



ELSEVIER

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde



Exponential elliptic boundary value problems on a solid torus in the critical of supercritical case

Athanasios Cotsoioli^{*}, Nikos Labropoulos

Department of Mathematics, University of Patras, Patras 26110, Greece

ARTICLE INFO

Article history:

Received 3 March 2010

Available online 11 May 2011

ABSTRACT

In this paper we investigate the behavior and the existence of positive and non-radially symmetric solutions to nonlinear exponential elliptic model problems defined on a solid torus \bar{T} of \mathbb{R}^3 , when data are invariant under the group $G = O(2) \times I \subset O(3)$. The model problems of interest are stated below:

$$\Delta v + \gamma = f(x)e^v, \quad v > 0 \text{ on } T, \quad v|_{\partial T} = 0 \quad (\mathbf{P}_1)$$

and

$$\begin{aligned} \Delta v + a + fe^v &= 0, \quad v > 0 \text{ on } T, \\ \frac{\partial v}{\partial n} + b + ge^v &= 0 \quad \text{on } \partial T. \end{aligned} \quad (\mathbf{P}_2)$$

We prove that exist solutions which are G -invariant and these exhibit no radial symmetries. In order to solve the above problems we need to find the best constants in the Sobolev inequalities in the exceptional case.

© 2011 Elsevier Inc. All rights reserved.

1. Introduction

In recent years, significant progress has been made on the analysis of a number of important features of nonlinear partial differential equations of elliptic and parabolic type. The study of these equations has received considerable attention, because of their special mathematical interest and because of practical applications of the torus in scientific research today. For example in Astronomy,

^{*} Corresponding author.

E-mail addresses: cotsioli@math.upatras.gr (A. Cotsoioli), nal@upatras.gr (N. Labropoulos).

investigators study the torus which is a significant topological feature surrounding many stars and black holes [26]. In Physics the torus is being explored at the National Spherical Torus Experiment (NSTX) at Princeton Plasma Physics Laboratory to test the fusion physics principles for the spherical torus concept at the MA level [36]. In Biology some investigators interested in circular DNA molecules detected in a large number of viruses, bacteria, and higher organisms. In this topologically very interesting type of molecule, superhelical turns are formed as the Watson–Crick double helix winds in a torus formation [25].

Let the solid torus be represented by the equation

$$\bar{T} = \{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - l)^2 + z^2 \leq r^2, l > r > 0\},$$

and the subgroup $G = O(2) \times I$ of $O(3)$. Note that the solid torus $\bar{T} \subset \mathbb{R}^3$ is invariant under the group G .

We consider the following nonlinear exponential elliptic boundary problems

$$\Delta v + \gamma = f(x)e^v, \quad v > 0 \text{ on } T, \quad v|_{\partial T} = 0 \tag{P1}$$

and

$$\begin{aligned} \Delta v + a + fe^v &= 0, \quad v > 0 \text{ on } T, \\ \frac{\partial v}{\partial n} + b + ge^v &= 0 \quad \text{on } \partial T, \end{aligned} \tag{P2}$$

where $\Delta v = -\nabla^i \nabla_i v$ is the Laplacian of v , $\frac{\partial}{\partial n}$ is the outer unit normal derivative, f, g are two smooth G -invariant functions and $\gamma, a, b \in \mathbb{R}$.

Clearly, a radially symmetric solution is a G -invariant solution, for any subgroup G of $O(n)$. The converse problem is considered in this paper, that is we prove that there exist positive solutions which are G -invariant and non-radially symmetrical if $G = O(2) \times I$.

Problems (P1) and (P2) own their origin to the “Nirenberg Problem” posed in 1969–1970 in the following way:

Given a (positive) smooth function f on (\mathbb{S}^2, g_0) (close to the constant function, if we want), is it the scalar curvature of a metric g conformal to g_0 ? (g_0 is the standard metric whose sectional curvature is 1) (see [3]).

Recall that, if we write g in the form $g = e^u g_0$, the problem is equivalent to solving the equation:

$$\Delta u + 2 = fe^u.$$

Nirenberg Problem has been studied extensively and is completely solved (see [2,8,45,37,19]). Further, we refer the reader to [14,15,13,11,12,39–41,9,10,33,38,7,30,1,16,43], in which the authors study this problem or its generalization.

Best constants in Sobolev inequalities are fundamental in the study of nonlinear PDEs on manifolds, because of their strong connection with the existence and the multiplicity of the solutions of the corresponding problems (see for example [46,5,47,35,34,29,6,7,22–24] and the references therein). It is also well known, that Sobolev embeddings can be improved in the presence of symmetries in the sense that we obtain continuous embeddings in higher L^p spaces, that it, allow us to solve equations with higher critical exponents (see for example [42,28,20,21,35,27,6,7,30,31,22–24] and the references therein). Especially, in our case we solve problems with the highest supercritical exponent (critical of supercritical).

Let:

$$\begin{aligned} C_{0,G}^\infty &= \{v \in C_0^\infty(T) : v \circ \tau = v, \forall \tau \in G\}, \\ C_G^\infty &= \{v \in C^\infty(T) : v \circ \tau = v, \forall \tau \in G\} \end{aligned}$$

and

$$L_G^p = \{v \in L^p(T) : v \circ \tau = v, \forall \tau \in G\}$$

that is, the spaces of all G -invariant functions under the action of the group $G = O(2) \times I$.

We define the Sobolev space $H_{1,G}^p(T)$, $p \geq 1$ as the completion of $C_G^\infty(T)$ with respect to the norm

$$\|v\|_{H_1^p} = \|\nabla v\|_p + \|v\|_p$$

and $\dot{H}_{1,G}^p(T)$ as the closure of $C_{0,G}^\infty(T)$ in $H_{1,G}^p(T)$.

In [22] we proved that for any $p \in [1, 2)$ real, the embedding $H_{1,G}^p(T) \hookrightarrow L_G^q(T)$ is compact for $1 \leq q < \frac{2p}{2-p}$, while the embedding $H_{1,G}^p(T) \hookrightarrow L_G^{\frac{2p}{2-p}}(T)$ is only continuous. Also, in [23] we proved that for any $p \in [1, 2)$ real, the embedding $H_{1,G}^p(T) \hookrightarrow L_G^q(\partial T)$ is compact for $1 \leq q < \frac{p}{2-p}$, while the trace embedding $H_{1,G}^p(T) \hookrightarrow L_G^{\frac{p}{2-p}}(\partial T)$ is only continuous. Additionally, we observe that if $\frac{3}{2} < p < 2$ then $q = \frac{2p}{2-p} > 6 = \frac{2 \cdot 3}{3-2}$ and $\tilde{q} > \frac{p}{2-p} > 4 = \frac{2(3-1)}{3-2}$, that is the exponents q and \tilde{q} are supercritical.

In this paper, we study the exceptional case when $p = n - k = 3 - 1 = 2$. In this case $H_{1,G}^2(T) \not\subset L_G^\infty(T)$, however, when $v \in H_{1,G}^2(T)$ we have $e^v \in L_G^1(T)$, $e^v \in L_G^1(\partial T)$ and the exponent $p = 2$ is the critical of supercritical.

This paper is organized as follows: In Section 2, we recall some definitions and we present the two lemmas on which are based the proofs of the theorems concerning the best constants. Proofs of these lemmas are in Section 6. Section 3 is devoted to the presentation of results of the paper. In Section 4, we determine the best constants μ and $\tilde{\mu}$ of the inequalities:

$$\int_T e^v dV \leq C \exp \left[\mu \|\nabla v\|_2^2 + \frac{1}{2\pi^2 r^2 l} \int_T v dV \right]$$

and

$$\int_{\partial T} e^v dS \leq C \exp \left[\tilde{\mu} \|\nabla v\|_2^2 + \frac{1}{4\pi^2 r l} \int_{\partial T} v dS \right].$$

In Section 5, we use the above two inequalities, in order to solve the nonlinear exponential elliptic problems (P_1) and (P_2) . Concerning problem (P_1) , we prove the existence of solutions of the associated variational problem. We study problem (P_2) in the same way as the (P_1) , except its last part (case 4 of Theorem 3.4), which is based upon the method of upper solutions and lower solutions.

2. Notations and preliminary results

For completeness we cite some background material and results from [23].

Let $\mathcal{A} = \{(\Omega_i, \xi_i) : i = 1, 2\}$ be an atlas on T defined by

$$\begin{aligned} \Omega_1 &= \{(x, y, z) \in T : (x, y, z) \notin H_{XZ}^+\}, \\ \Omega_2 &= \{(x, y, z) \in T : (x, y, z) \notin H_{XZ}^-\} \end{aligned}$$

where

$$H_{XZ}^+ = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y = 0\},$$

$$H_{XZ}^- = \{(x, y, z) \in \mathbb{R}^3 : x < 0, y = 0\}$$

and $\xi_i : \Omega_i \rightarrow I_i \times D, i = 1, 2$, with $I_1 = (0, 2\pi), I_2 = (-\pi, \pi)$,

$$D = \{(t, s) \in \mathbb{R}^2 : t^2 + s^2 < 1\}, \quad \partial D = \{(t, s) \in \mathbb{R}^2 : t^2 + s^2 = 1\},$$

$\xi_i(x, y, z) = (\omega_i, t, s), i = 1, 2$ with $\cos \omega_i = \frac{x}{\sqrt{x^2+y^2}}, \sin \omega_i = \frac{y}{\sqrt{x^2+y^2}}$, where

$$\omega_1 = \begin{cases} \arctan \frac{y}{x}, & x \neq 0, \\ \pi/2, & x = 0, y > 0, \\ 3\pi/2, & x = 0, y < 0, \end{cases} \quad \omega_2 = \begin{cases} \arctan \frac{y}{x}, & x \neq 0, \\ \pi/2, & x = 0, y > 0, \\ -\pi/2, & x = 0, y < 0, \end{cases}$$

and

$$t = \frac{\sqrt{x^2 + y^2} - l}{r}, \quad s = \frac{z}{r}, \quad 0 \leq t, s \leq 1.$$

The Euclidean metric g on $(\Omega, \xi) \in \mathcal{A}$ can be expressed as

$$(\sqrt{g} \circ \xi^{-1})(\omega, t, s) = r^2(l + rt).$$

For any G -invariant ν we define the functions $\phi(t, s) = (\nu \circ \xi^{-1})(\omega, t, s)$. Then we have:

$$\int_T e^\nu dV = 2\pi r^2 \int_D e^{\phi(t,s)} (l + rt) dt ds, \tag{1}$$

$$\|\nabla \nu\|_{L^2(T)}^2 = 2\pi \int_D |\nabla \phi(t, s)|^2 (l + rt) dt ds \tag{2}$$

and

$$\int_{\partial T} e^\nu dS = 2\pi r \int_{\partial D} e^{\phi(t,0)} (l + rt) dt, \tag{3}$$

where by ϕ we denote the extension of ϕ on ∂D .

Consider a finite covering $(T_j)_{j=1,\dots,N}$, where

$$T_j = \{(x, y, z) \in \bar{T} : (\sqrt{x^2 + y^2} - l_j)^2 + (z - z_j)^2 < \delta_j^2\}$$

is a tubular neighborhood (an open small solid torus) of the orbit O_{P_j} of P_j under the action of the group G . $P_j(x_j, y_j, z_j) \in \bar{T}$ and $l_j = \sqrt{x_j^2 + y_j^2}$ is the horizontal distance of the orbit O_{P_j} from the axis z/z and $\delta_j = l_j \varepsilon_j$ for any $\varepsilon_j > 0$.

Then the following lemmas hold:

Lemma 2.1.

1. For all $\varepsilon > 0$, there exists a constant C_ε , such that for all $v \in \dot{H}_{1,G}^2(T_j)$ the following inequality holds:

$$\int_{T_j} e^v dV \leq C_\varepsilon \exp \left[(1 + c\varepsilon) \frac{1}{32\pi^2 l_j} \|\nabla v\|_2^2 \right],$$

where $c > 0$.

2. For all $\varepsilon > 0$, there exist constants C_ε and D_ε , such that for all $v \in \dot{H}_{1,G}^2$ the following inequality holds:

$$\int_T e^v dV \leq C_\varepsilon \exp \left[\left(\frac{1}{32\pi^2(l-r)} + \varepsilon \right) \|\nabla v\|_2^2 + D_\varepsilon \|v\|_2^2 \right].$$

In addition the constant $\frac{1}{32\pi^2(l-r)}$ is the best constant for the above inequality.

Lemma 2.2. Let \bar{T} be the solid torus, $2\pi^2 r^2 l$ be the volume of T and $4\pi^2 r l$ be the volume of ∂T , then for all $\varepsilon > 0$ there exists a constant C_ε such that:

1. For all functions $v \in \mathcal{H}_G$ the following inequality holds:

$$\int_T e^v dV \leq C_\varepsilon \exp \left[(\mu + \varepsilon) \|\nabla v\|_2^2 + \frac{1}{2\pi^2 r^2 l} \int_T v dV \right]. \tag{4}$$

2. For all functions $v \in \mathcal{H}_G$ the following inequality holds:

$$\int_{\partial T} e^v dS \leq C_\varepsilon \exp \left[(\mu + \varepsilon) \|\nabla v\|_2^2 + \frac{1}{4\pi^2 r l} \int_{\partial T} v dS \right], \tag{5}$$

where, for the first inequality, $\mu = \frac{1}{32\pi^2(l-r)}$ if $\mathcal{H}_G = \dot{H}_{1,G}^2$ and $\mu = \frac{1}{16\pi^2(l-r)}$ if $\mathcal{H}_G = H_{1,G}^2$.

