



# Off-shell Noether current and conserved charge in Horndeski theory



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## ABSTRACT

We derive the off-shell Noether current and potential in the context of Horndeski theory, which is the most general scalar–tensor theory with a Lagrangian containing derivatives up to second order while yielding at most to second-order equations of motion in four dimensions. Then the formulation of conserved charges is proposed on basis of the off-shell Noether potential and the surface term got from the variation of the Lagrangian. As an application, we calculate the conserved charges of black holes in a scalar–tensor theory with non-minimal coupling between derivatives of the scalar field and the Einstein tensor.

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## 1. Introduction

Horndeski gravity theory, first formulated by Horndeski back in 1974 [1], is the most general scalar–tensor theory with a single scalar degree of freedom that possesses a Lagrangian containing higher than second order derivatives while yielding second-order equations of motion for the metric and the scalar field in four dimensions. In some cases, it recovers general relativity and a wide class of modified gravity models with a single scalar hair, such as Brans–Dicke theory,  $f(R)$  gravity,  $k$ -essence [25] and the covariant Galileon [26]. The Horndeski theory has received extensive attention since Deffayet et al. re-discovered independently it as the theory of generalized Galileon [2], which is equivalent to the original Horndeski theory in spite of a different formulation [3]. Up to now this theory has been extended and developed to investigate various aspects associated with gravitational theory, ranging from black hole physics to cosmology. In the context of solutions in the Horndeski theory, spherically symmetric solutions were investigated in [5–7] while rotating solutions were discussed in [21]. Owing to its complexity of the theory, it is difficult to find exact solutions in the full theory. Consequently, a lot of attention was drawn to seeking solutions [8–22] in a particular case of the Horndeski theory, where the Lagrangian only involves non-minimal coupling between derivatives of the scalar field and the Einstein tensor [11,24]. To further interpret thermodynamic property of the

black hole solutions, it is of great necessity to find a proper formulation of conserved charges in the full Horndeski theory, which is just our motivation in this work.

Recently, by making use of the Noether procedure, Kim, Kulka-rni and Yi put forward a formulation of quasi-local conserved charges in covariant theories of pure gravity [27], which can be seen as an off-shell generalization of the conventional Abbott–Deser–Tekin (ADT) formalism that is defined in terms of the Noether potential got through the linearized perturbation for the expression of the gravitational field equation in a fixed background metric satisfying the equation of motion in vacuum [39,40]. The main ideas of their method go as follows. Starting with the variation of the Lagrangian for the gravity system along the line of the covariant phase space approach [44,45], one reads off the expression for the equation of motion and surface term. Next, under the assumption that the variation is induced by a diffeomorphism symmetry generated by a smooth vector field, an off-shell Noether current and its corresponding potential with respect to the vector field are introduced in terms of the expression of the field equation and the surface term. Finally, by establishing the one-to-one relationship between the off-shell Noether potential and the ADT potential and following the method in [41–43] to incorporate a single parameter path in the space of solutions into the formalism, one can propose a quasi-local formulation of the conserved charges in the theories of gravity.

Inspired with the generalized formalism for the quasi-local conserved charges proposed in the work [27], the method therein has been generalized and developed to study conserved charges in various gravitational theories coupled with matter fields or not. In [28], Hyun, Jeong, Park and Yi developed the formalism in [27]

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by considering all the effect from gravitational field and matter fields in covariant theories of gravity, and they further showed that conserved charges via the modified ADT formalism coincide with those by the covariant phase space approach [44,45]. In [29], by directly varying the Bianchi identity for the expression of the equation of motion, we presented an off-shell Noether current in a different, but equivalent formulation compared with the one in [27]. Then we employed the generalized formulation to calculate the quasi-local conserved charges of black holes in four-dimensional conformal Weyl gravity and in arbitrary dimensional Einstein–Gauss–Bonnet gravity. Conserved charges of black holes with a sole scalar hair were taken into account in [30–32]. Other applications and developments of the modified ADT formalism can be found in [33–38].

In this letter, we focus on providing a systematic approach to calculate the conserved charge in the full Horndeski gravity theory. To do this, we derive the off-shell Noether currents and potentials of this theory and follow the works [27,28] to propose a formulation of the conserved charge through building the one-to-one correspondence between the off-shell Noether potential and the ADT potential. Then the generalized formalism is extended to a special scalar–tensor theory with non-minimal coupling to gravity [11,24], which has attracted much attention in the context of black hole physics. As concrete examples, we explicitly compute the mass and angular momentum of the three-dimensional rotating black hole with a sole scalar degree of freedom in [19] and the mass of the four-dimensional (charged) spherically symmetric black holes in [13,17].

The outline of this letter goes as follows. In section 2, we derive the off-shell Noether currents and potentials of the Horndeski theory and then present the formulation of conserved charge in this theory. In section 3, as an application in a particular case of the Horndeski theory, the off-shell Noether potential of the scalar–tensor theory with a single scalar field non-minimally coupled to the metric is derived. By using the formulation defined in terms of the potential and the surface term, we compute the mass and angular momentum of three-dimensional rotating black holes and the mass of four-dimensional charged spherically symmetric black holes in the special theory. The last section is our conclusions.

## 2. Off-shell Noether currents and the formulation of conserved charges

In this section, we derive the off-shell Noether currents and their corresponding potentials in the framework of the Horndeski theory along the line of the works [27,28]. By building the relationship between the off-shell Noether potential and the ADT potential [39,40], we further give the formulation of the conserved charge in the Horndeski gravity theory.

