is an algebraically closed field of characteristic 0, and \( G \) is a linear algebraic group defined over \( K \), then \( ZT(G) \), the “Zariski tangent space to \( G \) at the identity,” is defined as the common zero set of the linear maps \( (dF)_e \), where \( F \) ranges over polynomials generating the ideal of the variety \( G \) (and where the differential is defined formally). \( ZT(G) \) will again be a Lie subalgebra of \( gl(n, K) \) (defined as above using \( ad \), etc.). We will call it the Zariski Lie algebra of \( G \), \( Z \text{Lie}(G) \). If \( G \) is defined over \( k < K \) then we can choose the polynomials to be over \( k \) and so \( ZT(G) \) is defined over \( k \) too. Now suppose that \( K = R(i) \) where \( R \) is our real closed field. If \( G < GL(n, K) \) is a linear algebraic group defined over \( R \), then \( G(R) \) is a group quantifier-free definable in \( (R, +, \cdot) \) and the semialgebraic connected component \( G(R)^0 \) of \( G(R) \) is definable in \( (R, +, \cdot) \). Conversely if \( H < GL(n, R) \) is semialgebraic and semialgebraically connected, then there is a linear algebraic group \( G < GL(n, K) \) defined over \( R \) such that \( H = G(R)^0 \).

With this notation \((K = R(i), G < GL(n, K)\) a linear algebraic group defined over \( R \)) we have:

**LEMMA 2.4.** \((Z \text{Lie}(G))(R) = \text{Lie}(G(R)) = \text{Lie}(G(R)^0)\).

**Proof.** Let \( F_1, F_2, \ldots, F_t \) be polynomials over \( R \) generating the ideal of \( G \). Let \( U \) be an open neighbourhood of \( e \) in \( GL(n, R) \) and let \( f \) be a semialgebraic differentiable function from a suitable open subset \( V \) of \( R^m \) to suitable open \( W \) of \( R^{N-m} \) such that \( G(R) \cap U = G(R)^0 \cap U = \{ (x, f(x)) : x \in V \} \). So on \( U \), the solution set of \( F_1 = F_2 = \cdots = F_t = 0 \) agrees with the graph of \( f \). It is then easy to see that the common zero set in \( R^N \)
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of the formally defined \((dF)_e\) equals \(\{(a, df_e(a)): a \in R^n\}\). But the former is precisely the \(R\)-points of \(Z \text{Lie}(G)\) and the latter is \(\text{Lie}(G(R)) = \text{Lie}(G(R)^0)\).

Given a Lie algebra \(L\), \([L, L]\) denotes the subspace generated by the commutators \([a, b]\) for \(a, b \in L\). This is a Lie subalgebra, in fact an ideal, of \(L\). For the case \(R = \mathbb{R}\), the following is well known (see Section 4 of [4]) so we can transfer the results to the real-closed field \(R\) as all the data are semialgebraic.

**Lemma 2.5.** Suppose \(L\) is a Lie subalgebra of \(\text{gl}(n, R)\). Then \(L_1 = L \otimes_R K\) is a Lie subalgebra of \(\text{gl}(n, K)\) whose set of \(R\)-points is exactly \(L\). Moreover:

(i) \([L, L] = [L_1, L_1](R)\) and

(ii) \(L\) is semisimple iff \(L_1\) is semisimple.

Let \(L\) be a Lie subalgebra of \(\text{gl}(n, K)\) (\(K\) algebraically closed of characteristic 0 as above). In [1], \(L\) is said to be algebraic if there is a (connected) algebraic subgroup \(G\) of \(\text{GL}(n, K)\) such that \(L = Z \text{Lie}(G)\). For any subalgebra \(L < \text{gl}(n, K)\) there is a smallest algebraic Lie algebra containing \(L\). This is denoted \(a(L)\). The following crucial result is proved in [1, Sect. II.7] and is attributed to Cartan for \(K = \mathbb{C}\), and to Chevalley [2] in general.

**Fact 2.6.** Let \(L < \text{gl}(n, K)\) be an arbitrary Lie subalgebra. Then \([L, L]\) is an algebraic Lie algebra. Moreover, supposing \(a(L) = Z \text{Lie}(G)\) (where \(G < \text{GL}(n, K)\) is algebraic and connected), we have \([L, L] = Z \text{Lie}(G, G)\) (where \((G, G)\) is the commutator subgroup of \(G\), also a connected algebraic group).

3. COMMUTATIVE (AND NILPOTENT) GROUPS

\(R\) remains a real closed field and \(M\) an \(o\)-minimal expansion of \(R\). For the purposes of this section we let \(T\) denote the group \(\text{SO}(2, R)\). Also \(R^*_+, 0\) will denote the multiplicative group of positive elements of the field \(R\) and \(R^+\) the additive group. Let us remark now that each of these groups is definably connected and of dimension 1 in the \(o\)-minimal structure \(M\). Note that the torsion subgroup of \(T\) is infinite, so \(T\) has no proper definable subgroup containing all the torsion elements (otherwise \(T\) has a proper infinite definable subgroup, which must have dimension 1 and so of finite index, contradicting definable connectedness of \(T\)).

**Fact 3.1.** Let \(H\) be a commutative linear algebraic group defined over \(R\). Let \(G\) be the semialgebraic connected component of \(H(R)\). Then \(G\) is semialgebraically isomorphic to a group of the form \(T^m \times (R^*_+, 0)^l \times (R^+)^k\).
Proof. This is well known: \( H \) can be written as an almost direct product of three \( R \)-algebraic subgroups, \( H_u, H_d, \) and \( H_a \). \( H_a \) is an \( R \)-anisotropic algebraic torus; hence by transfer from \( R \) or by standard methods, \( H_a(R) \) is rationally, hence semi-algebraically, isomorphic to \( T^n \) for some \( m \). \( H_d \) is an \( R \)-split torus; hence \( H_d(R) \) is rationally isomorphic to a power of \( R^* \), the multiplicative group. \( H_u \) is unipotent and commutative, so \( H_u(R) \) is rationally isomorphic to a power of \( R \). Both \( T \) and \( R \) are semialgebraically connected. The semialgebraic connected component of \( R \) is as required.