For the second inequality $\mu > \frac{1}{8\pi^2(l-r)}$ for all $v \in H_{1,G}^2$.

The constant μ is the best constant for the above inequalities.

3. Statement of results

3.1. Best constants on the solid torus

We have the following theorem:

Theorem 3.1. Let \bar{T} be the solid torus, $2\pi^2 r^2 l$ be the volume of T and $4\pi^2 r l$ be the volume of ∂T , then there exists a constant C such that:

1. For all functions $v \in \mathcal{H}_G$ the following inequality holds:

$$\int_T e^v dV \leq C \exp \left[\mu \|\nabla v\|_2^2 + \frac{1}{2\pi^2 r^2 l} \int_T v dV \right]. \tag{6}$$

2. For all functions $v \in \mathcal{H}_G$ the following inequality holds:

$$\int_{\partial T} e^v dS \leq C \exp \left[\mu \|\nabla v\|_2^2 + \frac{1}{4\pi^2 r l} \int_{\partial T} v dS \right], \tag{7}$$

where, for the first inequality, $\mu = \frac{1}{32\pi^2(l-r)}$ if $\mathcal{H}_G = \dot{H}_{1,G}^2$ and $\mu = \frac{1}{16\pi^2(l-r)}$ if $\mathcal{H}_G = H_{1,G}^2$.

For the second inequality $\mu > \frac{1}{8\pi^2(l-r)}$ for all $v \in H_{1,G}^2$.

The constant μ is the best constant for the above inequalities.

Remark 3.1. In [32] Faget proved that for a compact 3-dimensional manifold without boundary the first best constant for inequality (6) is $\mu_3 = \frac{2}{81\pi}$ and the map: $H_1^3 \ni v \rightarrow e^v \in L^1$ is compact. Clearly, the best constant μ_3 depends only on the dimension 3 of the manifold. For the solid torus, we prove that the first best constant for the same inequality (6) is $\mu = \frac{1}{32\pi^2(l-r)}$ and the map: $H_{1,G}^2 \ni v \rightarrow e^v \in L_G^1$ is compact. In this case, the best constant μ depends on the geometry of the solid torus.

Corollary 3.1. For all $v \in \dot{H}_{1,G}^2$ such that $\|\nabla v\|_2^2 \leq 2\pi(l+r)$ and for all $\alpha \leq 4\pi$ the following holds:

$$\int_T e^{\alpha v^2} dV \leq C 2\pi^2 r^2 l \tag{8}$$

where the constant C is independent of $v \in \dot{H}_{1,G}^2$. The constant $\alpha \leq 4\pi$ is the best, in the sense that, if $\alpha > 4\pi$ the integral in the inequality is finite but it can be made arbitrary large by an appropriate choice of v .

Remark 3.2. Corollary 3.1 is a special case of the result of Moser [44].

3.2. Resolutions of the problems

For the problem

$$\Delta v + \gamma = f(x)e^v, \quad v > 0 \text{ on } T, \quad v|_{\partial T} = 0, \tag{P1}$$

we have the theorem:

Theorem 3.2. Consider a solid torus \bar{T} and the function f continuous and G -invariant.

Then the problem (P1) accepts a solution that belongs to C_G^∞ , if one of the following holds:

- (a) $\sup_T f < 0$ if $\gamma < 0$.
- (b) $\int_T f dV < 0$ and $\sup_T f > 0$ if $\gamma = 0$.
- (c) $\sup_T f > 0$ if $0 < \gamma < \frac{8(l-r)}{r^2}$.

For the problem

$$\begin{aligned} \Delta v + a + fe^v &= 0, \quad v > 0 \text{ on } T, \\ \frac{\partial v}{\partial n} + b + ge^v &= 0 \quad \text{on } \partial T, \end{aligned} \tag{P2}$$

we have the next theorem:

Theorem 3.3. Consider a solid torus \bar{T} and the smooth functions f, g G -invariant and not both identical 0. If $a, b \in \mathbb{R}$ and $R = 2\pi^2 r^2 l a + 4\pi^2 r l b$, the problem (P_2) accepts a solution that belongs to C_G^∞ in each one of the following cases:

1. If $a = b = 0$ the necessary and sufficient condition is f and g not both ≥ 0 and that $\int_T f \, dV + \int_{\partial T} g \, dS > 0$.
2. If $a \geq 0$ and $b \geq 0$, f, g not both ≥ 0 everywhere and $0 < R < 4\pi^2(l - r)$. Particularly, if $g = 0$ then we can substitute the last condition with $0 < R < 8\pi^2(l - r)$.
3. If $R > 0$ (respectively $R < 0$) it is necessary that f, g not both ≥ 0 everywhere (respectively ≤ 0). Then there exists a solution of the problem in each one of the following cases:
 - (a) $a < 0, b > 0, f < 0, g \leq 0$ and $b < \frac{l-r}{lr}$ if $g \not\equiv 0$ or $b < \frac{2(l-r)}{lr}$ if $g \equiv 0$.
 - (b) $a > 0, b < 0, f \leq 0, g < 0$ and $a < \frac{2(l-r)}{lr^2}$.
 - (c) $a > 0, b < 0, f \geq 0, g > 0$ and $a < \frac{2(l-r)}{lr^2}$ if $g \not\equiv 0$ or $a < \frac{4(l-r)}{lr^2}$ if $g \equiv 0$.
 - (d) $a < 0, b > 0, f > 0, g \geq 0$ and $b < \frac{l-r}{lr}$.
4. If $a \leq 0, b \leq 0$, not both $= 0$, it is necessary $\int_T f \, dV + \int_{\partial T} g \, dS > 0$. Then there exists a nonempty subset $S_{f,g}$ of $\mathbb{R}_-^2 = \{(a, b) \neq (0, 0) : a \leq 0, b \leq 0\}$ with the property that if $(c, d) \in S_{f,g}$ then $(c', d') \in S_{f,g}$ for any $c' \geq c, d' \geq d$ and such that the problem (P_2) has a solution if and only if $(a, b) \in S_{f,g}$. $S_{f,g} = \mathbb{R}_-^2$ if and only if the functions f, g are $\not\equiv 0$ and ≥ 0 . For all $(a, b) \in \mathbb{R}_-^2$ there exist functions f and g such that $\int_T f \, dV + \int_{\partial T} g \, dS > 0$ and $(a, b) \notin S_{f,g}$.

4. Proofs of the theorem concerning the best constants

Proof of Theorem 3.1. 1. We give a proof by contradiction based on Lemma 2.2.

Assume that for any C_α , there exist $v_\alpha \in \dot{H}_{1,G}^2$ with $\int_T v_\alpha \, dV = 0$ such that

$$\int_T e^{v_\alpha} \, dV > C_\alpha \exp\left(\mu \int_T |\nabla v_\alpha|^2 \, dV\right). \tag{9}$$

Set $\phi_\alpha(t, s) = (v_\alpha \circ \xi^{-1})(\omega, t, s)$. By (9) because of (1) and (2) we obtain sequentially

$$2\pi r^2 \int_D e^{\phi_\alpha} (l + rt) \, dt \, ds > C_\alpha \exp\left(2\pi \mu \int_D |\nabla \phi_\alpha|^2 (l + rt) \, dt \, ds\right),$$

$$2\pi r^2 (l + r) \int_D e^{\phi_\alpha} \, dt \, ds > C_\alpha \exp\left(2\pi \mu (l - r) \int_D |\nabla \phi_\alpha|^2 \, dt \, ds\right),$$

and since $\mu = \frac{1}{32\pi^2(l-r)}$ (see part 1 of Lemma 2.2) we have

$$\int_D e^{\phi_\alpha} \, dt \, ds > \frac{C_\alpha}{2\pi r^2(l+r)} \exp\left(\frac{1}{16\pi} \int_D |\nabla \phi_\alpha|^2 \, dt \, ds\right).$$

The last inequality means that for any c_α , there exists $\phi_\alpha \in \dot{H}_1^2(D)$ with $\int_D \phi_\alpha \, dt \, ds = 0$, such that

$$\int_D e^{\phi_\alpha} \, dt \, ds > c_\alpha \exp\left(\frac{1}{16\pi} \int_D |\nabla \phi_\alpha|^2 \, dt \, ds\right),$$

which is a contradiction (see Theorem 1 in [17]).

2. The proof of this part is similar to the proof of the first one. Let us sketch it. Assume that for any \tilde{C}_α , there exist $v_\alpha \in H^2_{1,G}$ with $\int_{\partial T} v_\alpha dS = 0$ such that

$$\int_{\partial T} e^{v_\alpha} dS > \tilde{C}_\alpha \exp\left(\mu \int_T |\nabla v_\alpha|^2 dV\right) \tag{10}$$

and define the function ϕ_α as in the first part.

By (10) because of (2) and (3) we take the inequality

$$\int_{\partial D} e^{\phi_\alpha} dt > \tilde{c}_\alpha \exp\left(\mu \int_D |\nabla \phi_\alpha|^2 dt ds\right),$$

where $\tilde{c}_\alpha = \frac{\tilde{C}_\alpha}{2\pi r(l+r)}$.

The last inequality is false (see Theorem 3 in [18]) and the theorem is proved. \square

Proof of Corollary 3.1. Given $\varepsilon > 0$, let $(T_j)_{j=1,\dots,N}$ be a finite covering of \bar{T} , where

$$T_j = \{Q \in \mathbb{R}^3: d(Q, O_{P_j}) < \delta_j, \delta_j = l_j \varepsilon_j \text{ and } \varepsilon_j \leq \varepsilon\}.$$

For \bar{T} we build a G -invariant partition of unity $(h_j)_{j=1,\dots,N}$ relative to the T_j 's. If we denote $\Phi = v \circ \xi_j^{-1}$, $\Phi \in \dot{H}^2_1(D)$, for all $v \in \dot{H}^2_{1,G}$, following the same argument as in Lemma 2.1 we obtain

$$\begin{aligned} \int_T e^{\alpha v^2} dV &= \int_T \left(\sum_{j=1}^N h_j\right) e^{\alpha v^2} dV \\ &= \sum_{j=1}^N \int_{T_j} h_j e^{\alpha v^2} dV \\ &= \sum_{j=1}^N \int_{I \times D} (h_j \circ \xi_j^{-1}) e^{\alpha v^2 \circ \xi_j^{-1}} (\sqrt{g} \circ \xi_j^{-1}) d\omega dt ds \\ &= 2\pi \sum_{j=1}^N \int_D (h_j \circ \xi_j^{-1}) e^{\alpha v^2 \circ \xi_j^{-1}} \delta_j^2 (l_j + \delta_j t) dt ds \\ &= \frac{1}{\pi} \sum_{j=1}^N \int_D (h_j \circ \xi_j^{-1}) e^{\alpha \Phi^2} 2\pi^2 \delta_j^2 l_j (1 + \varepsilon_j t) dt ds \\ &\leq (1 + \varepsilon) \frac{1}{\pi} \sum_{j=1}^N \int_D (h_j \circ \xi_j^{-1}) e^{\alpha \Phi^2} \text{Vol}(T_j) dt ds \\ &\leq (1 + \varepsilon) \frac{1}{\pi} \text{Vol}(T) \sum_{j=1}^N \int_D (h_j \circ \xi_j^{-1}) e^{\alpha \Phi^2} dt ds \end{aligned}$$

$$\begin{aligned}
 &= (1 + \varepsilon)2\pi r^2 l \int_D \sum_{j=1}^N (h_j \circ \xi_j^{-1}) e^{\alpha \Phi^2} dt ds \\
 &= (1 + \varepsilon)2\pi r^2 l \int_D e^{\alpha \Phi^2} dt ds
 \end{aligned}$$

or

$$\int_T e^{\alpha v^2} dV \leq (1 + \varepsilon)2\pi r^2 l \int_D e^{\alpha \Phi^2} dt ds. \tag{11}$$

Because of $\|\nabla v\|_2^2 \leq 2\pi(l+r)$ and (2) we obtain $\|\nabla \Phi\|_2 \leq 1$ and according to Theorem 2.47 of [3] for all $\Phi \in \dot{H}_1^2(D)$ with $\|\nabla \Phi\|_2 \leq 1$ and for any $\alpha \leq 4\pi$ the following inequality holds

$$\int_D e^{\alpha \Phi^2} dt ds \leq C\pi, \tag{12}$$

where the constant C is the same for all open and bounded subsets of \mathbb{R}^2 .

Thus, from inequalities (11) and (12) we obtain

$$\int_T e^{\alpha v^2} dV \leq (1 + \varepsilon)C2\pi^2 r^2 l. \tag{13}$$

Suppose now that inequality (13) does not hold for $\varepsilon = 0$. That is, there exists $v \in \dot{H}_{1,G}^2$ with $\|\nabla v\|_2^2 \leq 2\pi(l+r)$ and $\theta > 0$ such that the following inequality holds

$$\int_T e^{\alpha v^2} dV \geq (1 + \theta)C2\pi^2 r^2 l. \tag{14}$$

By (14), and because of (11) we obtain

$$(1 + \varepsilon)2\pi r^2 l \int_D e^{\alpha \Phi^2} dt ds \geq (1 + \theta)C2\pi^2 r^2 l. \tag{15}$$

Since (15) holds for any $\varepsilon > 0$ we can choose ε such that $\varepsilon < \theta$ and (15) yields

$$2\pi r^2 l \int_D e^{\alpha \Phi^2} dt ds \geq \frac{1 + \theta}{1 + \varepsilon} C2\pi^2 r^2 l$$

or

$$\int_D e^{\alpha \Phi^2} dt ds > C\pi. \tag{16}$$

But according to Theorem 2.47 of [3] for all $\Phi \in \dot{H}_1^2(D)$ the following

$$\int_D e^{\alpha\Phi^2} dt ds \leq C\pi$$

holds. Thus (16) is false and the corollary is proved. \square

5. Proofs of the theorems concerning the problems

Proof of Theorem 3.2. We see that if f is a constant the problem can be solved immediately. If $f = 0$ and $\gamma = 0$, solutions are all the constants. If $\gamma f > 0$ the constant $\ln(\gamma/f)$ is the solution.