As a starting point, we consider the Lagrangian for the Horndeski theory that takes the form

$$L = \sum_{i=0}^3 L_{(i)}, \quad (1)$$

where the components  $L_{(i)}$  are given by [1–3]

$$\begin{aligned} L_{(0)} &= \sqrt{-g} G^{(0)}(\phi, X), \\ L_{(1)} &= \sqrt{-g} G^{(1)}(\phi, X) (\nabla_\mu \nabla^\mu \phi), \\ L_{(2)} &= \sqrt{-g} [2G_{,X}^{(2)}(\phi, X) \delta_{v_1}^{[\mu_1} \delta_{v_2}^{\mu_2]} (\nabla_{\mu_1} \nabla^{v_1} \phi) (\nabla_{\mu_2} \nabla^{v_2} \phi) \\ &\quad + RG^{(2)}(\phi, X)], \\ L_{(3)} &= \sqrt{-g} [6G_{,X}^{(3)}(\phi, X) \delta_{v_1}^{[\mu_1} \delta_{v_2}^{\mu_2} \delta_{v_3}^{\mu_3]} (\nabla_{\mu_1} \nabla^{v_1} \phi) (\nabla_{\mu_2} \nabla^{v_2} \phi) \\ &\quad \times (\nabla_{\mu_3} \nabla^{v_3} \phi) - 6G_{\mu\nu} (\nabla^\mu \nabla^\nu \phi) G^{(3)}(\phi, X)]. \end{aligned} \quad (2)$$

In the above equation, the scalar curvature  $R$  is defined as the trace of the Ricci tensor  $R_{\mu\nu}$ ,  $G_{\mu\nu}$  denotes the Einstein tensor,  $X = -1/2 \nabla_\mu \phi \nabla^\mu \phi$ ,  $G_{,X}^{(i)}(\phi, X) = \partial G^{(i)}(\phi, X) / \partial X$ , and  $G^{(i)}(\phi, X)$  are arbitrary functions of the scalar field  $\phi$  and its kinetic term  $X$ . The components  $L_{(2,3)}$  contain terms involving non-minimal couplings to gravity, which result in the elimination of higher derivatives that might appear in the equations of motion. Consequently, in spite of the Lagrangian containing higher-order derivative terms, the field equations are of second-order [4]. The Horndeski gravity theory is a general scalar–tensor theory. It includes general relativity and all popular modified gravity theories with a single scalar field as special cases, for instance, the Lagrangian (1) reduces to the conventional Einstein–Hilbert Lagrangian when  $G^{(2)} = 1/2$  with  $G^{(0,1,3)} = 0$ , and in the work [21], it has been shown that the Horndeski theory recovers several well-known modified gravity models, such as Brans–Dicke theory,  $f(R)$  gravity,  $k$ -essence [25], the covariant Galileon [26] and the theory of Einstein–dilaton–Gauss–Bonnet gravity, with proper choice of the four free functions  $G^{(i)}$ .

The variation of the Lagrangian (1) is read off as

$$\begin{aligned} \delta L &= \sum_{i=0}^3 \delta L_{(i)} = \sqrt{-g} [\mathcal{T}_{\mu\nu} \delta g^{\mu\nu} + \mathcal{E}_{(\phi)} \delta \phi + \nabla_\mu \Theta^\mu(\delta g, \delta \phi)], \\ \mathcal{T}_{\mu\nu} &= \sum_{i=0}^3 \mathcal{T}_{\mu\nu}^{(i)}, \quad \mathcal{E}_{(\phi)} = \sum_{i=0}^3 \mathcal{E}_{(\phi)}^{(i)}, \\ \Theta^\mu(\delta g, \delta \phi) &= \sum_{i=0}^3 \Theta_{(i)}^\mu(\delta g, \delta \phi), \end{aligned} \quad (3)$$

where and in what follows, the quantity with the index “ $i$ ” is the one corresponding to the Lagrangian  $L_{(i)}$ . In Eq. (3), the expressions of the field equation  $\mathcal{T}_{\mu\nu}^{(0)}$ ,  $\mathcal{T}_{\mu\nu}^{(1)}$ ,  $\mathcal{E}_{(\phi)}^{(0)}$  and  $\mathcal{E}_{(\phi)}^{(1)}$  are given by

$$\begin{aligned} \mathcal{T}_{\mu\nu}^{(0)} &= -\frac{1}{2} (G^{(0)} g_{\mu\nu} + G_{,X}^{(0)} \Phi_{\mu\nu}), \\ \mathcal{T}_{\mu\nu}^{(1)} &= \frac{1}{2} g_{\mu\nu} \nabla_\sigma G^{(1)} \nabla^\sigma \phi - \nabla_{(\mu} G^{(1)} \nabla_{\nu)} \phi - \frac{1}{2} G_{,X}^{(1)} \square \phi \Phi_{\mu\nu}, \\ \mathcal{E}_{(\phi)}^{(0)} &= \nabla_\mu (G_{,X}^{(0)} \nabla^\mu \phi) + G_{,\phi}^{(0)}, \\ \mathcal{E}_{(\phi)}^{(1)} &= \nabla_\mu (G_{,X}^{(1)} \square \phi \nabla^\mu \phi) + G_{,\phi}^{(1)} \square \phi + \square G^{(1)}. \end{aligned} \quad (4)$$

Here and in the remainder of this work, the symmetric tensor  $\Phi_{\mu\nu}$  is defined through  $\Phi_{\mu\nu} = (\nabla_\mu \phi) (\nabla_\nu \phi)$ . The surface terms  $\Theta_{(0)}^\mu(\delta \phi)$  and  $\Theta_{(1)}^\mu(\delta g, \delta \phi)$  have the forms

$$\begin{aligned} \Theta_{(0)}^\mu &= -G_{,X}^{(0)} \nabla^\mu \phi \delta \phi, \\ \Theta_{(1)}^\mu &= \frac{1}{2} G^{(1)} (h \nabla^\mu \phi - 2h^{\mu\nu} \nabla_\nu \phi + 2\nabla^\mu \delta \phi) - \delta \phi G_{,X}^{(1)} (\square \phi) \nabla^\mu \phi \\ &\quad - \delta \phi \nabla^\mu G^{(1)}, \end{aligned} \quad (5)$$

where and in what follows

$$h_{\mu\nu} = \delta g_{\mu\nu}, \quad h^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} h_{\rho\sigma} = -\delta g^{\mu\nu}, \quad h = g^{\rho\sigma} \delta g_{\rho\sigma}.$$

The expressions of the field equations and surface terms associated with the Lagrangian components  $L_{(2,3)}$  are much more involved, so we present them in the Appendix A. We have proved that all the expressions for the equations of motion satisfy

$$2\nabla^\mu \mathcal{T}_{\mu\nu}^{(i)} + \mathcal{E}_{(\phi)}^{(i)} \nabla_\nu \phi = 0, \quad (6)$$

which result from the constraint that the Horndeski theory has to reserve diffeomorphism symmetry and directly lead to that

$$2\nabla^\mu \mathcal{T}_{\mu\nu} + \mathcal{E}_{(\phi)} \nabla_\nu \phi = 0. \quad (7)$$

In the absence of the scalar field, Eq. (7) reduces to the usual Bianchi identity  $\nabla^\mu G_{\mu\nu} = 0$  in general relativity. In this sense, Eq. (7) can be treated as a generalized Bianchi identity for the Horndeski theory.