**Lemma 3.2.** Each definable endomorphism \( f \) of \( T \) has the form \( f(x) = x^m \) for some \( m \in \mathbb{Z} \).

Proof. We identify \( T \) with the subgroup of the multiplicative group of \( R(i) \) consisting of elements of norm 1 (namely the circle group). Note \( f \) is continuous. Note also that \( f \) is either trivial (takes everything to 1) or surjective with finite kernel. First suppose \( f \) to be a definable automorphism of \( T \). Note that \( f(-1) = -1 \). Let \( \omega \) be a primitive 4th root of 1.

**Case I.** \( f(\omega) = \omega \). Note that \( T \setminus \{1, \omega\} \) has two definably connected components, one containing a unique (primitive) 8th root of 1, the other containing five 8th roots of unity. \( f \) being a definable homeomorphism of \( T \) must permute these components. But \( f \) is also a homomorphism. Thus \( f \) must fix each component setwise and so fixes some primitive 8th root of unity (and thus fixes all 8th roots of unity). Continuing this way we see that \( f \) fixes all \( 2^n \)th roots of unity for all \( n \). So \( \text{Fix}(f) \) is an infinite definable subgroup of \( T \) and so is therefore equal to \( T \). Namely \( f = \text{id} \).

**Case II.** \( f(\omega) \neq \omega \). It follows that \( f(\omega) = \omega^{-1} \). But then the composition of \( f \) with inversion \( (x \to x^{-1}) \) satisfies Case 1, whereby \( f \) is inversion.

We have shown:

Claim. If \( f \) is an automorphism of \( T \) then \( f \) is the identity map or inversion.

Now suppose that \( f \) has finite kernel, of size \( m \), say. Then \( \text{Ker}(f) \) must be the group of \( m \)th roots of unity. \( f \) induces a definable isomorphism \( f' \) from \( T/\text{Ker}(f) \) to \( T \). On the other hand, if \( g \) is the \( m \)th power map then \( g \) also induces a definable isomorphism \( g'^{*} \) from \( T/\text{Ker}(f) \) to \( T \). Thus \( g'^{*}(f')^{-1} \) is a definable automorphism of \( T \), which by the claim is either the identity or inversion. It follows that \( f(x) = x^m \) for all \( x \in T \) or \( f(x) = x^{-m} \) for all \( x \in T \). This completes the proof of the lemma.

**Corollary 3.3.** Any definable subgroup \( A \) of \( T^n \) is defined by a finite set of equations of the form \( x_1^{s_1} \cdots x_n^{s_n} = 1 \) where the \( s_i \) are in \( \mathbb{Z} \). Moreover
$A$ is, if definably connected, isomorphic to $T^t$ for some $t$ via a map of the same kind (namely a map whose coordinates have the form $x_1^{s_1} \cdots x_m^{s_m}$).

**Proof.** By induction on $m$, for $m = 1$ this is clear, as $T$ is definably connected and of dimension 1 in the $o$-minimal structure $M$. Assume this is proved for $m$, and we will prove it for $m + 1$. We may assume $A$ to be definably connected. By the induction hypothesis we may assume that the projection of $A$ on the first $m$ coordinates is surjective, namely isomorphic to $T^m$. We may also assume the projection of $A$ on the last coordinate is $T$. We may also assume that $B = \text{coker}(A) = \{ x \in T : (1, 1, \ldots, 1, x) \in A \}$ is finite, of size $d$ say. The $d$th power map takes $T$ onto itself with kernel $B$. Then $[(x, y^d) : (x, y) \in A]$ is the graph of a homomorphism $f$ from $T^m$ onto $T$. $f$ is of the form $f_1 \cdots f_m$ where each $f_i$ is a definable endomorphism of $T$. It is clear that $A$ is defined by $f_1(x_1) \cdots f_m(x_m) = x_1^{d}$ for some $s \in \mathbb{Z}$. The moreover clause is left to the reader.

**LEMMA 3.4.**

(i) Any definable subgroup $G$ of $(\mathbb{R}_>^*)^l$ is a finite intersection of subgroups defined by $f_1(x_1) \cdots f_l(x_l) = 1$, where the $f_i$ are definable endomorphisms of $\mathbb{R}_>^*$. In particular $G$ is definably isomorphic to some $(\mathbb{R}_{>0})^l$.

(ii) Any definable subgroup of $(\mathbb{R}^+)^k$ is semialgebraic.

**Proof.** (i) is proved as in Corollary 3.3. Note that $\mathbb{R}_>^*$ is torsion-free, so any definable endomorphism of it is either an automorphism or takes everything to 1. (ii) is well known.

**LEMMA 3.5.** If there is no definable isomorphism between $\mathbb{R}^+$ and $\mathbb{R}_>^*$ then any definable subgroup of $(\mathbb{R}_>^*)^l \times (\mathbb{R}^+)^k$ is of the form $A \times B$ where $A$ is a definable subgroup of $(\mathbb{R}_>^*)^l$ and $B$ is a definable subgroup of $(\mathbb{R}^+)^k$.