Consider the functional

$$I(v) = \int_T |\nabla v|^2 dV + 2\gamma \int_T v dV,$$

the set

$$A = \left\{ v \in H_{1,G}^2: \int_T f e^v dV = \gamma \text{Vol}(T) \right\}$$

and denote

$$v = \inf_{v \in A} I(v).$$

If $\gamma > 0$, in order $A \neq \emptyset$, it is necessary f to be somewhere positive, if $\gamma < 0$ it's necessary f to be somewhere negative, and if $\gamma = 0$ it is necessary f to change sign. In the following we accept that f satisfies the above necessary condition and it is not a constant.

(a) $\gamma < 0$ and f negative everywhere.

Combining Jensen's inequality:

$$\frac{1}{\text{Vol}(T)} \int_T v dV \leq \ln \left(\frac{1}{\text{Vol}(T)} \int_T e^v dV \right)$$

along with the following inequality:

$$\frac{1}{\text{Vol}(T)} \int_T e^v dV \leq \frac{1}{\text{Vol}(T) \sup f} \int_T f(x) e^v dV = \frac{\gamma}{\sup f}$$

we obtain

$$\int_T v dV \leq \text{Vol}(T) \ln \left(\frac{\gamma}{\sup f} \right) \tag{17}$$

and thus

$$I(v) \geq 2\gamma \text{Vol}(T) \ln \left(\frac{\gamma}{\sup f} \right).$$

From the last inequality we conclude that v is finite.

Let $\{v_i\} \in A$ be a minimizing sequence of I , that is $I(v_i) \rightarrow v$. If we take $I(v_i) \leq 1 + v$ we obtain

$$\int_T |\nabla v_i|^2 dV + 2\gamma \int_T v_i dV \leq 1 + v$$

thus

$$1 + v - 2\gamma \int_T v_i dV \geq \int_T |\nabla v_i|^2 dV \geq 0$$

and

$$\int_T v_i dV \leq \frac{1 + v}{2\gamma}. \tag{18}$$

By (17) and (18) we obtain $|\int_T v_i dV| < C$, where C is a constant.

In addition, we have

$$\int_T |\nabla v_i|^2 dV \leq 1 + v - 2\gamma \int_T v_i dV \leq 1 + v - 2\gamma C.$$

Thus $\{v_i\}$ is bounded in $H^2_{1,G}(T)$ and there exists a subsequence of v_i , denoted again by v_i and a function \bar{v} such that:

- (a) $\{v_i\} \rightharpoonup \bar{v}$ on $H^2_{1,G}(T)$ (by Banach’s theorem),
- (b) $\{v_i\} \rightarrow \bar{v}$ on $L^2_G(T)$ (by Kondrakov’s theorem),
- (c) $\{v_i\} \rightarrow \bar{v}$ a.e. (by Proposition 3.43 of [3]) and
- (d) $\{e^{v_i}\} \rightarrow e^{\bar{v}}$ on $L^1_G(T)$ (by Theorem 3.1).

From (c) arises that \bar{v} is G -invariant and so $\bar{v} \in A$, thus $I(\bar{v}) \geq v$.

From (d) we conclude that

$$\|\bar{v}\|_{H^2_1} \leq \liminf_{i \rightarrow \infty} \|v_i\|_{H^2_1} = v$$

and by definition of v we obtain $I(\bar{v}) = v$.

Using the variation method we can prove that \bar{v} is a weak solution of the corresponding Euler equation and, by the regularization Theorem of [48] and Theorem 3.54 of [3], we conclude that $\bar{v} \in C^\infty_G$.

(b) $\gamma = 0$ and f changes sign.

In this case we need the extra condition $\int_T f(x) dV < 0$, because if we multiply the equation of the problem by e^{-v} and integrate we obtain

$$\int_T f(x) dV = \gamma \int_T e^{-v} dV - \int_T e^{-v} |\nabla v|^2 dV,$$

the second part of this equality is negative.

Since $\gamma = 0$,

$$I(v) = \int_T |\nabla v|^2 dV$$

and considering $\int_T v dV = 0$, if we define the set

$$\tilde{A} = \left\{ v \in H^2_{1,G}: \int_T v dV = 0, \int_T f e^v dV = 0 \right\}$$

we will have

$$v = \inf_{v \in \tilde{A}} I(v) \geq 0.$$

In the following we work in the same way as in (a).

Thus, there exists a minimizing subsequence of v_i , denoted again by v_i that's converge on a function $\bar{v} \in \tilde{A}$.

If κ and λ are the Lagrange multipliers, the Euler equation is

$$\Delta \bar{v} + \kappa = \lambda f(x) e^{\bar{v}}.$$

Intergrating by parts, because of $\int_T f e^{\bar{v}} dV = 0$, we obtain $\kappa = 0$ and for the function \bar{v} holds

$$\Delta \bar{v} = \lambda f(x) e^{\bar{v}}. \tag{19}$$

By Eq. (19) we obtain that \bar{v} is not constant, because of $\int_T f(x) dV < 0$, and so $\lambda \neq 0$. In addition, multiplying the same equation by $e^{-\bar{v}}$ and integrating by parts we obtain $\lambda \int_T f(x) e^{\bar{v}} dV < 0$ and then $\lambda > 0$.

Finally, is easy to check that the solution of the equation is $\bar{v} - \ln \lambda$.

(c) $\gamma > 0$ and f somewhere positive.

Consider the same variation problem as in case (a) and suppose that f is somewhere positive, which is the necessary condition to be $A \neq \emptyset$, since $\sup_T f > 0$. We have

$$\gamma \text{Vol}(V) = \int_T f e^v dV \leq \sup_T f \int_T e^v dV. \tag{20}$$

In addition by Theorem 3.1 we have

$$\int_T e^v dV \leq C \exp \left\{ (\mu + \varepsilon) \int_T |\nabla v|^2 dV + \frac{1}{\text{Vol}(V)} \int_T v dV \right\}. \tag{21}$$

From (20) and (21) we obtain

$$\begin{aligned} \gamma \operatorname{Vol}(V) &\leq C \sup f \exp \left\{ (\mu + \varepsilon) \int_T |\nabla v|^2 dV + \frac{1}{\operatorname{Vol}(T)} \int_T v dV \right\}, \\ \frac{\gamma \operatorname{Vol}(V)}{C \sup f} &\leq \exp \left\{ (\mu + \varepsilon) \int_T |\nabla v|^2 dV + \frac{1}{\operatorname{Vol}(T)} \int_T v dV \right\}, \\ \ln \left(\frac{\gamma \operatorname{Vol}(T)}{C \sup f} \right) &\leq (\mu + \varepsilon) \int_T |\nabla v|^2 dV + \frac{1}{\operatorname{Vol}(T)} \int_T v dV, \\ 2\gamma \operatorname{Vol}(T) \ln \left(\frac{\gamma \operatorname{Vol}(T)}{C \sup f} \right) &\leq 2\gamma \operatorname{Vol}(T)(\mu + \varepsilon) \int_T |\nabla v|^2 dV + 2\gamma \int_T v dV, \\ 2\gamma \operatorname{Vol}(T) \ln \left(\frac{\gamma \operatorname{Vol}(T)}{C \sup f} \right) &\leq 2\gamma \operatorname{Vol}(T)(\mu + \varepsilon) \int_T |\nabla v|^2 dV + I(v) - \int_T |\nabla v|^2 dV, \\ I(v) &\geq 2\gamma \operatorname{Vol}(T) \ln \left(\frac{\gamma \operatorname{Vol}(T)}{C \sup f} \right) + [1 - 2\gamma \operatorname{Vol}(T)(\mu + \varepsilon)] \int_T |\nabla v|^2 dV \end{aligned}$$

or

$$I(v) \geq [1 - 2\gamma \operatorname{Vol}(T)(\mu + \varepsilon)] \int_T |\nabla v|^2 dV + C', \tag{22}$$

where $\mu = \frac{1}{32\pi^2(l-r)}$ and $C' = 2\gamma \operatorname{Vol}(T) \ln \left(\frac{\gamma \operatorname{Vol}(T)}{C \sup f} \right)$.

So, for $\gamma < \frac{8(l-r)}{l^2}$, we have that $I(v)$ is bounded below.

Thus if $v_i \in A$ is a minimizing sequence of I , by Eq. (22) we obtain that $\|\nabla v_i\|_2^2 \leq C_1$, and by Eqs. (20) and (21) that $\int_T v_i dV \geq C_2$, where C_1 and C_2 are constants. Since $v = \inf_{v \in A} I(v)$ and $\lim_{i \rightarrow \infty} I(v_i) = v$ we may assume that $I(v_i) < v + 1$ and so $\int_T v_i dV \leq C_3$, where C_3 is a constant. Thus $\{v_i\}$ is bounded in $H_{1,G}^2(T)$ and then the rest of the proof is the same as in case (a). \square

Proof of Theorem 3.3. Following [19], let $v \in C_G^\infty(\bar{T})$ be a solution of (P_2) . We observe that integration by parts yields

$$\begin{aligned} \int_T (\Delta v + a + fe^v) dV &= 0, \\ - \int_{\partial T} \frac{\partial v}{\partial n} dS + \int_T (a + fe^v) dV &= 0, \\ \int_{\partial T} (b + ge^v) dS + \int_T (a + fe^v) dV &= 0, \\ a \int_T dV + b \int_{\partial T} dS + \int_T fe^v dV + \int_{\partial T} ge^v dS &= 0, \\ a \operatorname{Vol}(T) + b \operatorname{Vol}(\partial T) + \int_T fe^v dV + \int_{\partial T} ge^v dS &= 0 \end{aligned}$$

namely

$$K(v) = a \operatorname{Vol}(T) + b \operatorname{Vol}(\partial T) + \int_T f e^v dV + \int_{\partial T} g e^v dS = 0. \quad (23)$$

Multiplying by e^{-v} and integrating by parts also implies

$$\begin{aligned} \int_T (e^{-v} \Delta v + a e^{-v} + f) dV &= 0, \\ - \int_{\partial T} e^{-v} \frac{\partial v}{\partial n} dS + \int_T (a e^{-v} + f) dV - \int_T e^{-v} |\nabla v|^2 dV &= 0, \\ \int_{\partial T} (e^{-v} b + g) dS + \int_T (a e^{-v} + f) dV - \int_T e^{-v} |\nabla v|^2 dV &= 0 \end{aligned}$$

namely

$$a \int_T e^{-v} dV + b \int_{\partial T} e^{-v} dS + \int_T f dV + \int_{\partial T} g dS - \int_T e^{-v} |\nabla v|^2 dV = 0. \quad (24)$$

Moreover, if $v \in H_{1,G}^2(T)$, according to [3,49,18] and because of Theorem 3.1, for any $q \geq 1$, $v \in L_G^q(T)$, $v \in L_G^q(\partial T)$ and $e^v \in L_G^q(T)$.

Set

$$I(v) = \frac{1}{2} \int_T |\nabla v|^2 dV + a \int_T v dV + b \int_{\partial T} v dS$$

and

$$A = \{v \in H_{1,G}^2: K(v) = 0\}.$$

Our aim is the minimization of $I(v)$ on A .

1. Case $a = b = 0$, $\int_T f dV + \int_{\partial T} g dS > 0$ and f and g not both ≥ 0 .

Since f and g are not both identically 0, the solutions of equation (P_2) are not constant functions. Hence if v is a solution we have

$$\int_T e^{-v} |\nabla v|^2 dV > 0 \quad (25)$$

and then by (24) and (25) yields

$$\int_T f dV + \int_{\partial T} g dS > 0.$$

Since $a = b = 0$ and $K(v) = \int_T f e^v dV + \int_{\partial T} g e^v dS$ in order $A = \{v \in H_1^2(T): K(v) = 0\} \neq \emptyset$ it's necessary f and g not to be both ≥ 0 .

Inversely, if f and g are not both ≥ 0 , we will prove that $A \neq \emptyset$.

Because of

$$\int_T f \, dV + \int_{\partial T} g \, dS > 0,$$

we have

$$\{f(T) \cup g(\partial T)\} \cap (0, +\infty) \neq \emptyset \quad \text{and} \quad \{f(T) \cup g(\partial T)\} \cap (-\infty, 0) \neq \emptyset.$$

Define a C^∞ function $\eta : [0, +\infty) \rightarrow [0, 1]$ such that $\eta = 1$ in $[0, 1/2]$, $\eta = 0$ in $[1, +\infty)$ and examine the following two cases:

(i) f changes sign on T .