Let us now consider the variation induced by a diffeomorphism generated by a smooth vector field  $\zeta^\mu$ . In other words, the variation of the field  $g_{\mu\nu}$  and  $\phi$  in Eq. (3) is replaced by their Lie derivative with respect to the vector  $\zeta^\mu$ . With help of Eqs. (6) and (7), we obtain an off-shell Noether current  $J^\mu$ , which is read off as

$$J^\mu = \frac{L}{\sqrt{-g}} \zeta^\mu + 2\mathcal{T}^{\mu\nu} \zeta_\nu - \Theta^\mu(\mathcal{L}_\zeta g, \mathcal{L}_\zeta \phi) = \sum_{i=0}^3 J_{(i)}^\mu, \quad (8)$$

$$J_{(i)}^\mu = \frac{L_{(i)}}{\sqrt{-g}} \zeta^\mu + 2\mathcal{T}_{(i)}^{\mu\nu} \zeta_\nu - \Theta_{(i)}^\mu(\mathcal{L}_\zeta g, \mathcal{L}_\zeta \phi).$$

The off-shell Noether potential  $K^{\mu\nu}$ , defined through  $J^\mu = \nabla_\nu K^{\mu\nu}$ , takes the form

$$K^{\mu\nu} = \sum_{i=0}^3 K_{(i)}^{\mu\nu}, \quad (9)$$

where the off-shell Noether potentials  $K_{(i)}^{\mu\nu}$ , which are associated with the Noether currents  $J_{(i)}^\mu$  through the relations  $J_{(i)}^\mu = \nabla_\nu K_{(i)}^{\mu\nu}$ , are presented by

$$K_{(0)}^{\mu\nu} = 0, \quad (10)$$

$$K_{(1)}^{\mu\nu} = 2G^{(1)} \zeta^{[\mu} \nabla^{\nu]} \phi,$$

$$K_{(2)}^{\mu\nu} = 4G_{,X}^{(2)} (\square \phi \zeta^{[\mu} \nabla^{\nu]} \phi - \zeta_\sigma \Psi^{\sigma[\mu} \nabla^{\nu]} \phi) + 4\zeta^{[\mu} \nabla^{\nu]} G^{(2)} + 2G^{(2)} \nabla^{[\mu} \zeta^{\nu]},$$

and

$$K_{(3)}^{\mu\nu} = 6G_{,X}^{(3)} [(\square \phi)^2 \zeta^{[\mu} \nabla^{\nu]} \phi - \Psi_{\alpha\beta} \Psi^{\alpha\beta} \zeta^{[\mu} \nabla^{\nu]} \phi - 2\square \phi \zeta_\sigma \Psi^{\sigma[\mu} \nabla^{\nu]} \phi + 2\zeta^\sigma \Psi_{\sigma\rho} \Psi^{\rho[\mu} \nabla^{\nu]} \phi] - 6[2\zeta^{[\mu} \nabla_\sigma (\Psi^{\nu]\sigma} G^{(3)}) - 2\zeta_\sigma \nabla^{[\mu} (\Psi^{\nu]\sigma} G^{(3)}) - 2\zeta^{[\mu} \nabla^{\nu]} (G^{(3)} \square \phi) + 2G^{(3)} \zeta_\sigma G^{\sigma[\mu} \nabla^{\nu]} \phi - 2G^{(3)} (\nabla_\sigma \zeta^{[\mu} \Psi^{\nu]\sigma} - G^{(3)} \square \phi \nabla^{[\mu} \zeta^{\nu]})]. \quad (11)$$

In Eq. (11) and what follows, for brevity, the symmetric tensor  $\Psi_{\mu\nu}$  is defined as  $\Psi_{\mu\nu} = \nabla_\mu \nabla_\nu \phi$ .

Comparing the off-shell Noether potentials  $K_{(i)}^{\mu\nu}$  and  $K^{\mu\nu}$  with the on-shell ones got through Wald's covariant phase space approach [44,45], one can find that they are equivalent although the Noether currents are different in both the cases.

Next, assume that the smooth vector field  $\zeta^\mu$  respects the symmetry of spacetime, achieved by a Killing vector  $\xi^\mu$ . We follow [28] to introduce the off-shell ADT current  $J_{ADT}^\mu$  associated with such a Killing vector by

$$J_{ADT}^\mu = \delta \mathcal{T}^{\mu\nu} \xi_\nu + \frac{1}{2} g^{\rho\sigma} \delta g_{\rho\sigma} \mathcal{T}^{\mu\nu} \xi_\nu + \mathcal{T}^{\mu\nu} \delta g_{\nu\sigma} \xi^\sigma + \frac{1}{2} \xi^\mu (\mathcal{E}_{(\phi)} \delta \phi + \mathcal{T}_{\rho\sigma} \delta g^{\rho\sigma}) = \nabla_\nu Q_{ADT}^{\mu\nu}, \quad (12)$$

where  $Q_{ADT}^{\mu\nu}$  is just the off-shell ADT potential corresponding to the ADT current. In terms of the variation of the Lagrangian (1)

and the definition of the off-shell Noether current  $J^\mu$ , the ADT potential which is in one-to-one correspondence with the off-shell Noether potential can be presented by

$$Q_{ADT}^{\mu\nu} = \frac{1}{2} \frac{1}{\sqrt{-g}} \delta(\sqrt{-g} K^{\mu\nu}(\xi)) - \xi^{[\mu} \Theta^{\nu]}(\delta g, \delta \phi) = \sum_{i=0}^3 Q_{(i)}^{\mu\nu}, \quad (13)$$

$$Q_{(i)}^{\mu\nu} = \frac{1}{2} \frac{1}{\sqrt{-g}} \delta(\sqrt{-g} K_{(i)}^{\mu\nu}(\xi)) - \xi^{[\mu} \Theta_{(i)}^{\nu]}(\delta g, \delta \phi).$$