**Proof.** Let $G$ be a definable subgroup of $(\mathbb{R}_>^*)^l \times (\mathbb{R}^+)^k$. We may assume that both projections of $G$ are surjective. If the cokernel of $G$ is all of $(\mathbb{R}^+)^k$ then $G$ is the direct product of these projections. Otherwise $G$ induces a definable homomorphism from $(\mathbb{R}_>^*)^l$ onto the quotient of $(\mathbb{R}^+)^k$ by this cokernel. But the latter quotient is then a nontrivial $R$-vector space, so one easily obtains a definable isomorphism of $\mathbb{R}_>^*$ with $\mathbb{R}^+$.

**LEMMA 3.6.** Suppose there is a definable isomorphism $e$ between $\mathbb{R}^+$ and $\mathbb{R}_>^*$. Let $G$ be a definable subgroup of $(\mathbb{R}_>^*)^l \times (\mathbb{R}^+)^k$. Then

(i) $G$ is definable in $(\mathbb{R}, +, \cdot, e)$.

(ii) $G$ is definably isomorphic to a semialgebraic group.
Proof. Applying the inverse of $e$ to $R^*_{>0}$ we may assume that $G$ is a definable subgroup of $(R^*)^{i+k}$. Now apply Lemma 3.4.

Lemma 3.7. Any connected definable subgroup $G$ of $T^m \times (R^*_{>0})^j \times (R^+)^k$ is of the form $A \times B$ where $A$ is the projection of $G$ on $T^m$ and $B$ is the projection of $G$ on $(R^*_{>0})^j \times (R^+)^k$.

Proof. Note that there can be no nontrivial definable homomorphism $f$ from $T$ to either $R^*_{>0}$ or $R^+$. $f$ would be trivial on the torsion elements of $T$ and so would be trivial everywhere. The lemma then follows as in the proof of Lemma 3.5.

Proposition 3.8. Let $G$ be a definable, definably connected, commutative subgroup of $GL(n, R)$. Then either

(i) 

(ii) there are definable automorphisms $f_1, \ldots, f_i$ of $R^*_{>0}$ such that $G$ is definable in $(R, +, \cdot, f_1, \ldots, f_i)$, or

(iii) there is a definable isomorphism $e$ between $R^+$ and $R^*_{>0}$ such that $G$ is definable in $(R, +, \cdot, e)$.

Moreover $G$ is definably isomorphic to a (linear) group definable in $(R, +, \cdot)$.

Proof. By looking at the Zariski closure of $G$ in $GL(n, R(i))$ and using definable connectedness of $G$ we find a commutative algebraic group $H < GL(n, R(i))$ defined over $R$ such that $G$ is contained in the semialgebraic connected component $H(R)^0$ of $H(R)$. The result then follows from Lemmas 3.3 to 3.7.

Finally in this section we generalize 3.8 to nilpotent groups. We will be brief. We need the following lemma This easily follows from the fact that commutative definable groups are products of one-dimensional groups, but we give a proof nevertheless.

Lemma 3.9. Let $G$ be a definable, definably connected, commutative subgroup of $GL(n, R)$, and let $H$ be a definable, definably connected subgroup of $G$. Then $H$ has a definable complement in $G$.

Proof. By Proposition 3.8 we may assume $G$ to be semialgebraic and so as given by Fact 3.1: $G = T^m \times (R^*_{>0})^j \times (R^+)^k$. By Lemma 3.7, $H$ is of the form $A \times B$ where $A$ is a definable definably connected subgroup of $T^m$ and $B$ is a definable (definably connected) subgroup of $(R^*_{>0})^j \times (R^+)^k$. By Corollary 3.3, $A$ is an algebraic subgroup of $T^m$ and it is well known that $A$ has an algebraic complement in $T^m$ (see III.8 of [1]). So we reduce to finding a definable complement of $B$ in $(R^*_{>0})^j \times (R^+)^k$. If there is a definable exponential function, then the latter group is definably isomorphic to a power of $R^+$, and by Lemma 3.4(ii), $B$ is a linear subspace
and so clearly it has a definable complement. If there is no definable exponential function, then by Lemma 3.5, $B$ is of the form $C \times D$, where $C$ is a definable subgroup of $(R^+_0)^\ell \times (R^+_0)^k$. As in the previous sentence, $D$ has a definable complement in $(R^+_0)^k$. So we are reduced to finding a definable complement for $C$ in $(R^+_0)^\ell$.

Changing notation, let $G = (R^+_0)^\ell$ and $H = C$. Let $B$ be a maximal product of the factors $R^+_0$ of $G$ which has trivial intersection with $H$. We claim that $G$ is the direct product of $B$ and $H$. If not, consider $\pi: G \to G/B$. $\pi$ is an isomorphism on $H$, and $G/B$ can be identified with the direct product of the remaining copies of $R^+_0$. By the maximality of $B$, each of these copies of $R^+_0$ has nontrivial intersection with $\pi(H)$ and thus is contained in $\pi(H)$ as $G/B$ is torsion-free and $R^+_0$ is one-dimensional. Thus $\pi(H) = G/B$ and $G = B \times H$ as required.

**Proposition 3.10.** Let $G < GL(n, R)$ be a definable, definably connected, nilpotent group. Then $G$ is definably isomorphic to a (linear) semialgebraic group.

**Proof.** Let $G_1$ be the semialgebraic connected component of the Zariski closure of $G$ in $GL(n, R)$. Then $G_1$ is nilpotent, and using Theorem 10.6(3) in [1], $G_1 = U \cdot T$ (direct product) where $U < GL(n, R)$ is unipotent and $T$ is semialgebraically isomorphic to a power of $R^+_0$ times a power of $T$. Let $\pi_1$ be the projection of $G_1$ on $U$ and let $\pi_2$ be the projection on $T$. Let $T_1 = G \cap T$. Let $T_2 = \pi_2(G)$. Then $T_2$ is commutative, definably connected and moreover contains $T_1$. Hence by Lemma 3.9, $T_2$ is the direct product of $T_1$ with some definable subgroup $T_3$ of $T_2$. Let $H = (\pi_2)^{-1}(T_3)$. Then $G$ is the direct product of $H$ with $T_1$. Moreover $T_1$ is precisely the kernel of the restriction of $\pi_1$ to $G$, whereby $\pi_1$ induces a definable isomorphism between $H$ and a definable subgroup $H_1$ of $U$. Then $H_1$ is (semi)algebraic.