There are two tori T_1 and T_2 contained in T such that $f > 0$ on T_1 and $f < 0$ on T_2 . Let the points P_i , $i = 1, 2$, belong to the central orbits O_{P_i} , $i = 1, 2$ of T_i , $i = 1, 2$, respectively and let

$$T_1 = \{(x, y, z) \in T : (\sqrt{x^2 + y^2} - l_{P_1})^2 + (z - z_{P_1})^2 < \delta^2\}$$

and

$$T_2 = \{(x, y, z) \in T : (\sqrt{x^2 + y^2} - l_{P_2})^2 + (z - z_{P_2})^2 < \delta^2\},$$

where $l_{P_i} = \sqrt{x_{P_i}^2 + y_{P_i}^2}$, $i = 1, 2$, the horizontal distance of the orbit O_{P_i} , $i = 1, 2$ from the axis z' .

Set

$$\alpha = \int_{T \setminus (T_1 \cup T_2)} f \, dV + \int_{\partial T} g \, dS,$$

and suppose that $\alpha \geq 0$. Then

$$\alpha_0 = \alpha + \int_{T_1} f \, dV > 0.$$

Consider the continuous function

$$\sigma(t) = \int_{T_2} f(P) \exp\left[t\eta\left(\frac{d(P, O_{P_2})}{\delta}\right)\right] dV, \quad t \in \mathbb{R},$$

where d is the Euclidean distance in \mathbb{R}^3 .

Since $\lim_{t \rightarrow +\infty} \sigma(t) = -\infty$ and $\lim_{t \rightarrow -\infty} \sigma(t) = 0$, there exists $t_0 \in \mathbb{R}$ such that $\sigma(t_0) = -\alpha_0$.

Hence if we define the function $\nu \in C_c^\infty(T)$ as

$$\nu(P) = \begin{cases} t_0 \eta(d(P, O_{P_2})/\delta), & P \in T_2, \\ 0, & P \notin T_2, \end{cases}$$

by definition of σ we obtain

$$\sigma(t_0) = \int_{\bar{T}_2} f(P)e^{\nu(P)} dV$$

and then

$$\int_T f(P)e^{\nu(P)} dV = -\alpha_0.$$

From the last equality we have

$$\begin{aligned} \int_{T_2} f e^\nu dV + \alpha + \int_{T_1} f dV &= 0, \\ \int_{T_2} f e^\nu dV + \int_{T \setminus (T_1 \cup T_2)} f dV + \int_{\partial T} g dS + \int_{T_1} f dV &= 0, \\ \int_{T_2} f e^\nu dV + \int_{T \setminus T_2} f dV - \int_{T_1} f dV + \int_{\partial T} g dS + \int_{T_1} f dV &= 0, \\ \int_{T_2} f e^\nu dV + \int_{T \setminus T_2} f dV + \int_{\partial T} g dS &= 0 \end{aligned}$$

and from this by definition of ν we obtain

$$\begin{aligned} \int_{T_2} f e^\nu dV + \int_{T \setminus T_2} f e^\nu dV + \int_{\partial T} g e^\nu dS &= 0, \\ \int_T f e^\nu dV + \int_{\partial T} g e^\nu dS &= 0. \end{aligned}$$

This means that $\nu \in A$ and hence $A \neq \emptyset$.

(ii) f does not change sign on T .

If $f \equiv 0$ and g changes sign, following arguments of the previous case, we construct a function $\nu \in C_G^\infty(\bar{T})$ such that $\int_{\partial T} g e^\nu dS = 0$, hence $K(\nu) = 0$ and $A \neq \emptyset$.

If $f \neq 0$, let us suppose that $f \geq 0$ and $K(\nu) = 0$. Then there exist $P_1 \in T$ and $P_2 \in \partial T$ such that $f(P_1) > 0$ and $g(P_2) < 0$.

Consider the tori

$$T_1 = \{(x, y, z) \in T : (\sqrt{x^2 + y^2} - l_{P_1})^2 + (z - z_{P_1})^2 < \delta^2\}$$

and

$$T_2 = \{(x, y, z) \in T : (\sqrt{x^2 + y^2} - l_{P_2})^2 + (z - z_{P_2})^2 < \delta^2\},$$

where δ is small enough, such that $\bar{T}_1 \cap \bar{T}_2 = \emptyset$, $f > 0$ a.e. in T_1 and $g < 0$ a.e. in $T_2 \cap \partial T$.

Set

$$\beta = \int_{T \setminus T_1} f \, dV + \int_{\partial T \setminus T_2} g \, dS + \int_{\partial T \cap T_2} g(P) \exp \left[t \eta \left(\frac{2d(P, O_{P_2})}{\delta} \right) \right] dS$$

and choose t large enough such that $\beta < 0$.

Denote

$$T_{2(\delta/2)} = \left\{ (x, y, z) \in T : (\sqrt{x^2 + y^2} - l_{P_2})^2 + (z - z_{P_2})^2 < \left(\frac{\delta}{2} \right)^2 \right\}$$

and define a function $\vartheta \in C^\infty(\bar{T})$, $0 \leq \vartheta(P) \leq 1$ such that $\vartheta = 1$ in a neighborhood of $\partial T \cap T_{2(\delta/2)}$, $\vartheta = 0$ out of T_2 and its support K to have small enough measure such that the following holds:

$$\gamma = \int_K f(P) \exp \left[t \vartheta(P) \eta \left(\frac{2d(P, O_{P_2})}{\delta} \right) \right] dV - \int_K f(P) dV < -\beta.$$

Consider now the continuous function

$$\hat{\sigma}(t) = \int_{T_1} f(P) \exp \left[t \eta \left(\frac{d(P, O_{P_1})}{\delta} \right) \right] dV, \quad t \in \mathbb{R}.$$

Since $f \geq 0$, $f \not\equiv 0$, $\lim_{t \rightarrow -\infty} \hat{\sigma}(t) = 0$ and $\lim_{t \rightarrow +\infty} \hat{\sigma}(t) = +\infty$ there exists $t' \in \mathbb{R}$ such that $\hat{\sigma}(t') = -(\beta + \gamma)$, that is

$$\int_{T_1} f(P) \exp \left[t' \eta \left(\frac{d(P, O_{P_1})}{\delta} \right) \right] dV = -(\beta + \gamma) > 0. \tag{26}$$

Define now the function $v \in C^\infty(\bar{T})$ by

$$v(P) = \begin{cases} t' \eta(d(P, O_{P_1})/\delta), & P \in T_1, \\ t \vartheta(P) \eta(2d(P, O_{P_2})/\delta), & P \in T_2, \\ 0, & P \notin T_1 \cup T_2. \end{cases}$$

We have

$$\begin{aligned} \beta &= \int_{T \setminus T_1} f \, dV + \int_{\partial T \setminus T_2} g \, dS + \int_{\partial T \cap T_2} g e^v \, dS \\ &= \int_{T \setminus T_1} f \, dV + \int_{\partial T \setminus T_2} g e^v \, dS + \int_{\partial T \cap T_2} g e^v \, dS \\ &= \int_{T \setminus T_1} f \, dV + \int_{\partial T} g e^v \, dS \end{aligned} \tag{27}$$

and

$$\begin{aligned}
 \gamma &= \int_{T_2} f(P) \exp \left[t\vartheta(P) \eta \left(\frac{2d(P, O_{P_2})}{\delta} \right) \right] dV \\
 &\quad - \int_{T_2 \setminus K} f(P) \exp \left[t\vartheta(P) \eta \left(\frac{2d(P, O_{P_2})}{\delta} \right) \right] dV - \int_K f(P) dV \\
 &= \int_{T_2} f e^\nu dV - \int_{T_2 \setminus K} f dV - \int_K f dV.
 \end{aligned} \tag{28}$$

By (26)–(28) we now obtain

$$\begin{aligned}
 \int_{T \setminus T_1} f dV - \int_{T_2 \setminus K} f dV - \int_K f dV + \int_{T_1} f e^\nu dV + \int_{T_2} f e^\nu dV + \int_{\partial T} g e^\nu dS &= 0, \\
 \int_{T \setminus T_1} f dV - \int_{T_2} f dV + \int_{T_1} f e^\nu dV + \int_{T_2} f e^\nu dV + \int_{\partial T} g e^\nu dS &= 0, \\
 \int_{T \setminus (T_1 \cup T_2)} f dV + \int_{T_1 \cup T} f e^\nu dV + \int_{\partial T} g e^\nu dS &= 0, \\
 \int_{T \setminus (T_1 \cup T_2)} f e^\nu dV + \int_{T_1 \cup T_2} f e^\nu dV + \int_{\partial T} g e^\nu dS &= 0, \\
 \int_T f e^\nu dV + \int_{\partial T} g e^\nu dS &= 0.
 \end{aligned}$$

Hence $\nu \in A$ and $A \neq \emptyset$.

We observe that if $K(\nu) = 0$ then $K(\nu + c) = 0$ for any constant c . So we can suppose that $\int_T \nu dV = 0$ for any $\nu \in A$.

Set

$$\mu = \inf_{\nu \in A} \left\{ \int_T |\nabla \nu|^2 dV : \int_T \nu dV = 0 \right\} \geq 0.$$

Let $\{v_i\}$ be a minimizing sequence. Since $\sup_i (\|\nabla v_i\|^2) < +\infty$, this is bounded in $H^2_{1,G}(T)$. Thus there exists a subsequence $\{v_i\}$ and a function $\nu \in H^2_{1,G}(T)$ such that:

- (a) $\{v_i\} \rightharpoonup \nu$ on $H^2_{1,G}(T)$ (by Banach’s theorem),
- (b) $\{v_i\} \rightarrow \nu$ on $L^q_G(T)$, $q \geq 1$ (by Kondrakov’s theorem),
- (c) $\{v_i\} \rightarrow \nu$ a.e. (by Proposition 3.43 of [3]),
- (d) $\{e^{v_i}\} \rightarrow e^\nu$ (by Theorem 3.1) and
- (e) $\{v_i\} \rightarrow \nu$ a.e., on ∂T and $\{e^{v_i}\} \rightarrow e^\nu$ on $L^q_G(\partial T)$,

where by v_i and ν we denote the trace of v_i and ν on ∂T , respectively (by Theorem 4 of [18]).

The latter implies

$$\lim_{i \rightarrow \infty} \int_T v_i dV = \int_T \nu dV$$

and

$$\lim_{i \rightarrow \infty} \left(\int_T f e^{v_i} dV + \int_{\partial T} g e^{v_i} dS \right) = \int_T f e^v dV + \int_{\partial T} g e^v dS = 0.$$

From the last two equalities along with (c) arises that $v \in A$ and $\int_T v dV = 0$, hence, by definition of μ , $\|\nabla v\|_2^2 \geq \mu$.

From (b) and using Theorem 3.17 of [3] we obtain

$$\|\nabla v\|_2^2 \leq \liminf_{i \rightarrow \infty} \|\nabla v_i\|_2^2 = \mu.$$

Thus, by definition of μ , $\|\nabla v\|_2^2 = \mu$ and the $\inf \|\nabla v_i\|_2^2$ is attained, where $v_i \in A$ and $\int_T v_i dV = 0$.

If κ and λ are the Lagrange multipliers, the Euler equation is

$$\int_T \nabla^i v \nabla_i h dV + \kappa \left(\int_T f e^v h dV + \int_{\partial T} g e^v h dS \right) + \lambda \int_T h dV = 0, \tag{29}$$

for all $h \in H^2_1(T)$.

Since $K(v) = 0$, for $h = 1$ arises $\lambda = 0$, and for $h = v$, $\kappa \neq 0$. (If $\kappa = 0$, $\|\nabla v\|_2 = 0$ and since $\int_T v dV = 0$, $v = 0$ a.e. thus $K(v) > 0$, which is false.)

According to Theorem 1 of [18] the solution $v \in H^2_{1,C}$ of (29) is C^∞ and if satisfies:

$$\left. \begin{aligned} \Delta v + \kappa f e^v &= 0 && \text{in } T, \\ \frac{\partial v}{\partial n} + \kappa g e^v &= 0 && \text{on } \partial T \end{aligned} \right\}. \tag{30}$$

Setting $h = e^{-v}$ in (29) we find

$$\kappa = \left(\int_T f dV + \int_{\partial T} g dS \right)^{-1} \int_T |\nabla v|^2 e^{-v} dV > 0 \tag{31}$$

and then $v - \ln \kappa$ is a solution of **(P₂)**.

2. Case $a \geq 0, b \geq 0$, not both $\equiv 0$ and

$$f^{-1}((-\infty, 0)) \neq \emptyset \quad \text{or} \quad g^{-1}((-\infty, 0)) \neq \emptyset.$$

In this case we have

$$R = a \text{Vol}(T) + b \text{Vol}(\partial T) > 0$$

and by (23)

$$\int_T f e^v dV + \int_{\partial T} g e^v dS < 0.$$

Then, if f, g are not both $\geq 0, A \neq \emptyset$.

By Theorem 3.1 arises that, for all $\varepsilon > 0$, there exists a constant C_ε such that

$$\int_T e^\nu dV \leq C_\varepsilon \exp \left[(1 + \varepsilon) \frac{1}{16\pi^2(l-r)} \|\nabla \nu\|_2^2 + \frac{1}{\text{Vol}(T)} \int_T \nu dV \right] \tag{32}$$

and

$$\int_{\partial T} e^\nu dS \leq C_\varepsilon \exp \left[(1 + \varepsilon) \frac{1}{8\pi^2(l-r)} \|\nabla \nu\|_2^2 + \frac{1}{\text{Vol}(\partial T)} \int_{\partial T} \nu dS \right], \tag{33}$$

for all $\nu \in H_{1,G}^2$.