In Eq. (13), the Killing vector  $\xi^\mu$  is treated as a fixed background, namely,  $\delta \xi^\mu = 0$ , and the quantities  $Q_{(i)}^{\mu\nu}$  denote the contributions from the Lagrangian  $L_{(i)}$  respectively. For the variation of the off-shell Noether potentials  $K_{(i)}^{\mu\nu}$ , see the equations (B.1) and (B.4) in the Appendix B.

Finally, by following the approach in [41–43] to incorporate a single parameter path characterized by a parameter  $s$  ( $s \in [0, 1]$ ) in the space of solutions, we define the covariant formulation of conserved charges associated with the Noether potential  $Q_{ADT}^{\mu\nu}$  in Eq. (13) by [27,28]

$$Q = \frac{1}{8\pi} \int_0^1 ds \int d\Sigma_{\mu\nu} Q_{ADT}^{\mu\nu}(g, \phi; s), \quad (14)$$

where  $d\Sigma_{\mu\nu} = \frac{1}{2} \frac{1}{(D-2)!} \epsilon^{\mu\nu\mu_1\mu_2\cdots\mu_{(D-2)}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{(D-2)}}$  with  $\epsilon_{012\cdots(D-1)} = \sqrt{-g}$  and  $D$  is the dimension of spacetime. Eq. (14) can be a proposal of the formalism for the conserved charge, defined in the interior region or at the asymptotical infinity, for the most general Horndeski theory with the Lagrangian (1) whenever its integration is well-defined.

### 3. Conserved charges in a scalar–tensor theory with non-minimal derivative coupling

As an application of the off-shell Noether current and the formulation of the conserved charge, in the present section, we give a derivation of the formulation of the conserved charge in the context of a scalar–tensor theory with the prescription of non-minimal coupling between derivatives of a scalar field and the Einstein tensor, and then explicitly compute the mass and angular momentum of (rotating) black holes in such a theory. We start with the Lagrangian [11,24]

$$L_{(s)} = \sqrt{-g} [\lambda(R - 2\Lambda) - \eta \nabla_\mu \phi \nabla^\mu \phi + \beta G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi], \quad (15)$$

where  $(\lambda, \Lambda, \eta, \beta)$  are constants. In fact, the Lagrangian (15) can be seen as a subclass of the Lagrangian (1) for the full Horndeski theory in addition to a total divergence term, by setting

$$G^{(0)} = -2\lambda\Lambda + 2\eta X, \quad G^{(1)} = 0, \quad G^{(2)} = \lambda, \quad G^{(3)} = \frac{\beta}{6} \phi, \quad (16)$$

or [11]

$$G^{(0)} = -2\lambda\Lambda + 2\eta X, \quad G^{(2)} = \lambda + \beta X, \quad G^{(1)} = G^{(3)} = 0. \quad (17)$$

As a consequence, we only need to substitute the above  $G^{(i)}$  into the expressions for the equations of motion, surface terms and off-shell Noether potentials associated to the most general Lagrangian (1) to get the corresponding quantities for the Lagrangian (15). The expressions of the field equations are read off as

$$T_{\mu\nu}^{(s)} = \lambda G_{\mu\nu} - \eta \Phi_{\mu\nu} + g_{\mu\nu} (\lambda\Lambda - \eta X) + \frac{\beta}{2} [4R^\sigma_{(\mu} \Phi_{\nu)\sigma} - 2\nabla^\sigma \nabla_{(\mu} \Phi_{\nu)\sigma} + \square \Phi_{\mu\nu}]$$

$$\begin{aligned}
& + g_{\mu\nu}(\nabla^\rho\nabla^\sigma\Phi_{\rho\sigma} - R^{\rho\sigma}\Phi_{\rho\sigma} + 2\Box X) \\
& + 2XG_{\mu\nu} - 2\nabla_\mu\nabla_\nu X - R\Phi_{\mu\nu}], \\
\mathcal{E}_{(\phi)}^{(s)} = & 2\nabla_\mu[(\eta g^{\mu\nu} - \beta G^{\mu\nu})\nabla_\nu\phi]. \tag{18}
\end{aligned}$$

The surface term  $\Theta_{(s)}^\mu(\delta g, \delta\phi)$  for the Lagrangian (15) has the form

$$\begin{aligned}
\Theta_{(s)}^\mu = & -2\eta\nabla^\mu\phi\delta\phi + 2\lambda g^{\rho[\mu}\nabla^{\sigma]}h_{\rho\sigma} \\
& + \frac{\beta}{2}[2\Phi_{\sigma\rho}\nabla^\sigma h^{\rho\mu} - 2h_{\rho\sigma}\nabla^\rho\Phi^{\sigma\mu} - \Phi^{\rho\sigma}\nabla^\mu h_{\rho\sigma} \\
& + h_{\rho\sigma}\nabla^\mu\Phi^{\rho\sigma} - \Phi^{\mu\nu}\nabla_\nu h + h\nabla_\nu\Phi^{\mu\nu} - 2h^{\rho\mu}\nabla_\rho X \\
& + 4Xg^{\rho[\mu}\nabla^{\sigma]}h_{\rho\sigma} + 2h\nabla^\mu X + 4\delta\phi G^{\mu\nu}\nabla_\nu\phi], \tag{19}
\end{aligned}$$

while the off-shell Noether potentials are

$$\begin{aligned}
K_{(s)}^{\mu\nu} = & 2\lambda\nabla^{[\mu}\zeta^{\nu]} \\
& + 2\beta(2\zeta^{[\mu}\nabla^{\nu]}X + X\nabla^{[\mu}\zeta^{\nu]} + \zeta^{[\mu}\nabla_\sigma\Phi^{\nu]\sigma} - \zeta_\sigma\nabla^{[\mu}\Phi^{\nu]\sigma} \\
& + \Phi^{\sigma[\mu}\nabla_\sigma\zeta^{\nu]}). \tag{20}
\end{aligned}$$