Explanation: Let $U_n$ be the group of upper triangular matrices in $GL(n, R)$ with 1’s on the diagonal. Let $N_n$ be the set of upper triangular $n \times n$ matrices over $R$ with 0’s on the diagonal. This is an additive subgroup (and even a Lie subalgebra) of $M(n, R)$. Any element $Y \in U_n$ has the form $I + X$ where $I$ is the identity matrix and $X \in N_n$. Consider the following polynomials in the unknown matrix $X$: $e^X = I + X + X^2/2 + \cdots + X^{n-1}/(n-1)!$, and $\log(I + X) = X - X^2/2 + \cdots + (-1)^nX^{n-1}/(n-1)$. A straightforward computation shows that $e^X$ establishes a bijection between $N_n$ and $U_n$ whose inverse is $\log(-)$. Let $L_1 = \text{Lie}(H_1)$. So $L_1$ is a Lie subalgebra of $gl(n, R)$. Let $H_2$ be the image of $L_1$ under $e^{(-)}$. By 3.6 in [4], together with Lemma 2.4, $H_2$ is the group of $R$-points of an algebraic group defined over $R$, and $\text{Lie}(H_2) = L_1$. By Fact 2.3, $H_1 = H_2$. 
Hence \( H \) is definably isomorphic to a (semi)algebraic linear group. By Proposition 3.8, \( T_1 \) is also definably isomorphic to a semi-algebraic linear group, and thus so is \( G \).

4. MAIN RESULTS

As above \( M \) will be an \( o \)-minimal expansion of a real closed field, and \( G < GL(n, R) \) will be a definably connected group definable in \( M \).

**Theorem 4.1.** There are semialgebraic \( G_1, G_2 < GL(n, R) \) such that \( G_2 < G < G_1 \). \( G_2 \) is normal in \( G_1 \) and \( G_1/G_2 \) is abelian. Moreover there are abelian, definable, definably connected subgroups \( A_1, \ldots, A_k \) of \( G \) such that \( G = G_2 \cdot A_1 \cdot \cdots \cdot A_k \).

**Proof.** Let \( L = \text{Lie}(G) \). Let \( K = R(i) \) and let \( L_0 = L \otimes_R K \). Let \( L_1 = \text{a}(L_0) \) (which recalls is the smallest “algebraic” Lie algebra containing \( L_0 \)), and let \( H < GL(n, K) \) be a connected algebraic group such that \( Z \text{Lie}(H) = L_1 \).

Note that \( L_1 \) and \( H \) are defined over \( R \). By Fact 2.6, \( (H, H) \), the commutator subgroup of \( H \), also a connected algebraic group defined over \( R \), has Zariski Lie algebra \( L_2 = [L_0, L_0] \). Let \( G_1 = H(R)^0 \) and \( G_2 = (H, H)(R)^0 \). By Lemma 2.4, \( \text{Lie}(G_1) = L_1(R) \) and \( \text{Lie}(G_2) = L_2(R) \). Note that \( L < L_1(R) \). On the other hand by Lemma 2.5(i), \( L_2(R) = [L, L] \) and so \( L_2(R) < L \). By Fact 2.3 we conclude that \( G_2 < G < G_1 \). From the definition of \( G_1 \) and \( G_2 \), we see that the abstract commutator subgroup of \( G_1 \) is contained in \( G_2 \) whereby \( G_2 \) is normal in \( G_1 \) and the quotient is abelian.

For the moreover clause, we can find \( a \in G/G_2 \) with infinite exponent. Let \( b \in G \) be such that \( b/G_2 = a \). Then \( b \) has infinite exponent, and \( \langle b \rangle \) is an infinite commutative subgroup of \( G \) with trivial intersection with \( G_2 \). Let \( B \) be an infinite definable abelian subgroup of \( G \) containing \( \langle b \rangle \) (such as the centre of the centralizer of \( B \)), and let \( A_1 \) be its definably connected component. Then \( A_1 \) is not contained in \( G_2, G_2 \cdot A_1 \) is definable, normal in \( G \), and of dimension strictly greater than \( \text{dim}(G_2) \). We continue to find \( A_2, \ldots, A_k \) as required.

**Corollary 4.2.** Either \( G \) is semialgebraic, or there are are definable automorphisms \( f_1, \ldots, f_k \) of \( R^+_{>0} \) such that \( G \) is definable in the structure \( (R, +, \cdot, f_1, \ldots, f_k) \) or there is a definable isomorphism \( \epsilon \) between \( R^+ \) and \( R^+_{>0} \) such that \( G \) is definable in the structure \( (R, +, \cdot, \epsilon) \).

**Proof.** By Theorem 4.1 and Proposition 3.8.
Note that if \( G \) is an arbitrary group definable in \( M \), then by [5], \( G/Z(G) \) definably embeds in some \( GL(n, R) \) and so its image satisfies the conclusion of Corollary 4.2.