From the definitions of $K(\nu)$ and R and by (23) we obtain

$$R = \left| \int_T f e^\nu dV + \int_{\partial T} g e^\nu dS \right| \leq \left(\max_{\bar{T}} |f| \right) \int_T e^\nu dV + \left(\max_{\partial T} |g| \right) \int_{\partial T} e^\nu dS$$

and using (32), (33) we obtain

$$\begin{aligned} R &\leq \left(\max_{\bar{T}} |f| \right) C_\varepsilon \exp \left[(1 + \varepsilon) \frac{1}{16\pi^2(l-r)} \|\nabla \nu\|_2^2 + \frac{1}{\text{Vol}(T)} \int_T \nu dV \right] \\ &\quad + \left(\max_{\partial T} |g| \right) C_\varepsilon \exp \left[(1 + \varepsilon) \frac{1}{8\pi^2(l-r)} \|\nabla \nu\|_2^2 + \frac{1}{\text{Vol}(\partial T)} \int_{\partial T} \nu dS \right] \\ &\leq \left(\max_{\bar{T}} |f| \right) C_\varepsilon \exp \left[(1 + \varepsilon) \frac{1}{8\pi^2(l-r)} \|\nabla \nu\|_2^2 + \frac{1}{\text{Vol}(T)} \int_T \nu dV \right] \\ &\quad + \left(\max_{\partial T} |g| \right) C_\varepsilon \exp \left[(1 + \varepsilon) \frac{1}{8\pi^2(l-r)} \|\nabla \nu\|_2^2 + \frac{1}{\text{Vol}(\partial T)} \int_{\partial T} \nu dS \right]. \end{aligned}$$

The last inequality gives

$$\inf_{\nu \in A} \left\{ \int_T \nu dV + (1 + \varepsilon) \frac{1}{8\pi^2(l-r)} \text{Vol}(T) \|\nabla \nu\|_2^2 \right\} = c_T(\varepsilon) > -\infty \tag{34}$$

and

$$\inf_{\nu \in A} \left\{ \int_{\partial T} \nu dS + (1 + \varepsilon) \frac{1}{8\pi^2(l-r)} \text{Vol}(\partial T) \|\nabla \nu\|_2^2 \right\} = c_{\partial T}(\varepsilon) > -\infty. \tag{35}$$

By (34), (35) we obtain

$$\begin{aligned}
 I(v) &= \frac{1}{2} \int_T |\nabla v|^2 dV + a \int_T v dV + b \int_{\partial T} v dS \\
 &\geq \left[\frac{1}{2} - (1 + \varepsilon) \frac{1}{8\pi^2(l-r)} (a \text{Vol}(T) + b \text{Vol}(\partial T)) \right] \|\nabla v\|_2^2 + ac_T(\varepsilon) + bc_{\partial T}(\varepsilon) \\
 &= \left[\frac{1}{2} - (1 + \varepsilon) \frac{1}{8\pi^2(l-r)} R \right] \|\nabla v\|_2^2 + ac_T(\varepsilon) + bc_{\partial T}(\varepsilon). \tag{36}
 \end{aligned}$$

If we assume $R < 4\pi^2(l-r)$ and if we choose $\varepsilon > 0$ such that $c = \frac{1}{2} - (1 + \varepsilon) \frac{1}{8\pi^2(l-r)} R > 0$, by (36) we conclude that $\mu = \inf_{v \in A} I(v) > -\infty$.

Let $\{v_i\}_{i \in \mathbb{N}}$, $v_i \in A$ be a minimizing sequence of $I(v)$ such that

$$\mu \leq I(v_i) \leq \mu + 1 \tag{37}$$

for any $i \in \mathbb{N}$ (36) and (37) yield

$$0 \leq \|\nabla v_i\|_2^2 \leq \frac{I(v_i) - ac_T(\varepsilon) - bc_{\partial T}(\varepsilon)}{c} \leq \frac{\mu + 1 - ac_T(\varepsilon) - bc_{\partial T}(\varepsilon)}{c} < +\infty.$$

By (34), (35) and (36) we also obtain

$$\int_T v_i dV \geq c_T(\varepsilon) - (1 + \varepsilon) \frac{1}{8\pi^2(l-r)} \text{Vol}(T)(\mu + 1) = C_T \tag{38}$$

and

$$\int_{\partial T} v_i dS \geq c_{\partial T}(\varepsilon) - (1 + \varepsilon) \frac{1}{8\pi^2(l-r)} \text{Vol}(\partial T)(\mu + 1) = C_{\partial T}. \tag{39}$$

By the definition of $I(v)$ and because of (37) yields

$$a \int_T v_i dV + b \int_{\partial T} v_i dS \leq I(v_i) \leq \mu + 1.$$

The last relation, because of (38), (39) gives us

$$\int_T v_i dV \leq \frac{\mu + 1}{a} - C_{\partial T} \quad \text{if } a \neq 0 \tag{40}$$

and

$$\int_{\partial T} v_i dS \leq \frac{\mu + 1}{b} - C_T \quad \text{if } b \neq 0. \tag{41}$$

By (37), (38), (40) and (41) we have

$$\left| \int_T v_i dV \right| \leq C_1 \tag{42}$$

and

$$\left| \int_{\partial T} v_i dS \right| \leq C_2. \tag{43}$$

Since the inequality

$$\int_T \phi^2 dV \leq C \left(\int_T |\nabla \phi|^2 dV + \left| \frac{1}{\text{Vol}(T)} \int_T \phi dV \right|^2 \right) \tag{44}$$

holds for any $\phi \in H^2_{1,G}(T)$, taking into account that (37) and (38) also hold, it follows that $\{v_i\}_{i \in \mathbb{N}}$, $v_i \in A$ is bounded in $L^2_G(T)$. Moreover, since (38) holds we conclude that $\sup_{i \in \mathbb{N}} (\|v_i\|_{H^2_{1,G}}) < \infty$. Hence, as in the previous case there exists $v \in A$ such that $I(v) = \mu$.

Recall that, if v is the Lagrange multiplier, the Euler equation is

$$\int_T \nabla^i v \nabla_i h dV + a \int_T h dV + b \int_{\partial T} h dS = v \left(\int_T f e^v dV + \int_{\partial T} g e^v dS \right), \tag{45}$$

for all $h \in H^2_1(T)$.

For $h = 1$ since $K(v) = 0$ we find

$$v = -(a \text{Vol}(T) + b \text{Vol}(\partial T)) \left(\int_T f e^v dV + \int_{\partial T} g e^v dS \right)^{-1} = 1.$$

Using the same arguments as in case 1, we prove that $v \in C^\infty(\bar{T})$ and that is a solution of (P_2) .

If $g \equiv 0$ we have

$$\begin{aligned} R &= \left| \int_T f e^v dV \right| \leq \left(\max_T |f| \right) \int_T e^v dV \\ &\leq \left(\max_T |f| \right) C_\varepsilon \exp \left[(1 + \varepsilon) \frac{1}{16\pi^2(l-r)} \|\nabla v\|_2^2 + \frac{1}{\text{Vol}(T)} \int_T v dV \right]. \end{aligned}$$

Hence, if $R < 8\pi^2(l-r)$, following the same process as above we prove that (P_2) has a solution.

3. Suppose that $R > 0$ and a, b not both ≥ 0 (the case $R < 0$ and a, b not both ≤ 0 can be treated in the same way).

By (23) it is necessary that f, g are not both ≥ 0 everywhere. Then $A \neq \emptyset$.

(a) $a < 0, b > 0, f < 0, g \leq 0$ and $b \text{Vol}(\partial T) < 4\pi^2(l-r)$ if $g \not\equiv 0$ or $b \text{Vol}(\partial T) < 8\pi^2(l-r)$ if $g \equiv 0$.

Since $f \in C^\infty(\bar{T})$ is negative everywhere and \bar{T} is compact, there exists $\delta > 0$ such that $|f| \geq \delta > 0$.

If $v \in A$ we have

$$|R| = \left| \int_T f e^v dV + \int_{\partial T} g e^v dS \right| = \int_T |f| e^v dV + \int_{\partial T} |g| e^v dS \tag{46}$$

and by elementary inequality $e^x \geq 1 + x$, $x \in \mathbb{R}$ we obtain

$$|R| \geq \int_T |f| e^v dV \geq \delta \int_T e^v dV \geq \delta \int_T (1 + v) dV = \delta \text{Vol}(T) + \delta \int_T v dV.$$

Since $a < 0$ we finally obtain

$$a \int_T v dV \geq a \left(\frac{|R|}{\delta} - \text{Vol}(T) \right). \tag{47}$$

By (33) implies that for any $\varepsilon > 0$ there exists a constant \tilde{C}_ε such that

$$|R| \leq \tilde{C}_\varepsilon \exp \left[(1 + \varepsilon) \frac{1}{8\pi^2(l-r)} \|\nabla v\|_2^2 + \frac{1}{\text{Vol}(\partial T)} \int_{\partial T} v dS \right]. \tag{48}$$

By (48) we obtain

$$b \int_{\partial T} v dS \geq b \text{Vol}(\partial T) \ln \frac{|R|}{\tilde{C}_\varepsilon} - (1 + \varepsilon) \frac{b \text{Vol}(\partial T)}{8\pi^2(l-r)} \|\nabla v\|_2^2. \tag{49}$$

By the definition of $I(v)$ and (47), (49) we obtain

$$I(v) \geq \left[\frac{1}{2} - (1 + \varepsilon) \frac{b \text{Vol}(\partial T)}{8\pi^2(l-r)} \right] \|\nabla v\|_2^2 + b \text{Vol}(\partial T) \ln \frac{|R|}{\tilde{C}_\varepsilon} + a \left(\frac{|R|}{\delta} - \text{Vol}(T) \right). \tag{50}$$

If $b \text{Vol}(\partial T) < 4\pi^2(l-r)$, that is $b < \frac{l-r}{l^r}$ and ε is chosen small enough, (50) implies that $I(v)$ is bounded below for all $v \in A$ and we can prove the existence of a solution of (P_2) as in the previous cases.

If $g \equiv 0$, it suffices to assume that $b < \frac{2(l-r)}{l^r}$ and then by (48) we obtain

$$|R| = \int_T |f| e^v dV \leq \tilde{C}_\varepsilon \exp \left[(1 + \varepsilon) \frac{1}{16\pi^2(l-r)} \|\nabla v\|_2^2 + \frac{1}{\text{Vol}(\partial T)} \int_{\partial T} v dS \right]$$

and we continue as above.

(b) $a > 0$, $b < 0$, $f \leq 0$, $g < 0$ and $a \text{Vol}(T) < 4\pi^2(l-r)$.

We work as in the previous case and, supposing that $a < \frac{2(l-r)}{l^r}$ we conclude the existence of a solution of (P_2) .

Cases (c) and (d) are similar to (b).

4. Case $a \leq 0$, $b \leq 0$, not both = 0.

By (24) it is necessary to assume that $\int_T f dV + \int_{\partial T} g dS > 0$ and by (23) arises that f, g are not both ≤ 0 a.e.

The proof of this case is based upon the method of upper solutions and lower solutions and is the same as the one in Theorem 2, case (iv) of [19].

Let us sketch the proof: It suffices to find functions $v_-, v_+ \in C_G^\infty(\bar{T})$ such that $v_+ \geq v_-$ which satisfy the equations

$$\left. \begin{aligned} \Delta v_+ + a + f e^{v_+} &\geq 0 && \text{in } T, \\ \frac{\partial v_+}{\partial n} + b + g e^{v_+} &\geq 0 && \text{on } \partial T \end{aligned} \right\} \tag{51}$$

and

$$\left. \begin{aligned} \Delta v_- + a + f e^{v_-} &\leq 0 && \text{in } T, \\ \frac{\partial v_-}{\partial n} + b + g e^{v_-} &\leq 0 && \text{on } \partial T \end{aligned} \right\} \tag{52}$$

respectively.

We denote by $P_{(a,b)}$ the nonlinear problem (P_2) and solve this case in four steps. More precisely, we prove that:

1. For any $u \in C_G^0(\bar{T})$, $P_{(a,b)}$ accepts a lower solution v_- such that $v_- \leq u$.
2. For any $u \in C_G^0(\bar{T})$, $P_{(a,b)}$ accepts an upper solution v_+ such that $v_+ \geq u$.
3. Choosing f, g appropriately, the set $S_{f,g}$ can be contained in $\mathbb{R}_-^2 = \{(a, b) \neq (0, 0) : a \leq 0, b \leq 0\}$ strictly.
4. If f, g are $\neq 0$ and nonnegative everywhere then $S_{f,g} = \mathbb{R}_-^2$. \square

6. Proofs of the lemmas

Proof of Lemma 2.1. 1. Let $\varepsilon_0 > 0$ and $(T_j)_{j=1,\dots,N}$ be a finite covering of \bar{T} , where

$$T_j = \{Q \in \mathbb{R}^3 : d(Q, O_{P_j}) < \delta_j, \delta_j = l_j \varepsilon_j \text{ and } \varepsilon_j \leq \varepsilon_0\}.$$

Then for any $v \in C_{0,G}^\infty(T_j)$ by (1) we obtain

$$\begin{aligned} \int_{T_j} e^v dV &= \int_{I \times D} e^{v \circ \xi_j^{-1}} (\sqrt{g} \circ \xi_j^{-1}) d\omega dt ds = 2\pi l_j \delta_j^2 \int_D e^\phi \left(1 + \frac{\delta_j}{l_j} t\right) dt ds \\ &\leq 2\pi l_j \delta_j^2 (1 + \varepsilon_0) \int_D e^\phi dt ds. \end{aligned}$$

From this and by Theorem 1 of [17] we have

$$\int_{T_j} e^v dV \leq 2\pi l_j \delta_j^2 C(1 + \varepsilon_0) \exp\left(\mu_2 \int_D |\nabla \phi|^2 dt ds\right),$$

where $\mu_2 = \frac{1}{16\pi}$ is the best constant of Sobolev inequality

$$\int_D e^f dV \leq C \exp[\mu_2 \|\nabla f\|_2^2],$$

with $f \in \dot{H}_1^2(D)$.