Note that  $K_{(s)}^{\mu\nu}$  in the above equation is equivalent to the on-shell Noether potential obtained through Wald's covariant phase space approach in [8], where the Noether potential was adopted to calculate the mass of static black holes. For the variation of the Noether potential (20) see Eq. (B.8) in the Appendix B. Substituting the expressions for  $K_{(s)}^{\mu\nu}$  and  $\Theta_{(s)}^\mu$  into Eq. (14) in the condition that  $\zeta^\mu$  is a Killing vector, one can further propose a formulation of the conserved charge in the scalar-tensor theory described by the Lagrangian (15).

Till now, it has been extensively studied to seek solutions of the Lagrangian (15), for instance, in the contexts of static solutions with various asymptotical structures [8–18], rotating black holes in three dimensions [19,20], and slowly rotating black holes in four dimensions [21,22]. Thermodynamics of this theory was investigated in [23]. The formulation (14) provides another avenue to obtain conserved charges of these black holes. As an explicit example, we now pay attention to computing mass and angular momentum of the BTZ-type black hole with a single scalar degree of freedom in [19], where the authors only utilized the method of Euclidean action to calculate the mass of the black hole in the static case. The corresponding Lagrangian of the black hole solution is the one in Eq. (15) with  $\lambda = 1$ . The line element and the scalar field that is only dependent on the radial coordinate take the forms

$$\begin{aligned}
ds^2 = & -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\varphi - \frac{a}{2r^2}dt\right)^2, \\
\left(\frac{d\phi}{dr}\right)^2 = & -\frac{\ell^2\Lambda + 1}{\beta f(r)}, \tag{21}
\end{aligned}$$

where

$$f(r) = \frac{r^2}{\ell^2} - m + \frac{a^2}{4r^2}, \quad \ell^2 = \frac{\beta}{\eta}, \tag{22}$$

and the constants  $(m, a)$  correspond to the mass and angular momentum respectively.

The mass  $M$  of the BTZ-type black hole (21) can be treated as a Noether charge with respect to time translational symmetry reflected by the Killing vector  $\xi_{(t)}^\mu = (-1, 0, 0)$ . On the other hand, the perturbations of the fields are achieved by letting the parameters fluctuate as  $m \rightarrow m + dm$  and  $a \rightarrow a + da$ . Under such conditions, the  $(t, r)$  component of the ADT potential corresponding to the Killing vector  $\xi_{(t)}^\mu$  is read off as

$$\begin{aligned}
Q_{ADT}^{tr} = & \frac{1}{2} \frac{1}{\sqrt{-g}} \delta(\sqrt{-g}K_{(s)}^{tr}(\xi_{(t)})) - \xi_{(t)}^l \Theta_{(s)}^r_l \\
= & \frac{(1 - \Lambda\ell^2)}{4r} dm. \tag{23}
\end{aligned}$$

Substituting Eq. (23) into the formulation (14), we have

$$M = \frac{1}{16}(1 - \Lambda\ell^2)m. \tag{24}$$

The angular momentum  $J$  of the black hole can be obtained in a similar manner as we perform to compute the mass when the Killing vector is chosen as  $\xi_{(\varphi)}^\mu = \delta_{\varphi}^\mu$ . It is presented by

$$J = \frac{1}{16}(1 - \Lambda\ell^2)a. \tag{25}$$

Both the mass  $M$  and the angular momentum  $J$  satisfy the first law of thermodynamics. In particular, if  $\ell^2 = -\Lambda^{-1}$ , the scalar field  $\phi$  vanishes.  $M$  and  $J$  reduce to the mass and angular momentum of the conventional BTZ black hole, respectively. In the work [19], the authors also constructed solutions of black holes with a planar horizon in arbitrary dimensions. Making use of the formulation (14) to compute the mass of these black holes, we get their mass that is consistent with the one derived via the method of Euclidean action.

Next, we calculate the mass of the four-dimensional charged spherically symmetric black hole with an asymptotically locally AdS structure in [17]. The Lagrangian associated with this black hole is  $L_{(s)}(\lambda = 1, \eta \rightarrow \alpha/2, \beta \rightarrow \eta/2) + L_{em}$ , where  $L_{em} = -\sqrt{-g}F_{\mu\nu}F^{\mu\nu}/4$ . The black hole solution is given by<sup>1</sup>

$$\begin{aligned}
ds^2 = & -F(r)dt^2 + G(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \\
A = & A_t(r)dt, \\
\left(\frac{d\phi}{dr}\right)^2 = & -\frac{1}{32} \frac{\alpha^2[4(\alpha + \Lambda\eta)r^4 + \eta q^2][4(\alpha - \Lambda\eta)r^4 + 8\eta r^2 - \eta q^2]^2}{r^6\eta(\alpha - \Lambda\eta)^2(\alpha r^2 + \eta)^3 F(r)}. \tag{26}
\end{aligned}$$

In the above equation,

$$\begin{aligned}
F(r) = & \frac{r^2}{l^2} + \frac{\sqrt{\eta\alpha}[4\eta(\alpha + \Lambda\eta) + \alpha^2 q^2]^2 \arctan(\frac{\sqrt{\eta\alpha}r}{\eta})}{16\eta^2(\alpha - \Lambda\eta)^2 r} \\
& - \frac{m}{r} + \frac{3\alpha + \Lambda\eta}{\alpha - \Lambda\eta} \\
& + \frac{\alpha^2 q^2}{48\eta(\alpha - \Lambda\eta)^2} \frac{3(\alpha q^2 + 16\eta)r^2 - q^2\eta}{r^4}, \quad l^2 = \frac{3\eta}{\alpha}, \\
G(r) = & \frac{1}{16} \frac{\alpha^2[4(\alpha - \Lambda\eta)r^4 + 8\eta r^2 - \eta q^2]^2}{r^4(\alpha - \Lambda\eta)^2(\alpha r^2 + \eta)^2 F(r)}, \tag{27}
\end{aligned}$$

and the  $t$  component of the U(1) gauge field

$$\begin{aligned}
A_t(r) = & \frac{q\sqrt{\eta\alpha}[4\eta(\alpha + \Lambda\eta) + \alpha^2 q^2] \arctan\left(\frac{\sqrt{\eta\alpha}r}{\eta}\right)}{4\eta^2(\alpha - \Lambda\eta)} \\
& + \frac{\alpha(8\eta + \alpha q^2)q}{4\eta(\alpha - \Lambda\eta)r} - \frac{\alpha}{12(\alpha - \Lambda\eta)} \frac{q^3}{r^3}. \tag{28}
\end{aligned}$$