**Theorem 4.3.** If \( G \) is semisimple, then \( G \) is semialgebraic.

**Proof.** As in the proof of 4.1 this depends on 2.6 which says that the commutator of a Lie algebra is algebraic. Let \( L = \text{Lie}(G) \). By Fact 2.3(iv), \( L \) is semisimple. Let \( L_0 = L \otimes_R K \). By 2.5(ii), \( L_0 \) is semisimple. But a semisimple Lie algebra is its own commutator, whereby \( L_0 \) is algebraic; namely there is an algebraic subgroup \( H \) of \( GL(n, K) \) defined over \( R \) such that \( L_0 = Z \text{Lie}(H) \). By Lemma 2.4, \( \text{Lie}(H^0) = L \). By 2.3, \( H(R)^0 = G \).

**Remark 4.4.** The proof of Theorem 4.3 shows that any semisimple Lie subalgebra of \( gl(n, R) \) is the Lie algebra of a (unique semisimple) semialgebraic subgroup of \( GL(n, R) \).

The next result yields the “Levi decomposition” (see Chapter 6 of [4]) in our context.

**Theorem 4.5.** \( G \) is the almost semidirect product of a normal solvable definable subgroup \( N \) and a semialgebraic semisimple subgroup \( H \).

**Proof.** Let \( L = \text{Lie}(G) \). Let \( N \) be the maximal definable, definably connected, solvable normal subgroup of \( G \). Let \( \mathcal{N} = \text{Lie}(N) \). Then \( \mathcal{N} \) is solvable by 2.3(ii). Note that the definable group \( G/N \) is semisimple (that is, it has no normal, infinite, definable abelian subgroup). Let \( H_1 \) be the quotient of \( G/N \) by its finite centre. Then \( H_1 \) is centreless. So \( H_1 \) definably embeds in some \( GL(m, R) \) (by Claim 2.26 of [6] for example). So we obtain a definable homomorphism \( f \) from \( G \) into \( GL(m, R) \) such that \( f(G) \) is semisimple, and \( N \) is the definably connected component of \( \ker(f) \). \( df \) is then a definable homomorphism of Lie algebras, taking \( L \) onto \( \text{Lie}(\text{Im}(f)) \) with kernel \( \mathcal{N} \). By Fact 2.3(iv), \( \text{Lie}(\text{Im}(f)) \) is semisimple. It follows that \( \mathcal{N} \) is a maximal solvable subalgebra of \( L \). By Levi's theorem [4] \( L \) is the semidirect product of \( \mathcal{N} \) with a semisimple Lie subalgebra \( L_1 \). By 4.4, \( L_1 = \text{Lie}(H) \) where \( H \) is a semialgebraic semisimple subgroup of \( G \). Clearly \( G \) is an almost semidirect product of \( N \) and \( H \).

**Theorem 4.6.** Suppose \( G \) is bounded and closed (in \( R^n \)). Then \( G \) is semialgebraic.

**Proof.** Any definable, definably connected, abelian subgroup \( A \) of \( G \) will be also closed and bounded. The results of Section 3 imply that there is a semialgebraic map embedding \( A \) in some \( T^m \). By 3.3, the image of \( A \) is semialgebraic, and hence so is \( A \). 4.1 implies that \( G \) is semialgebraic.
Remark 4.7. The results above can be adapted to obtain information about arbitrary closed (so Lie) subgroups of $GL(n, \mathbb{R})$. For example, we have:

(i) Suppose $G$ is a closed subgroup of $GL(n, \mathbb{R})$. Suppose moreover that every closed subgroup of $G$ definable in the structure $(\mathbb{R}, +, \cdot, G)$ is connected-by-finite. Then $G$ is definable in the structure $(\mathbb{R}, +, \cdot, \exp)$.

We also recover the following classical result:

(ii) Suppose $G$ is a compact subgroup of $GL(n, \mathbb{R})$. Then $G$ is semialgebraic.

Proof. (i) Let $M$ be the structure $(\mathbb{R}, +, \cdot, G)$. We may assume $G$ to be connected. As $G$ has a well-defined Lie algebra $L$, the first part of the proof of 4.1 (as well as classical Lie group/Lie algebra theory) adapts to finding a semialgebraic normal subgroup $G_2$ of $G$ such that $G/G_2$ is abelian. So $G/G_2$ is a connected abelian Lie group and thus has an element $a$ of infinite exponent. Again let $b \in G$ be such that $b/G_2 = a$. $b$ has infinite exponent. The centre of the centraliser of $b$ in $G$, say $A_1$, is an infinite closed (commutative) subgroup of $G$, definable in $M$ and so connected-by-finite. It follows that the closed normal subgroup $G_2 \cdot A$ of $G$ has dimension (as a Lie group) strictly greater than that of $G_2$. We continue to find closed commutative subgroups $A_2, \ldots, A_m$ of $G$ all definable in $M$ such that $G = G_2 \cdot A_1 \cdot \cdots \cdot A_m$. Each $A_i$ is connected-by-finite. The proofs in Section 3 yield that each $A_i$ is definable in $(\mathbb{R}, +, \cdot)$ together with some continuous automorphisms of $(\mathbb{R}, >0, \cdot)$ and/or some continuous isomorphisms between $(\mathbb{R}, +)$ and $(\mathbb{R}, >0, \cdot)$. But any such automorphism and/or isomorphism is definable in $(\mathbb{R}, +, \cdot, \exp)$. Thus $G$ is definable in $(\mathbb{R}, +, \cdot, \exp)$, as required.

(ii) As $G$ is compact, any closed subgroup of $G$ will be connected-by-finite, whereby (i) applies. The commutative closed subgroups $A_i$ obtained there will be compact. As in the proof of 4.6 each $A_i$ is semialgebraic and hence so is $G$.