Moreover from (2) we obtain

$$\int_{T_j} |\nabla v|^2 dV = 2\pi \int_D |\nabla \phi|^2 (l_j + \delta_j t) dt ds \geq 2\pi l_j (1 - \varepsilon_0) \int_D |\nabla \phi|^2 dt ds,$$

thus

$$\int_D |\nabla \phi|^2 dt ds \leq \frac{1}{2\pi l_j} \frac{1}{1 - \varepsilon_0} \int_{T_j} |\nabla v|^2 dV.$$

Finally, we have

$$\int_{T_j} e^v dV \leq C_{\varepsilon_0} \exp \left[(1 + c\varepsilon_0) \frac{\mu_2}{2\pi l_j} \int_{T_j} |\nabla v|^2 dt ds \right],$$

where $C_{\varepsilon_0} = 2\pi l_j \delta_j^2 C(1 + \varepsilon_0)$ and $\frac{1}{1 - \varepsilon_0} = 1 + c\varepsilon_0, c > 0$.

2. Let us choose $\delta > 0$ such that the torus \bar{T} is covered by N open subsets

$$T_{j,\delta/2} = \{Q \in T: d(Q, O_{P_j}) < \delta/2\}.$$

We consider the decreasing real valued C^∞ function $\Psi(r)$, which equals 1 for $0 \leq r \leq \delta/2$ and 0 for $r \geq \delta$ and we note $\psi_j(Q) = \Psi(d(Q, O_{P_j}))$.

The ψ_j 's defined on $T_j = \{Q \in T: d(Q, O_{P_j}) < \delta\}$ are G -invariant, but they are not a partition of unity.

Let $v \in C_{0,G}^\infty(T)$. Then $(v\psi_j) \in C_{0,G}^\infty(T_j)$ and from the first part of this lemma we obtain

$$\int_{T_j} e^{v\psi_j} dV \leq C \exp \left[(1 + c\varepsilon_0) \frac{\mu_2}{2\pi l_j} \|\nabla(v\psi_j)\|_2^2 \right]. \tag{53}$$

Because of the following relation:

$$\|\nabla(v\psi_j)\|_2^2 \leq \|\psi_j \nabla v\|_2^2 + 2\|\psi_j \nabla v\| \|v \nabla \psi_j\| + \|v \nabla \psi_j\|_2^2$$

and since for all $\varepsilon_0 > 0$ a constant D_{ε_0} exists such that

$$\|\psi_j \nabla v\| \|v \nabla \psi_j\| \leq \varepsilon_0 \|\psi_j \nabla v\|_2^2 + D_{\varepsilon_0} \|v \nabla \psi_j\|_2^2,$$

we obtain

$$\|\nabla(v\psi_j)\|_2^2 \leq (1 + 2\varepsilon_0) \|\nabla v\|_2^2 + \tilde{D} \|v\|_2^2,$$

where $\tilde{D} = (2D_{\varepsilon_0} + 1)(\sup_T |\nabla \psi_j|^2)$.

From (53) because of the last inequality we have

$$\int_{T_j} e^{v\psi_j} dV \leq C \exp \left[(1 + c\varepsilon_0)(1 + 2\varepsilon_0) \frac{\mu_2}{2\pi l_j} \|\nabla v\|_2^2 + D \|v\|_2^2 \right], \tag{54}$$

where $D = (1 + c\varepsilon_0) \frac{\mu_2}{2\pi l_j} \tilde{D}$.

Since $\inf l_j = l - r$ given $\varepsilon > 0$ we can choose ε_0 small enough such that from (54) we obtain

$$\begin{aligned} \int_T e^v dV &\leq \sum_{i=1}^N \int_{T_{j,\delta/2}} e^v dV \leq \sum_{i=1}^N \int_{T_{j,\delta}} e^{v\psi_j} dV \\ &\leq C \exp\left[\left(\frac{\mu_2}{2\pi(l-r)} + \varepsilon\right) \|\nabla v\|_2^2 + D \|v\|_2^2\right], \end{aligned}$$

and so we have the desired inequality.

Now we need to prove that the constant $\frac{\mu_2}{2\pi(l-r)}$ is the best constant μ such that the inequality

$$\int_T e^v dV \leq C \exp[(\mu + \varepsilon) \|\nabla v\|_2^2 + D \|v\|_2^2]$$

holds for all $v \in \dot{H}_{1,G}^2$.

For that purpose, for all ε , we need to find a sequence $(v_\alpha) \in \dot{H}_{1,G}^2$, such that for all $\Delta, E \in \mathbb{R}$ the following holds:

$$\lim_{\alpha \rightarrow 0} \frac{\|\nabla v_\alpha\|_2^2 + \Delta \|v_\alpha\|_2^2 + E}{\ln \int_T e^{v_\alpha} dV} \leq \frac{2\pi(l-r)}{\mu_2} + \varepsilon. \tag{55}$$

Let us consider the orbit O_{inf} of minimum length $2\pi(l-r)$. For any $\varepsilon_0 > 0$, let

$$T_{j_0} = \{Q \in \mathbb{R}^3 : d(Q, O_{inf}) < \delta, \delta = \varepsilon_0(l-r)\},$$

where $d(Q, O_{inf})$ denotes the distance from Q to the orbit O_{inf} .

It is easy to prove that

$$d(Q, O_P) = \delta d_D(\xi_{j_0}(Q), O) = \delta \sqrt{t^2 + s^2}, \tag{56}$$

where d_D denotes the distance in the disc D centered on O .

For all $\alpha > 0$ define the functions (v_α) by

$$v_\alpha(Q) = \begin{cases} -2 \ln(\alpha + d^2(Q, O_{inf})) + 2 \ln(\alpha + \delta^2), & \text{if } Q \in T \cap T_{j_0}, \\ 0, & \text{if } Q \in T \setminus T_{j_0}. \end{cases}$$

Since v_α depends only on the distance to O_{inf} , $v_\alpha \in \dot{H}_{1,G}^2(T_{j_0})$.

Setting $\phi_\alpha = v_\alpha \circ \xi_{j_0}^{-1}$ we obtain

$$\begin{aligned} \int_T e^{v_\alpha} dV &= \int_{I \times D} e^{v_\alpha \circ \xi_{j_0}^{-1}} (\sqrt{g} \circ \xi_{j_0}^{-1}) d\omega dt ds \\ &= \int_{I \times D} e^{\phi_\alpha} \delta^2 ((l-r) + \delta t) d\omega dt ds \end{aligned}$$

$$\begin{aligned}
 &= 2\pi(l-r)\delta^2 \int_D e^{\phi_\alpha} \left(1 + \frac{\delta}{l-r}t\right) dt ds \\
 &\geq 2\pi(l-r)\delta^2(1-\varepsilon_0) \int_D e^{\phi_\alpha} dt ds.
 \end{aligned}
 \tag{57}$$

Hence, by definition of v_α and because of (56) for all $\xi_{j_0}(Q) = (t, s) \in D$ we obtain

$$\phi_\alpha(\xi_j(Q)) = \ln\left(\frac{\alpha + \delta^2}{\alpha + \delta^2(t^2 + s^2)}\right)^2,$$

thus

$$\int_D e^{\phi_\alpha} dt ds = \int_D \left(\frac{\alpha + \delta^2}{\alpha + \delta^2(t^2 + s^2)}\right)^2 dt ds.$$

Changing variables in the latter equality we obtain

$$\begin{aligned}
 \int_D e^{\phi_\alpha} dt ds &= \int_0^{2\pi} \int_0^1 \frac{(\alpha + \delta^2)^2 r}{(\alpha + \delta^2 r^2)^2} dr d\theta \\
 &= \frac{(\alpha + \delta^2)^2 \pi}{\delta^2} \int_0^1 \frac{(\alpha + \delta^2 r^2)'}{(\alpha + \delta^2 r^2)^2} dr \\
 &= \frac{(\alpha + \delta^2)\pi}{\alpha}.
 \end{aligned}
 \tag{58}$$

By (57) and (58) we have

$$\begin{aligned}
 \int_I e^{v_\alpha} dV &\geq (1-\varepsilon_0)2\pi(l-r)\delta^2 \frac{(\alpha + \delta^2)\pi}{\alpha} \\
 &\geq (1-\varepsilon_0)2\pi^2(l-r)\delta^4 \frac{1}{\alpha} \\
 &= C_{\varepsilon_0} \frac{1}{\alpha},
 \end{aligned}$$

and then

$$\ln \int_I e^{v_\alpha} dV \geq \ln C_{\varepsilon_0} + \ln \frac{1}{\alpha},
 \tag{59}$$

where $C_{\varepsilon_0} = (1-\varepsilon_0)2\pi^2(l-r)\delta^4$.

Moreover, because of (2) we have

$$\begin{aligned} \|\nabla u_\alpha\|_2^2 &= 2\pi \int_D |\nabla \phi_\alpha(t, s)|^2 ((l-r) + \delta t) dt ds \\ &\leq (1 + \varepsilon_0) 2\pi (l-r) \int_D |\nabla \phi_\alpha(t, s)|^2 dt ds. \end{aligned} \quad (60)$$

Since

$$\begin{aligned} |\nabla \phi_\alpha(t, s)|^2 &= |\nabla[-2 \ln(\alpha + \delta^2(t^2 + s^2)) + 2 \ln(\alpha + \delta^2)]|^2 \\ &= |-2 \nabla[\ln(\alpha + \delta^2(t^2 + s^2))]|^2 \\ &= 4 \left| \left(\frac{2\delta^2 t}{\alpha + \delta^2(t^2 + s^2)}, \frac{2\delta^2 s}{\alpha + \delta^2(t^2 + s^2)} \right) \right|^2 \\ &= \frac{16\delta^4(t^2 + s^2)}{[\alpha + \delta^2(t^2 + s^2)]^2}, \end{aligned}$$

we have

$$\begin{aligned} \int_D |\nabla \phi_\alpha(t, s)|^2 dt ds &= \int_D \frac{16\delta^4(t^2 + s^2)}{[\alpha + \delta^2(t^2 + s^2)]^2} dt ds \\ &= 2\pi \int_0^1 \frac{16\delta^4 r^2}{(\alpha + \delta^2 r^2)^2} r dr. \end{aligned}$$

Changing variables we obtain

$$\begin{aligned} \int_D |\nabla \phi_\alpha(t, s)|^2 dt ds &= 16\pi \int_0^{\delta^2} \frac{\tau}{(\alpha + \tau)^2} d\tau \\ &= \frac{1}{\mu_2} \int_0^{\delta^2} \frac{\tau}{(\alpha + \tau)^2} d\tau. \end{aligned} \quad (61)$$

We further define the function

$$h(\alpha) = \ln \frac{1}{\alpha} - \int_0^{\delta^2} \frac{\tau}{(\alpha + \tau)^2} d\tau, \quad \alpha > 0,$$

and changing the variable we obtain

$$\begin{aligned}
 h(\alpha) &= \ln \frac{1}{\alpha} - \int_0^{\delta^2} \frac{\tau}{\alpha^2(1 + \frac{\tau}{\alpha})^2} d\tau \\
 &= \int_{\delta^2}^{\delta^2/\alpha} \frac{1}{u} du - \int_0^{\delta^2/\alpha} \frac{u}{(1+u)^2} du \\
 &= \int_{\delta^2}^{\delta^2/\alpha} \left(\frac{1}{u} - \frac{u}{(1+u)^2} \right) du - \int_0^{\delta^2} \frac{u}{(1+u)^2} du \\
 &= \int_{\delta^2}^{\delta^2/\alpha} [1 - (1+u^{-1})^{-2}] u^{-1} du - \int_0^{\delta^2} \frac{u}{(1+u)^2} du
 \end{aligned}$$

and because of

$$\begin{aligned}
 [1 - (1+u^{-1})^{-2}] u^{-1} &= \left[1 - \left(1 + \frac{1}{u} \right)^{-2} \right] \frac{1}{u} = \left[1 - \left(\frac{u}{u+1} \right)^2 \right] \frac{1}{u} \\
 &= \frac{1}{u+1} \cdot \frac{2u+1}{u+1} \cdot \frac{1}{u} < \frac{2}{u+1} \cdot \frac{1}{u} < \frac{2}{u^2}
 \end{aligned}$$

we finally obtain

$$\begin{aligned}
 \lim_{\alpha \rightarrow 0} h(\alpha) &= \int_{\delta^2}^{\infty} [1 - (1+u^{-1})^{-2}] u^{-1} du - \int_0^{\delta^2} \frac{u}{(1+u)^2} du \\
 &\leq \int_{\delta^2}^{\infty} 2u^{-2} du - \int_0^{\delta^2} \frac{u}{(1+u)^2} du = C_0.
 \end{aligned}$$

Thus, for any $\alpha > 0$ close to 0 the following holds:

$$\int_0^{\delta^2} \frac{\tau}{(\alpha + \tau)^2} d\tau = \ln \frac{1}{\alpha} + C_1. \tag{62}$$