The constants  $(m, q)$  denote the mass and electric charge respectively and the parameter  $\Lambda$  is assumed to satisfy that  $\Lambda < 0$ . When  $q = 0$ , the black hole (26) reduces to the neutral one in [13].

<sup>1</sup> We have set  $\kappa = 1$  in comparison with the solution in [17]. Therein the  $t$  component of the gauge field  $A_0(r)$  has several typos.



To get the mass of the black hole, the infinitesimal variation of the fields is determined by letting the constants  $(m, q)$  change as  $m \rightarrow m + dm$  and  $q \rightarrow q + dq$ , and the Killing vector  $\xi_{(t)}^\mu = -\delta_t^\mu$ . By using Eq. (13), one can get the ADT potential related to the gravitational field and the scalar field. Besides, since the theory includes gauge field  $A_\mu$ , one has to consider the contribution to the potential from the Lagrangian  $L_{em}$ , which is read off as [28]

$$Q_{em}^{\mu\nu} = \frac{1}{4}\xi^\sigma A_\sigma (hF^{\mu\nu} + 4h^{\rho[\mu}F^{\nu]}\rho + 4g^{\alpha[\mu}g^{\nu]\beta}\partial_\alpha\delta A_\beta) + \frac{1}{2}F^{\mu\nu}\xi^\sigma\delta A_\sigma + \xi^{[\mu}F^{\nu]\sigma}\delta A_\sigma. \quad (29)$$

Therefore, the ADT potential corresponding to the Lagrangian  $L_{(s)} + L_{em}$  is  $Q_{ADT}^{\mu\nu} \rightarrow Q_{total}^{\mu\nu} = Q_{ADT}^{\mu\nu} + Q_{em}^{\mu\nu}$ . The  $(t, r)$  component is

$$\sqrt{-g}Q_{total}^{tr} = \frac{3 - \Lambda l^2}{6} \sin\theta d(m), \quad (30)$$

whose integration yields the mass

$$\mathcal{M} = \frac{3 - \Lambda l^2}{12} m. \quad (31)$$

When  $q = 0$  and  $\alpha = -\eta\Lambda$ , the black hole (26) reduces to the well-known four-dimensional Schwarzschild-AdS black hole. In such a case,  $\mathcal{M} = m/2$  is just the mass of the Schwarzschild-AdS black hole. In the work [13], the authors also computed the mass of the neutral spherically symmetric black hole through the method of Euclidean action, which is different from the mass  $\mathcal{M}$  here and does not recover the mass of the Schwarzschild-AdS black hole.

Finally, we have applied the method in the present work to calculate the conserved charges of Warped-AdS<sub>3</sub> black holes with a scalar field in [20]. Unfortunately, both the mass and angular momentum are zero. This maybe arise from the fact that the formulation (14) for the conserved charge is covariant and the warped-AdS<sub>3</sub> black hole is locally equivalent to the warped-AdS<sub>3</sub> space, while the mass and angular momentum of the latter vanish. In order to get sensible results, we shall take into account this point in the future work.

#### 4. Summary

We obtain the off-shell Noether current (8) and its corresponding potential (9) in the context of the full Horndeski gravity theory described by the Lagrangian (1). To achieve this, we first derive the surface terms and equations of motion from the variation of the Lagrangian. By lifting the conventional ADT potential to the off-shell level, we further give a proposal on the formulation (14) of the conserved charge in terms of the off-shell ADT potential, which is actually equivalent to the Noether potential via the covariant phase space approach. Our derivation provides a general and systematic method to compute the conserved charge in the Horndeski theory. Because of the generality of the Horndeski theory, it is feasible to extend the formulation (14) to various well-known scalar-tensor theories with a single scalar degree of freedom.

As an application of the general formalism, we derive the off-shell Noether potential of a specific subclass of the full Horndeski gravity theory depicted by the Lagrangian (15), namely, the scalar-tensor theory with non-minimal coupling between derivatives of a scalar field and the Einstein tensor. In terms of the off-shell Noether potential and the surface term of this special theory, we first explicitly compute both the mass and angular momentum of the BTZ-type black hole (21), as well as the mass of the four-dimensional charged spherically symmetric black hole (26).

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#### Appendix A. The variation of the terms $L_{(2,3)}$

In this appendix, we shall derive the expressions of the field equations and surface terms from the variation of the Lagrangian terms  $L_{(2)}$  and  $L_{(3)}$ . In what follows, note that we use the notations  $(\Psi_{\alpha\beta})^2 = (\nabla_\alpha\nabla_\beta\phi)(\nabla^\alpha\nabla^\beta\phi)$  and  $(\Psi_{\alpha\beta})^3 = (\nabla_\alpha\nabla_\beta\phi)(\nabla^\alpha\nabla_\gamma\phi)(\nabla^\gamma\nabla^\beta\phi)$ .