5. GROUPS ELEMENTARILY EQUIVALENT TO SIMPLE LIE GROUPS

We characterise the definably simple groups definable in o-minimal structures as the groups elementarily equivalent to simple Lie groups. As usual by a simple group we mean a non-abelian group which has no proper nontrivial normal subgroup.
THEOREM 5.1. Let $G$ be any infinite group. The following are equivalent:

1. $G$ has no proper nontrivial normal subgroup definable in $(G, \cdot)$, and $(G, \cdot)$ is definable in some $\alpha$-minimal structure.

2. $G$ is (abstractly) isomorphic to a semialgebraically simple, semialgebraic subgroup of $GL(n, R)$ for some real closed field $R$.

3. $(G, \cdot)$ is elementarily equivalent to $(H, \cdot)$ for some simple (centreless) Lie group $H$.

Proof. Assume (3). $H$ being centreless is embeddable in some $GL(n, R)$ as a closed subgroup via the adjoint representation. So we assume $H < GL(n, R)$. By simplicity of $H$ and the first part of the proof of Remark 4.7(i), $H$ is semialgebraic. By Theorem 1.1 of [7], the structure $(H, \cdot)$ is bi-interpretable with the structure $(\mathbb{R}, +, \cdot)$ or with the structure $(\mathbb{C}, +, \cdot)$. Let us assume the first case. So, over certain parameters from $H$, there is an ordered field (isomorphic to $\mathbb{R}$) definable in $(H, \cdot)$ and (definably in $(H, \cdot)$) there is an embedding of $H$ into some general linear group over this field, as a semialgebraic subgroup. $G$ is elementarily equivalent to $(H, \cdot)$ there are, definable in $(G, \cdot)$ with parameters, an ordered field $(k, +, \cdot, <)$ and an embedding of $G$ into some $GL(n, k)$ as a subgroup quantifier-free definable in $(k, +, \cdot, <)$. By transfer $k$ must be real closed (any ordered field definable in $(H, \cdot)$ must be isomorphic to the real field, hence real closed). Note that $H$ (considered as a semialgebraic subgroup of $GL(n, R)$) has no proper nontrivial normal semialgebraic subgroups. This transfers to $G$ considered as a semialgebraic subgroup of $GL(n, k)$ and proves (2) in the case that $(H, \cdot)$ is bi-interpretable with $(\mathbb{R}, +, \cdot)$.

When $(H, \cdot)$ is bi-interpretable with $(\mathbb{C}, +, \cdot)$, we see in a similar manner that $G$ is isomorphic to a simple algebraic subgroup $G_1$ of $GL(n, K)$ for some algebraically closed field $K$ of characteristic 0. Let $R$ be a real closed subfield of $K$ such that $K = R(i)$. So $G_1$ is a (semi)algebraic subgroup of $GL(2n, R)$, yielding again (2).

Clearly (2) implies (1).

(1) implies (2) is the main result in [6, 7]. (In [6], a group satisfying (1) is shown to be isomorphic to a semialgebraic linear group over a real closed field $R$. In [7], the group $(G, \cdot)$ is shown moreover to be bi-interpretable with $(R, +, \cdot)$ so we get semialgebraic simplicity too.)

Now assume (2). We want to prove (3). Let us assume $G$ to be a semialgebraic, semialgebraically simple subgroup of $GL(n, R)$ with $R$ a real closed field. Let $L = \text{Lie}(G)$ as defined in Section 2. By Theorem 2.36 of [6], $L$ is a simple Lie algebra over $R$. The classification of simple (and even semisimple) Lie algebras over $R$ yields that there are only finitely many simple subalgebras of $gl(n, R)$, up to isomorphism. So there is some natural number $r$ and a sentence $\sigma$ the language of fields which is true in
(\mathbb{R}, +, \cdot) and “says” “there are are simple Lie subalgebras $L_1, \ldots, L_r$ of $\text{gl}(n, \mathbb{R})$, such that any simple subalgebra of $\text{gl}(n, \mathbb{R})$ is isomorphic to one of the $L_i$.” (Quantification over Lie subalgebras of $\text{gl}(n, \mathbb{R})$ may seem at first sight to be second order, but actually by considering bases of these Lie algebras it is first order.) \sigma is then true in $(\mathbb{R}_{\text{alg}}, +, \cdot)$ where $\mathbb{R}_{\text{alg}}$ is the field of real algebraic numbers (another real closed field). It follows that our simple Lie subalgebra $L$ of $\text{gl}(n, \mathbb{R})$ is isomorphic to some Lie subalgebra $L_1$ say of $\text{gl}(n, \mathbb{R})$ which is defined over $\mathbb{R}_{\text{alg}}$; namely $L$ has a basis consisting of matrices with real algebraic coordinates. By the proof of Theorem 2.37 of [6] $G$ is semialgebraically isomorphic to $\text{Aut}(L)$ (the semialgebraic connected component of $\text{Aut}(L)$). Thus $G$ is isomorphic, even semialgebraically, to $\text{Aut}(L_1)^0 = G_1$, a semialgebraic linear group defined over $\mathbb{R}_{\text{alg}}$. So $G_1$ is also semialgebraically simple. Let $G_2$ be the linear group defined in $(\mathbb{R}, +, \cdot)$ by the same formula (over $\mathbb{R}_{\text{alg}}$) defining $G_1$ in $(\mathbb{R}, +, \cdot)$. Clearly $(G_2, \cdot)$ is elementarily equivalent to $(G_1, \cdot)$ and thus to $(G, \cdot)$. Moreover $G_2$, being semialgebraically simple, is a simple centreless real Lie group (well known, or alternatively see Section 6). This completes the proof.