From (60)–(62) we obtain

$$\|\nabla v_\alpha\|_2^2 \leq \frac{(1 + \varepsilon_0)2\pi(l-r)}{\mu_2} \ln \frac{1}{\alpha} + C. \tag{63}$$

On the other hand we have

$$\begin{aligned}
 \|v_\alpha\|_2^2 &= 2\pi \delta^2 \int_D |\phi_\alpha|^2 (l-r) + \delta t \, dt \, ds \\
 &\leq (1 + \varepsilon_0) 2\pi (l-r) \delta^2 \int_D |\phi_\alpha|^2 \, dt \, ds \\
 &= C_0 \int_D \left| 2 \ln \left(\frac{\alpha + \delta^2}{\alpha + \delta^2(t^2 + s^2)} \right) \right|^2 \, dt \, ds \\
 &= 4C_0 \int_D (\ln(\alpha + \delta^2) - \ln(\alpha + \delta^2(t^2 + s^2)))^2 \, dt \, ds \\
 &\leq 8\pi C_0 \int_D [2 \ln^2(\alpha + \delta^2) + 2 \ln^2(\alpha + \delta^2(t^2 + s^2))] \, dt \, ds \\
 &= 8\pi C_0 \left(\int_0^1 2 \ln^2(\alpha + \delta^2) r \, dr + \int_0^1 2 \ln^2(\alpha + \delta^2 r^2) r \, dr \right) \\
 &= 8\pi C_0 \ln^2(\alpha + \delta^2) \int_0^1 2r \, dr + \frac{8\pi C_0}{\delta^2} \int_0^1 \ln^2(\alpha + \delta^2 r^2) 2\delta^2 r \, dr \\
 &= 8\pi C_0 \ln^2(\alpha + \delta^2) + \frac{8\pi C_0}{\delta^2} \int_0^1 \ln^2(\alpha + \delta^2 r^2) (\alpha + \delta^2 r^2)' \, dr \\
 &= C_1 + C_2 \int_\alpha^{\alpha+\delta^2} \ln^2 \zeta \, d\zeta \\
 &= C_1 + C_2 [\zeta (\ln^2 \zeta - 2 \ln \zeta + 2)]_\alpha^{\alpha+\delta^2} \\
 &= C_1 + C_2 [(\alpha + \delta^2) (\ln^2(\alpha + \delta^2) - 2 \ln(\alpha + \delta^2) + 2)] - C_2 \alpha (\ln^2 \alpha - 2 \ln \alpha + 2),
 \end{aligned}$$

and since $\lim_{\alpha \rightarrow 0^+} (\alpha \ln \alpha) = \lim_{\alpha \rightarrow 0^+} (\alpha \ln^2 \alpha) = 0$ we have

$$\|v_\alpha\|_2^2 \leq C_1 + C_2 C_3 = C. \tag{64}$$

Finally, from (59), (63) and (64) for any $\Delta, E \in \mathbb{R}$ the following holds:

$$\frac{\|\nabla v_\alpha\|_2^2 + \Delta \|v_\alpha\|_2^2 + E}{\ln \int_T e^{v_\alpha} \, dV} \leq \frac{\frac{(1+\varepsilon_0)2\pi(l-r)}{\mu_2} \ln \frac{1}{\alpha} + C}{\ln \frac{1}{\alpha} + \ln C_{\varepsilon_0}},$$

thus,

$$\lim_{\alpha \rightarrow 0} \frac{\|\nabla v_\alpha\|_2^2 + \Delta \|v_\alpha\|_2^2 + E}{\ln \int_T e^{v_\alpha} \, dV} \leq (1 + \varepsilon_0) \frac{2\pi(l-r)}{\mu_2}. \tag{65}$$

For any $\varepsilon > 0$ consider $\varepsilon_0 > 0$ such that $(1 + \varepsilon_0) \frac{2\pi(l-r)}{\mu_2} \leq \frac{2\pi(l-r)}{\mu_2} + \varepsilon$ and so from (65) we obtain our result. \square

Proof of Lemma 2.2. Following arguments similar to those in [4] and [30] we prove the first and second part of the lemma, respectively.

1. Our aim here is to find a constant C_ε , such that for any $\varepsilon > 0$ and for all functions $v \in \mathcal{H}_G$, with $\int_T v \, dV = 0$ the following inequality holds

$$\int_T e^v \, dV \leq C_\varepsilon \exp[(\mu + \varepsilon) \|\nabla v\|_2^2],$$

where $\mu = \frac{1}{16\pi L}$ if $\mathcal{H}_G = \dot{H}_{1,G}^2$ and $\mu = \frac{1}{8\pi L}$ if $\mathcal{H}_G = H_{1,G}^2$.

(i) Let $v \in C_{0,G}^\infty(T)$ with $\int_T v \, dV = 0$ and $\check{v} = \sup(v, 0)$.

Then $\check{v} \in \dot{H}_{1,G}^2(T)$ and

$$\int_{\mathbb{R}^3} \check{v} \, dx = \frac{1}{2} \int_{\mathbb{R}^3} v \, dx, \quad \int_{\mathbb{R}^3} |\nabla \check{v}| \, dx \leq \int_{\mathbb{R}^3} |\nabla v| \, dx.$$

For any $t \in \mathbb{R}$, denote by $m_t(v)$ the measure of the set

$$\Omega_t(v) = \{x \in T : v(x) \geq t\}.$$

Given $v \in C_{0,G}^\infty(T)$, $m_t(v)$ is a decreasing function of t , not necessarily continuous. Let $m > 0$ depending on ε . Then, for a given $v \in C_{0,G}^\infty(T)$ two different cases can occur: whether there exists $s \geq 0$ such that $m_s(v) \geq m$ or not.

(a) Suppose there exists $s \geq 0$ such that $m_s(v) \geq m$.

If we denote

$$S = \sup\{s \in \mathbb{R} : m_s(v) \geq m\},$$

we will have $S \geq 0$, $m_{S+1}(v) < m$ and $m_{S/2}(v) \geq m$.

According to Lemma 2.1 we have the following:

$$\begin{aligned} \int_T e^v \, dV &= e^{S+1} \int_T e^{v-(S+1)} \, dV \leq e^{S+1} \int_T e^{\widehat{v-(S+1)}} \, dV \\ &\leq e^{S+1} C_{\varepsilon/2} \exp\left[\left(\mu + \frac{\varepsilon}{2}\right) \|\nabla v\|_2^2 + D_{\varepsilon/2} \|\widehat{v-(S+1)}\|_2^2\right]. \end{aligned} \tag{66}$$

Since $\int_T v \, dV = 0$, and $\|\check{v}\|_1 = \frac{1}{2} \|v\|_1$ by Poincaré inequality there exists a constant C_1 such that

$$\|\check{v}\|_1 = \frac{1}{2} \|v\|_1 \leq C_1 \|\nabla v\|_2, \tag{67}$$

and since $S + 1 > 0$ we obtain

$$\|\widehat{v-(S+1)}\|_1 \leq \|\check{v}\|_1 \leq C_1 \|\nabla v\|_2. \tag{68}$$

From the last two inequalities, by Hölder’s inequality and the Sobolev continuous and compact embedding of $\dot{H}_{1,G}^2$ in $L_G^p(T)$, we obtain

$$\begin{aligned} \|v - \widehat{(S+1)}\|_2^2 &\leq m_{S+1}^{1/2}(v) \|v - \widehat{(S+1)}\|_4^2 \\ &< m^{1/2} \|v - \widehat{(S+1)}\|_4^2 \\ &\leq m^{1/2} C_2 \|\nabla v\|_2^2, \end{aligned} \tag{69}$$

where C_2 is a constant independent of v and μ .

By the definition of $\Omega_t(v)$ we have that

$$\Omega_0(v) = \{x \in T : v(x) \geq 0\}$$

and

$$\Omega_{S/2}(v) = \{x \in T : v(x) \geq S/2\}.$$

Thus

$$\|\check{v}\|_1 = \int_{\Omega_0(v)} v \, dV \geq \int_{\Omega_{S/2}(v)} v \, dV \geq \int_{\Omega_{S/2}(v)} \frac{S}{2} \, dV = \frac{S}{2} m_{S/2}(v). \tag{70}$$

From (67) and (70), since $m_{S/2}(v) \geq m$, we obtain

$$S \leq \frac{2}{m_{S/2}(v)} \|\check{v}\|_1 \leq \frac{2}{m} C_1 \|\nabla v\|_2. \tag{71}$$

The elementary inequality

$$x < Sx^2 + \frac{1}{S}, \quad x \in \mathbb{R}, \quad S > 0,$$

with $x = \|\nabla v\|_2$ yields

$$\|\nabla v\|_2 < S\|\nabla v\|_2^2 + \frac{1}{S}, \quad S > 0. \tag{72}$$

From (71), and because of (72), we obtain

$$S \leq \frac{2C_1}{m} \left(S\|\nabla v\|_2^2 + \frac{1}{S} \right),$$

and with $\frac{2C_1}{m} = \frac{m}{S}$ we obtain

$$S \leq m\|\nabla v\|_2^2 + 4C_1^2 m^{-3} = m\|\nabla v\|_2^2 + C_3 m^{-3}. \tag{73}$$

Thus, from (66), and because of (69) and (73), we obtain

$$\int_T e^v dV \leq C_\varepsilon \exp\left[\left(\frac{\mu_2}{L} + \frac{\varepsilon}{2} + D_{\varepsilon/2} C_2 m^{1/2} + m\right) \|\nabla v\|_2^2\right] \tag{74}$$

where $C_\varepsilon = C_{\varepsilon/2} \exp(C_3 m^{-3} + 1)$.

(b) Suppose now that $m_s(v) < m$ for any $s \geq 0$.

By Lemma 2.1 we have the following:

$$\int_T e^v dV \leq \int_T e^{\check{v}} dV \leq C_{\varepsilon/2} \exp\left[\left(\mu + \frac{\varepsilon}{2}\right) \|\nabla \check{v}\|_2^2 + D_{\varepsilon/2} \|\check{v}\|_2^2\right]$$

or

$$\int_T e^v dV \leq C_{\varepsilon/2} \exp\left[\left(\mu + \frac{\varepsilon}{2}\right) \|\nabla v\|_2^2 + D_{\varepsilon/2} \|v\|_2^2\right]. \tag{75}$$

In this case, $m_0(v) \leq m$ and so

$$\|\check{v}\|_2^2 \leq m_0^{1/2}(v) \|\check{v}\|_4^2 < m^{1/2} \|\check{v}\|_4^2 \leq m^{1/2} C_2 \|\nabla v\|_2^2. \tag{76}$$

From (75), and because of (76) we obtain

$$\int_T e^v dV \leq C_{\varepsilon/2} \exp\left[\left(\mu + \frac{\varepsilon}{2} + D_{\varepsilon/2} C_2 m^{1/2} + m\right) \|\nabla v\|_2^2\right]. \tag{77}$$

In both cases we have to choose $m > 0$ such that

$$D_{\varepsilon/2} C_2 m^{1/2} + m < \frac{\varepsilon}{2}$$

and

$$C_\varepsilon = C_{\varepsilon/2} \exp\left(\frac{2C_1}{m^2} + 1\right)$$

so, for all $v \in \dot{H}_{1,G}^2$ with $\int_T v dV = 0$ the following inequality holds

$$\int_T e^v dV \leq C_\varepsilon \exp[(\mu + \varepsilon) \|\nabla v\|_2^2], \tag{78}$$

where $\mu = \frac{1}{16\pi L}$.

(ii) Let now $v \in H_{1,G}^2$. Following the same steps as in the first part of Lemma 2.1 by Theorem 3 of [18], for all $v \in C_G^\infty(T)$, we obtain

$$\int_{T_j} e^v dV \leq C \exp\left[(1 + c\varepsilon) \frac{1}{16\pi^2 l_j} \int_{T_j} |\nabla v|^2 dV\right]. \tag{79}$$

Consequently, since $C_G^\infty(T)$ is dense in $H_{1,G}^2$ and (79) holds for any $j = 1, 2, \dots, N$, by the second part of Lemma 2.1, we conclude that, for all $\varepsilon > 0$, there are constants C_ε and D_ε such that for all $v \in H_{1,G}^2$ the following holds:

$$\int_T e^v dV \leq C_\varepsilon \exp[(\mu + \varepsilon)\|\nabla v\|_2^2 + D_\varepsilon\|v\|_2^2],$$

where $\mu = \frac{1}{8\pi l}$ is the best constant for this inequality.

Following the same steps as in the first part of this lemma we derive that for any $\varepsilon > 0$ and for all functions $v \in H_{1,G}^2$, with $\int_T v dV = 0$ the following inequality holds

$$\int_T e^v dV \leq C_\varepsilon \exp[(\mu + \varepsilon)\|\nabla v\|_2^2], \tag{80}$$

where $\mu = \frac{1}{8\pi l}$.

By parts (i) and (ii) of the lemma we conclude that, for all $\varepsilon > 0$, there exists constant C_ε such that for all $v \in \dot{H}_{1,G}^2$ or $v \in H_{1,G}^2$ with $\int_T v dV = 0$, inequalities (78) and (80) hold respectively.