The variation of the Lagrangian terms  $L_{(2)}$  and  $L_{(3)}$  with respect to both the fields  $g_{\mu\nu}$  and  $\phi$  is presented as

$$\delta L_{(j)} = \sqrt{-g}[\mathcal{T}_{\mu\nu}^{(j)}\delta g^{\mu\nu} + \mathcal{E}_{(\phi)}^{(j)}\delta\phi + \nabla_\mu\Theta_{(j)}^\mu(\delta g, \delta\phi)], \quad j = 2, 3. \quad (A.1)$$

For the Lagrangian  $L_{(2)}$ , the expressions of the field equation  $\mathcal{T}_{\mu\nu}^{(2)}$  and  $\mathcal{E}_{(\phi)}^{(2)}$  are given by

$$\begin{aligned} \mathcal{T}_{\mu\nu}^{(2)} = & \frac{1}{2}g_{\mu\nu}G_{,X}^{(2)}(\square\phi)^2 + g_{\mu\nu}\nabla_\sigma(G_{,X}^{(2)}\square\phi)\nabla^\sigma\phi \\ & - 2\nabla_{(\mu}(G_{,X}^{(2)}\square\phi)\nabla_{\nu)}\phi + G^{(2)}G_{\mu\nu} \\ & + \frac{1}{2}G_{,XX}^{(2)}((\Psi_{\alpha\beta})^2 - (\square\phi)^2)\Phi_{\mu\nu} + 2\nabla^\sigma(G_{,X}^{(2)}\Psi_{\sigma(\mu}\nabla_{\nu)}\phi \\ & - \frac{1}{2}RG_{,X}^{(2)}\Phi_{\mu\nu} + \frac{1}{2}g_{\mu\nu}G_{,X}^{(2)}(\Psi_{\alpha\beta})^2 - \nabla_\sigma(G_{,X}^{(2)}\Psi_{\mu\nu}\nabla^\sigma\phi) \\ & - \nabla_\mu\nabla_\nu G^{(2)} + g_{\mu\nu}\square G^{(2)}, \end{aligned} \quad (A.2)$$

$$\begin{aligned} \mathcal{E}_{(\phi)}^{(2)} = & 2\square(G_{,X}^{(2)}\square\phi) - \nabla_\mu[G_{,XX}^{(2)}((\Psi_{\alpha\beta})^2 - (\square\phi)^2)\nabla^\mu\phi] \\ & + \nabla_\mu(G_{,X}^{(2)}R\nabla^\mu\phi) - G_{,X\phi}^{(2)}((\Psi_{\alpha\beta})^2 - (\square\phi)^2) \\ & - 2\nabla_\mu\nabla_\nu(G_{,X}^{(2)}\Psi^{\mu\nu}) + G_{,\phi}^{(2)}R, \end{aligned} \quad (A.3)$$

and the surface term  $\Theta_{(2)}^\mu(\delta g, \delta\phi)$  takes the form

$$\begin{aligned} \Theta_{(2)}^\mu = & G_{,X}^{(2)}\square\phi(h\nabla^\mu\phi - 2h^{\mu\nu}\nabla_\nu\phi) + 2G_{,X}^{(2)}\square\phi\nabla^\mu\delta\phi \\ & - 2\nabla^\mu(G_{,X}^{(2)}\square\phi)\delta\phi + G_{,XX}^{(2)}((\Psi_{\alpha\beta})^2 - (\square\phi)^2)\nabla^\mu\phi\delta\phi \\ & + G_{,X}^{(2)}(2\Psi^{\mu\rho}\nabla^\sigma\phi - \Psi^{\rho\sigma}\nabla^\mu\phi)h_{\rho\sigma} - 2G_{,X}^{(2)}\Psi^{\mu\nu}\nabla_\nu\delta\phi \\ & + 2\nabla_\nu(G_{,X}^{(2)}\Psi^{\mu\nu})\delta\phi - h^{\mu\nu}\nabla_\nu G^{(2)} + G^{(2)}\nabla_\nu h^{\mu\nu} \\ & + h\nabla^\mu G^{(2)} - G^{(2)}\nabla^\mu h - G_{,X}^{(2)}R\nabla^\mu\phi\delta\phi. \end{aligned} \quad (A.4)$$

For the Lagrangian  $L_{(3)}$ , the expression for the equation of motion  $\mathcal{T}_{\mu\nu}^{(3)}$  is presented by

$$\mathcal{T}_{\mu\nu}^{(3)} = \mathcal{T}_{\mu\nu}^{(31)} + \mathcal{T}_{\mu\nu}^{(32)}, \quad (A.5)$$

where

$$\begin{aligned} \mathcal{T}_{\mu\nu}^{(31)} = & -\frac{1}{2}G_{,XX}^{(3)}[(\square\phi)^3 - 3\square\phi(\Psi_{\alpha\beta})^2 + 2(\Psi_{\alpha\beta})^3]\Phi_{\mu\nu} \\ & + 3\nabla_\sigma(G_{,X}^{(3)}\Psi_{\mu\rho}\Psi_{\nu}{}^\rho\nabla^\sigma\phi) \\ & - \frac{1}{2}g_{\mu\nu}\{3\nabla_\sigma[G_{,X}^{(3)}((\Psi_{\alpha\beta})^2 - (\square\phi)^2)]\nabla^\sigma\phi \\ & + 2G_{,X}^{(3)}((\Psi_{\alpha\beta})^3 - (\square\phi)^3)\} \\ & + 3\nabla_{(\mu}[G_{,X}^{(3)}((\Psi_{\alpha\beta})^2 - (\square\phi)^2)]\nabla_{\nu)}\phi \end{aligned}$$

$$\begin{aligned}
& + 6\nabla^\sigma (G_{,X}^{(3)} \square \phi \Psi_{\sigma(\mu)} \nabla_{\nu)} \phi - 3\nabla_\sigma (G_{,X}^{(3)} \square \phi \Psi_{\mu\nu} \nabla^\sigma \phi) \\
& - 6\nabla_\sigma (G_{,X}^{(3)} \Psi^{\sigma\rho} \Psi_{\rho(\mu)} \nabla_{\nu)} \phi, \tag{A.6}
\end{aligned}$$

and the component  $\mathcal{T}_{\mu\nu}^{(32)}$ , which is the contribution from the second term  $-6\sqrt{-g}G^{(3)}G_{\mu\nu}\Psi^{\mu\nu}$  of  $L_{(3)}$  in Eq. (2), is read off as