6. ABSTRACT SIMPLICITY OF SEMIALGEBRAICALLY SIMPLE GROUPS

Suppose $R$ is a real closed field and $G$ is a semialgebraic group over $R$ which is semialgebraically simple, that is, nonabelian and without proper nontrivial semialgebraic normal subgroups. What can be said concerning the simplicity of $G$ as an abstract group? This kind of question is quite common both in model theory and algebraic group theory. For example, it is well known that a definably simple group of finite Morley rank is simple as an abstract group. A special case of this is that for algebraic groups over algebraically closed fields, algebraic simplicity and abstract simplicity coincide. On the other hand considerable effort has gone into trying to understand the normal subgroup structure of groups of the form $G(R)$ where $G$ is a simple algebraic group defined over a ring $R$.

Our current problem is in fact of this nature: if $G$ is a semialgebraically simple, semialgebraic group in a real closed field $R$, then we may assume that $G$ is the semialgebraically connected component $H(R)^0$ of the set $H(R)$ of $R$-points of a connected linear algebraic group $H$ defined over $R$. Moreover we may assume $H$ to be $R$-simple, in the sense that $H$ has no proper nontrivial normal algebraic group defined over $R$. So we want to understand when and whether such groups $H(R)^0$ are abstractly simple. In the case when $R$ is the real field $\mathbb{R}$, it is well known that $G = H(R)^0$ is
abstractly simple. See Proposition 3.6 in [11] for a proof of this. A model-theoretic proof appears in [10]. On the other hand we cannot expect the same result for arbitrary real closed fields $R$: Suppose that $H$ is a simple algebraic group defined over $\mathbb{R}$, and suppose that the group $H(R)$ is compact. If $R$ is a real closed field properly containing $\mathbb{R}$ then $H(R)$ is not abstractly simple: the elements of $H(R)$ infinitely close to the identity will be a proper normal infinite subgroup. Our main result here (Theorem 6.1) is that this is the only obstruction.

Let $G$ be a semialgebraic, semialgebraically simple group over a real closed field $R$. By Theorem 5.1, $(G, \cdot)$ is elementarily equivalent to a (semialgebraic) simple linear real Lie group $(G_1, \cdot)$. We will say that $G$ is of compact type if $G_1$ can be chosen to be compact.

We will prove in this section:

**Theorem 6.1.** *Let $G$ be a semialgebraic, semialgebraically simple group over a real closed field $R$. Suppose $G$ is not of compact type. Then $G$ is abstractly simple.***

Let us connect “compact type” with other well-known properties. We first recall a notion from algebraic group theory. Let $H$ be a connected linear algebraic group defined over a perfect field $k$. $H$ is said to be $k$-isotropic if $H$ has an algebraic subgroup $T$ defined over $k$ and isomorphic over $k$ to some power of the multiplicative group. $T$ is what is called a $k$-split torus. $H$ is said to be $k$-anisotropic if it is not $k$-isotropic. See Section 2.1 of [11] for more on this notion and also for background on other notions from algebraic group theory which we will be using below.

**Remark 6.2.** Let $R$ be a real closed field, and let $H$ be an $R$-simple algebraic group defined over $R$. Let $G = H(R)^0$. The following are equivalent:

- (i) $G$ is of compact type.
- (ii) $G$ is closed and bounded (so definably compact in the language of [8]).
- (iii) $H$ is $R$-anisotropic.

**Proof.** As in the proof of the “(2) implies (3)” direction of Theorem 5.1, we may assume $H$ to be defined over the real algebraic numbers. Then $H$ is $R$-isotropic if and only if $H$ is $\mathbb{R}$-isotropic. By 24.6 in [1], $H$ is $\mathbb{R}$-anisotropic if and only if $H(\mathbb{R})$ is connected and compact. This is enough.

Before going into the proof of Theorem 6.1 we will give some restatements and consequences. We will leave the proofs to the interested readers. The first depends on the notion “definable compactness” in arbitrary $o$-minimal structures [8].
Corollary 6.3. Let $M$ be an o-minimal structure, and let $G$ be a definably simple non-definably-compact group definable in $M$. Then $G$ is simple as an abstract group.

Corollary 6.4. Let $G$ be a simple noncompact Lie group. Then there is $k < \omega$ such that for each $a \in G$ with $a \neq e$, $G = (a^{\pm 1})^G \cdots (a^{\pm 1})^G$ ($k$ times), where $a^\pm$ means $a$ or $a^{-1}$.

The final statement is in the language of nonstandard analysis. For any structure $X$, $^*X$ denotes the nonstandard extension in the nonstandard universe $^*V$.

Corollary 6.5. Let $G$ be a simple Lie group. Then $^*G$ is simple as an abstract group iff $G$ is noncompact.

Our proof of Theorem 6.1 will go through another restatement; the Kneser–Tits conjecture for real closed fields. This latter result does not appear explicitly in the literature but Platonov has informed us that he has an unpublished proof. We will state the Kneser–Tits conjecture, give a proof in the real-closed field case, and then deduce Theorem 6.1.

We first repeat some of the definitions given earlier. Let $K$ be a perfect field, and let $G$ be a connected (linear say) algebraic group defined over $K$. (We identify $G$ with its set of $\bar{K}$ points, where $\bar{K}$ is the algebraic closure of $K$.) $G$ is said to be almost $K$-simple if it has no nontrivial connected normal algebraic subgroup defined over $K$. $G$ is said to be simply connected if there is no connected algebraic group $H$ and isogeny from $H$ onto $G$ (it is enough to consider $H$ defined over $K$). $G$ is said to be $K$-isotropic if it has some nontrivial $K$-split torus defined over $K$. Let us assume $G$ to be almost $K$-simple. $G(K)^+$ is defined to be the subgroup of $G(K)$ generated by the unipotent elements, clearly a normal subgroup of $G(K)$. The following will be crucial. (i) is due to Tits [13] and (ii) is in [2].