Finally, we observe that if $\tilde{v} = v - \frac{1}{2\pi^2 r^2 l} \int_T v dV$ we have

$$\begin{aligned} \int_T \tilde{v} dV &= \int_T \left(v - \frac{1}{2\pi^2 r^2 l} \int_T v dV \right) dV \\ &= \int_T v dV - \frac{1}{2\pi^2 r^2 l} \int_T v dV \int_T dV \\ &= \int_T v dV - \frac{1}{\text{Vol}(T)} \text{Vol}(T) \int_T v dV \\ &= 0, \end{aligned}$$

and so, rewriting (78) and (80) with $\tilde{v} = v - \frac{1}{2\pi^2 r^2 l} \int_T v dV$ we obtain:

$$\begin{aligned} \int_T e^{v - \frac{1}{2\pi^2 r^2 l} \int_T v dV} dV &\leq C_\varepsilon \exp\left[(\mu + \varepsilon)\left\| \nabla \left(v - \frac{1}{2\pi^2 r^2 l} \int_T v dV \right) \right\|_2^2\right], \\ e^{-\frac{1}{2\pi^2 r^2 l} \int_T v dV} \int_T e^v dV &\leq C_\varepsilon \exp[(\mu + \varepsilon)\|\nabla v\|_2^2] \end{aligned}$$

or

$$\int_T e^v dV \leq C_\varepsilon \exp\left[(\mu + \varepsilon)\|\nabla v\|_2^2 + \frac{1}{2\pi^2 r^2 l} \int_T v dV\right],$$

and the first part of the lemma is proved.

2. Let $v \in C_G^\infty(T)$, with $\int_{\partial T} v dS = 0$, $\phi = v \circ \xi^{-1}$ and n the outward unit normal.

By Stoke’s theorem we have

$$\begin{aligned}
 \int_{\partial T} e^v dS &= 2\pi r^2 \int_{\partial D} e^\phi (l + rt) d\sigma_D \\
 &\leq 2\pi r^2 (l + r) \int_{\partial D} e^\phi d\sigma_D \\
 &= 2\pi r^2 (l + r) \int_D \operatorname{div}(e^\phi n) dt ds \\
 &= 2\pi r^2 (l + r) \int_D [\operatorname{div} n + n(\phi)] e^\phi dt ds \\
 &\leq 2\pi r^2 (l + r) \left[C_0 \int_D e^\phi dt ds + \int_D |\nabla\phi| e^\phi dt ds \right], \tag{81}
 \end{aligned}$$

where $C_0 = \sup_D (|\operatorname{div} n|)$.

By (81), and because of Theorem 4 of [17] arises

$$\int_{\partial T} e^v dS \leq 2\pi r^2 (l + r) \left[C_0 \tilde{C} \exp(\mu \|\nabla\phi\|_2^2) + \int_D |\nabla\phi| e^\phi dt ds \right]. \tag{82}$$

By Hölder’s inequality and by Theorem 3 of [18] we obtain

$$\int_D |\nabla\phi| e^\phi dt ds \leq \|\nabla\phi\|_2 \left(\int_D e^{2\phi} dt ds \right)^{1/2} \leq \tilde{C} \|\nabla\phi\|_2 \exp(2\tilde{\mu} \|\nabla\phi\|_2^2), \tag{83}$$

where $\tilde{\mu}$ is a constant greatest than $1/8\pi$.

From the elementary inequality $t \leq C_1 \exp(\varepsilon_0 t^2)$, $t \geq 0$, $\varepsilon_0 > 0$ and C_1 a constant with arbitrary $\varepsilon_0 > 0$ and $t = \|\nabla\phi\|_2$ we obtain

$$\|\nabla\phi\|_2 \leq C_1 \exp(\varepsilon_0 \|\nabla\phi\|_2^2). \tag{84}$$

Combining inequalities (82)–(84) we obtain

$$\begin{aligned}
 \int_{\partial T} e^v dS &\leq 2\pi r^2 (l + r) [C_0 \tilde{C} \exp(\tilde{\mu} \|\nabla\phi\|_2^2) + \tilde{C} \|\nabla\phi\|_2 \exp(2\tilde{\mu} \|\nabla\phi\|_2^2)] \\
 &\leq 2\pi r^2 (l + r) [C_0 \tilde{C} \exp(\tilde{\mu} \|\nabla\phi\|_2^2) + \tilde{C} C_1 \exp(\varepsilon_0 \|\nabla\phi\|_2^2) \exp(2\tilde{\mu} \|\nabla\phi\|_2^2)] \\
 &\leq 2\pi r^2 (l + r) \tilde{C} (C_0 + C_1) \exp[(2\tilde{\mu} + \varepsilon_0) \|\nabla\phi\|_2^2].
 \end{aligned}$$

Since

$$\|\nabla\phi\|_2^2 = \int_D |\nabla\phi|^2 dt ds \leq \frac{1}{L} \int_T |\nabla v|^2 dV,$$

the latter inequality becomes

$$\begin{aligned} \int_{\partial T} e^v dS &\leq 2\pi r^2(l+r)\tilde{C}(C_0 + C_1) \exp\left(\frac{2\tilde{\mu} + \varepsilon_0}{L} \int_T |\nabla v|^2 dV\right) \\ &\leq C \exp\left(\frac{2\tilde{\mu} + \varepsilon_0}{L} \int_T |\nabla v|^2 dV\right). \end{aligned}$$

Given $\varepsilon > 0$, we can choose $\varepsilon_0 > 0$ such that

$$\frac{2\tilde{\mu} + \varepsilon_0}{L} < \frac{2\tilde{\mu}}{L} + \varepsilon = \mu + \varepsilon,$$

and the last inequality yields

$$\int_{\partial T} e^v dS \leq C \exp\left((\mu + \varepsilon) \int_T |\nabla v|^2 dV\right). \quad (85)$$

Rewriting (85) with $\tilde{v} = v - \frac{1}{4\pi^2 r^2 l} \int_{\partial T} v dS$ yields the second inequality of the lemma. \square

References

- [1] Adimurthi, J. Giacomoni, Bifurcation problems for superlinear elliptic indefinite equations with exponential growth, *NoDEA Nonlinear Differential Equations Appl.* 12 (2005) 1–20.
- [2] Th. Aubin, Métrique riemannienne et courbure, *J. Differential Geom.* 4 (1970) 383–424.
- [3] Th. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer, Berlin, 1998.
- [4] Th. Aubin, Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire, *J. Funct. Anal.* 32 (1979) 148–174.
- [5] Th. Aubin, Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, *J. Math. Pures Appl.* 55 (1976) 269–296.
- [6] Th. Aubin, A. Cotsiolis, Equations elliptiques non linéaires sur S_n dans le cas supercritique, *Bull. Sci. Math.* 123 (1999) 33–45.
- [7] Th. Aubin, A. Cotsiolis, Equations non linéaires avec le p -Laplacien et la fonction exponentielle sur les variétés Riemanniennes compactes, *Bull. Sci. Math.* 124 (1) (2000) 1–19.
- [8] M.S. Berger, On Riemannian structure of prescribed Gaussian curvature for compact 2-manifolds, *J. Differential Geom.* 5 (1971) 325–332.
- [9] G. Bianchi, Non-existence and symmetry of solutions to the scalar curvature equation, *Comm. Partial Differential Equations* 21 (1–2) (1996) 229–234.
- [10] G. Bianchi, The scalar curvature equation on \mathbf{R}^n and S^n , *Adv. Differential Equations* 1 (5) (1996) 857–880.
- [11] H. Brezis, F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Comm. Partial Differential Equations* 16 (1991) 1223–1253.
- [12] S.-Y.A. Chang, M.J. Gursky, P.C. Yang, The scalar curvature equation on 2- and 3-spheres, *Calc. Var. Partial Differential Equations* 1 (2) (1993) 205–229.
- [13] W.X. Chen, Scalar curvatures on S^n , *Math. Ann.* 283 (3) (1989) 353–365.
- [14] W.X. Chen, W. Ding, Scalar curvatures on S^2 , *Trans. Amer. Math. Soc.* 303 (1) (1987) 365–382.
- [15] W.X. Chen, W. Ding, A problem concerning the scalar curvature on S^2 , *Kexue Tongbao (English Ed.)* 33 (7) (1988) 533–537.
- [16] W.X. Chen, C. Li, A priori estimate for the Nirenberg problem, *Discrete Contin. Dyn. Syst. Ser. S* 1 (2) (2008) 225–233.
- [17] P. Cherrier, Une inégalité de Sobolev sur les variétés Riemanniennes, *Bull. Sci. Math.* 103 (1979) 353–374.
- [18] P. Cherrier, Meilleures constantes dans des inégalités relatives aux espaces de Sobolev, *Bull. Sci. Math.* 108 (1984) 255–262.
- [19] P. Cherrier, Problèmes de Neumann non linéaires sur les variétés riemanniennes, *J. Funct. Anal.* 57 (2) (1984) 154–206.
- [20] A. Cotsiolis, Problème de Neumann avec exposant sur-critique et une non-linéarité sur-critique sur le bord, *C. R. Acad. Sci. Paris* 322 (1996) 139–142.
- [21] A. Cotsiolis, D. Iliopoulos, Equations elliptiques non linéaires à croissance de Sobolev sur-critique, *Bull. Sci. Math.* 119 (1995) 419–431.
- [22] A. Cotsiolis, N. Labropoulos, Dirichlet problem on a solid torus in the critical of supercritical case, *Bull. Greek Math. Soc.* 53 (2007) 39–57.
- [23] A. Cotsiolis, N. Labropoulos, A Neumann problem with the q -Laplacian on a solid torus in the critical of supercritical case, *Electron. J. Differential Equations* 2007 (164) (2007) 1–18.

- [24] A. Cotsiolis, N. Labropoulos, Best constants in Sobolev inequalities on manifolds with boundary in the presence of symmetries and applications, *Bull. Sci. Math.* 132 (2008) 562–574.
- [25] M. Cox, J. Battista, Deinococcus radiodurans – the consummate survivor, *Nature* 3 (2005) 882–892.
- [26] K. Davidson, N. Smith, Astronomy: A massive cool dust torus around Carinae, *Nature* 405 (2000) 532.
- [27] F. Demengel, B. Nazaret, On some nonlinear partial differential equations involving the p -Laplacian and critical Sobolev traces maps, *Asymptot. Anal.* 23 (2000) 135–156.
- [28] W. Ding, On a conformally invariant elliptic equation on R^n , *Comm. Math. Phys.* 107 (1986) 331–335.
- [29] O. Druet, Generalized scalar curvature type equations on compact Riemannian manifolds, *Proc. Roy. Soc. Edinburgh Sect. A* 130 (2000) 767–788.
- [30] Z. Faget, Best constants in Sobolev inequalities on Riemannian manifolds in the presence of symmetries, *Potential Anal.* 17 (2002) 105–124.
- [31] Z. Faget, Optimal constants in critical Sobolev inequalities on Riemannian manifolds in the presence of symmetries, *Ann. Global Anal. Geom.* 24 (2003) 161–200.
- [32] Z. Faget, Optimal constants in the exceptional case of Sobolev inequalities on Riemannian manifolds, *Trans. Amer. Math. Soc.* 360 (2008) 2303–2325.
- [33] Z. Han, Y.Y. Li, A note on the Kazdan–Warner type condition, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 13 (3) (1996) 283–292.
- [34] E. Hebey, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Lecture Notes in Math., vol. 5, Courant Institute of Mathematical Sciences, 1999.
- [35] E. Hebey, M. Vaugon, Sobolev spaces in the presence of symmetries, *J. Math. Pures Appl.* 76 (1997) 859–881.
- [36] S.M. Kaye, Operational limits in the national spherical torus experiment, in: *Proc. of 27th EPS Conference on Contr. Fusion and Plasma Phys.*, Budapest, 12–16 June 2000, ECA, vol. 24B, 2000, pp. 1437–1440.
- [37] J. Kazdan, F. Warner, Curvature functions for compact 2-manifolds, *Ann. of Math.* 99 (1974) 14–47.
- [38] M. Li, Bifurcation in Nirenberg’s problem, *C. R. Acad. Sci. Paris Ser. I* 326 (1998) 583–588.
- [39] Y.Y. Li, Prescribing scalar curvature on S^n and related problems, I, *J. Differential Equations* 120 (2) (1995) 319–410.
- [40] Y.Y. Li, The Nirenberg problem in a domain with boundary, *Topol. Methods Nonlinear Anal.* 6 (2) (1995) 309–329.
- [41] Y.Y. Li, Prescribing scalar curvature on S^n and related problems, II. Existence and compactness, *Comm. Pure Appl. Math.* 49 (6) (1996) 541–597.
- [42] P.L. Lions, The concentration compactness principle in the calculus of variations. The limit case, parts I, II, *Rev. Mat. Iberoam.* 1 (1–2) (1985) 145–201, 45–121.
- [43] L. Mei, A priori bounds for an indefinite superlinear elliptic equation with exponential growth, *J. Math. Anal. Appl.* 339 (2008) 1294–1304.
- [44] J. Moser, A sharp form of an inequality by N. Trüdinger, *Indiana Univ. Math. J.* 20 (1971) 1077–1092.
- [45] J. Moser, On a nonlinear problem in differential geometry, in: *Dynamical Systems*, Proc. Sympos., Univ. Bahia, Salvador, 1971, Academic Press, New York, 1973, pp. 273–280.
- [46] M. Obata, The conjectures on conformal transformations of Riemannian manifolds, *J. Differential Geom.* 6 (1971) 247–258.
- [47] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, *J. Differential Geom.* 20 (1984) 479–495.
- [48] P. Tolksdorf, On Dirichlet problem for quasilinear equations in domain with critical boundary points, *Comm. Partial Differential Equations* 8 (7) (1983) 773–817.
- [49] N.S. Trüdinger, On imbeddings into Orlicz spaces and some applications, *J. Math. Phys.* 17 (5) (1967) 473–483.