Fact 6.6. Suppose $G$ is almost $K$-simple and $K$-isotropic. Then

(i) $G(K)^+$ has no infinite normal subgroups.

(ii) If $G$ is simply connected then $G(K)^+$ contains $T(K)$ for every $K$-split torus $T$ of $G$.

The Kneser–Tits conjecture is:

Conjecture 6.7. Let $G$ be a simply connected, almost $K$-simple, $K$-isotropic algebraic group defined over $K$. Then $G(K)^+ = G(K)$; hence (by Fact 6.6(ii)) $G(K)$ has no normal infinite subgroups.

See [11] for more information on this conjecture. It is mentioned there that Platonov gave a counterexample.
**Proposition 6.8.** Conjecture 6.7 is true for \( K \) a real closed field.

**Proof.** Let \( G \) be an almost \( R \)-simple, simply connected, \( R \)-isotropic algebraic group defined over the real closed field \( R \). We have to prove that \( G(R) \) has no infinite proper normal subgroups.

**Claim 1.** We may assume that \( R \) is a saturated real closed field (hence contains \( R \)).

**Proof of Claim 1.** Let \( R \) be an elementary extension of \( R \). Suppose that \( H \) is an infinite proper normal subgroup of \( G(R) \) and let \( d \in G(R) \backslash H \). Let \( a_1, \ldots, a_n \in H \). So \( d \notin a_1^{G(R)} \cdots a_n^{G(R)} \). This latter statement transfers to \( R_1 \) to show that \( d \) is not contained in the normal subgroup \( H_1 \) of \( G(R_1) \) generated by \( H \), whereby \( H_1 \) is a proper infinite normal subgroup of \( G(R_1) \).

As in the proof of Theorem 5.1 (2 implies 3), we may assume that \( G \) is defined over \( R \), and note that \( G \) is \( R \)-isotropic.

**Claim 2.** \( G(R) \) has no infinite normal subgroups.

**Proof.** Proposition 7.6 of [11] affirms the truth of the Kneser–Tits conjecture for \( R \). Alternatively we could use Remark 6.2 (which has a model-theoretic proof) once we know that \( G(R) \) is semialgebraically connected (or equivalently connected as a Lie group). We do not know a direct proof for this latter fact.

By a *finite* element of \( G(R) \) we mean an element of \( G(R) \) with a standard part in \( G(R) \), equivalently an element whose distance in the real closed field \( R \) to the identity element of \( G(R) \) is less than some natural number.

**Claim 3.** \( G(R)^+ \) contains all finite elements of \( G(R) \).

**Proof.** Let \( X \) be the set of unipotent elements of \( G(R) \). This is an infinite semialgebraic normal subset of \( G(R) \). By Claim 2, \( X \) generates \( G(R) \). On the other hand, by [10] for example, some finite product \( X \cdot X \cdots X \) contains an open nonempty subset of \( G(R) \). Thus for every \( a \in G(R) \) there is \( n_a \in \mathbb{N} \), such that some open neighbourhood of \( a \) in \( G(R) \) is contained in \( X \cdot X \cdots X \) (\( n_a \) times). Thus, if \( C \) is any compact semialgebraic subset of \( G(R) \) there is some \( n_c \) such that \( C \) is contained in \( X \cdot X \cdots X \) (\( n_c \) times). Now any finite element \( b \) of \( G(R) \) is contained in \( C^R \) (the interpretation of \( C \) in \( R \)) for some compact semialgebraic subset \( C \) of \( G(R) \). So by transfer, \( b \) is a product of at most \( n_c \) unipotent elements of \( G(R) \). We have proved Claim 3.

At this point the proof of Proposition 7.6 of [11] could be adapted to yield the desired conclusion. We give a somewhat simpler proof.
CLAIM 4. $G(R)^+ \text{ contains all semisimple elements of } G(R)$.

Proof. Let $g \in G(R)$ be semisimple. Then there is some algebraic torus $S < G$ defined over $R$ containing $g$ (see [12]). $S$ decomposes as $S' \cdot S''$ where $S'$ is $R$-split and $S''$ is $R$-anisotropic. By Fact 6.6(ii), $S'(R)$ is contained in $G(R)^+$. On the other hand, $S'(R)$ is conjugate by some $x \in G(R)$ to a subgroup $T$ consisting of finite elements of $G(R)$. (In the Lie group $G(R)$ all maximal compact subgroups are conjugate. Fix a maximal compact $H \subset G(R)$ $H$ is real algebraic. Any closed, bounded and hence conjugate to a subgroup $T$ consisting of finite elements of $G(R)$ in the Lie group $G(R)$ all maximal compact subgroups are conjugate. Fix a maximal compact $H \subset G(R)$ $H$ is real algebraic. Any closed, bounded and hence conjugate to a subgroup $T$ consisting of finite elements of $G(R)$.)

We now conclude the proof. Let $Y$ be the set of semisimple elements of $G$. It is well known that $Y$ is Zariski-dense in $G$ (by 11.10 and 13.17 of [1] the union of the conjugates of a maximal torus of $G$ is Zariski-dense). So if $g \in G(R)$ then $g \cdot Y \cap Y$ is Zariski-dense in $G$ and defined over $R$. As $G(R)$ is Zariski-dense in $G$, $(g \cdot Y \cap Y) \cap G(R)$ is nonempty. Thus $g$ is a product of two semisimple elements of $G(R)$. By Claim 4, $G(R)^+ = G(R)$. (Alternatively we could use the fact that every element of $G(R)$ is a product of a semisimple element and a unipotent element, both also in $G(R)$.)
of 6.3, 6.4, and 6.5 appropriate to their respective levels of generality, in particular not making use of the object $G^+$ or of the structure theory of algebraic groups.

REFERENCES