$$\begin{aligned}
\mathcal{T}_{\mu\nu}^{(32)} = & 3G_{,X}^{(3)} G_{\rho\sigma} \Psi^{\rho\sigma} \Phi_{\mu\nu} \\
& + 3G^{(3)} (R\Psi_{\mu\nu} + R_{\mu\nu}\square\phi - 4R_{\rho(\mu}\Psi_{\nu)}{}^\rho) - 3\square(G^{(3)}\Psi_{\mu\nu}) \\
& + 3g_{\mu\nu}[\square(G^{(3)}\square\phi) - \nabla_\rho\nabla_\sigma(G^{(3)}\Psi^{\rho\sigma}) + G^{(3)}G_{\rho\sigma}\Psi^{\rho\sigma}] \\
& - 3\nabla_\mu\nabla_\nu(G^{(3)}\square\phi) + 6\nabla^\rho\nabla_{(\mu}(\Psi_{\nu)\rho}G^{(3)}) \\
& + 3\nabla^\sigma[G^{(3)}(2G_{\sigma(\mu}\nabla_{\nu)}\phi - G_{\mu\nu}\nabla_\sigma\phi)]. \tag{A.7}
\end{aligned}$$

The expression  $\mathcal{E}_{(\phi)}^{(3)}$  of the equation of motion for the matter field  $\phi$  is given by

$$\begin{aligned}
\mathcal{E}_{(\phi)}^{(3)} = & \nabla_\mu[G_{,XX}^{(3)}((\square\phi)^3 - 3\square\phi(\Psi_{\alpha\beta})^2 + 2(\Psi_{\alpha\beta})^3)\nabla^\mu\phi] \\
& - 6G_{,\phi}^{(3)}G^{\rho\sigma}\Psi_{\rho\sigma} - 6\nabla_\mu\nabla_\nu(G_{,X}^{(3)}\square\phi\Psi^{\mu\nu}) \\
& + 6\nabla_\mu\nabla^\nu(G_{,X}^{(3)}\Psi^{\mu\sigma}\Psi_{\nu\sigma}) - 6G^{\rho\sigma}\nabla_\rho\nabla_\sigma G^{(3)} \\
& + 3\square[G_{,XX}^{(3)}(\square\phi)^2] - 3\square[G_{,X}^{(3)}(\Psi_{\alpha\beta})^2] \\
& - 6\nabla_\mu(G_{,X}^{(3)}G^{\rho\sigma}\Psi_{\rho\sigma}\nabla^\mu\phi) + G_{,X\phi}^{(3)}((\square\phi)^3 \\
& - 3\square\phi(\Psi_{\alpha\beta})^2 + 2(\Psi_{\alpha\beta})^3). \tag{A.8}
\end{aligned}$$

The surface term  $\Theta_{(3)}^\mu(\delta g, \delta\phi)$  is also split in two components, namely,

$$\Theta_{(3)}^\mu(\delta g, \delta\phi) = \Theta_{(31)}^\mu(\delta g, \delta\phi) + \Theta_{(32)}^\mu(\delta g, \delta\phi), \tag{A.9}$$

where the component  $\Theta_{(31)}^\mu$  coming from the contribution of the variation of the first term in  $L_{(3)}$  has the form

$$\begin{aligned}
\Theta_{(31)}^\mu = & -\frac{3}{2}G_{,X}^{(3)}(\Psi_{\alpha\beta})^2(h\nabla^\mu\phi - 2h^{\mu\nu}\nabla_\nu\phi) \\
& + 3h_{\rho\sigma}G_{,X}^{(3)}\square\phi(2\Psi^{\sigma\mu}\nabla^\rho\phi - \Psi^{\sigma\rho}\nabla^\mu\phi) \\
& + \frac{3}{2}G_{,X}^{(3)}(\square\phi)^2(h\nabla^\mu\phi + 2\nabla^\mu\delta\phi - 2h^{\mu\nu}\nabla_\nu\phi) \\
& - 3\nabla^\mu[G_{,X}^{(3)}(\square\phi)^2]\delta\phi - 3G_{,X}^{(3)}(\Psi_{\alpha\beta})^2\nabla^\mu\delta\phi \\
& + 3\delta\phi\nabla^\mu[G_{,X}^{(3)}(\Psi_{\alpha\beta})^2] - 6G_{,X}^{(3)}\square\phi\Psi^{\mu\nu}\nabla_\nu\delta\phi \\
& + 6\delta\phi\nabla_\nu(G_{,X}^{(3)}\square\phi\Psi^{\mu\nu}) + 6G_{,X}^{(3)}\Psi^{\mu\sigma}\Psi_{\nu\sigma}\nabla^\nu\delta\phi \\
& - 6\delta\phi\nabla^\nu(G_{,X}^{(3)}\Psi^{\mu\sigma}\Psi_{\nu\sigma}) \\
& - G_{,XX}^{(3)}((\square\phi)^3 - 3\square\phi(\Psi_{\alpha\beta})^2 + 2(\Psi_{\alpha\beta})^3)\nabla^\mu\phi\delta\phi \\
& - 3G_{,X}^{(3)}\Psi^\nu{}_\sigma(2\Psi^{\sigma\mu}\nabla^\rho\phi - \Psi^{\sigma\rho}\nabla^\mu\phi)h_{\rho\nu}, \tag{A.10}
\end{aligned}$$

and the component  $\Theta_{(32)}^\mu$ , which is just the surface term from the variation of the second term  $-6\sqrt{-g}G^{(3)}G_{\mu\nu}\Psi^{\mu\nu}$  in  $L_{(3)}$ , is presented by

$$\begin{aligned}
\Theta_{(32)}^\mu = & 6h_{\rho\sigma}\nabla^\sigma(G^{(3)}\Psi^{\mu\rho}) - 6G^{(3)}\Psi_{\sigma\rho}\nabla^\sigma h^{\rho\mu} \\
& + 3G^{(3)}\Psi_{\rho\sigma}\nabla^\mu h^{\rho\sigma} - 3h^{\rho\sigma}\nabla^\mu(G^{(3)}\Psi_{\rho\sigma}) \\
& + 3G^{(3)}\Psi^{\mu\nu}\nabla_\nu h - 3h\nabla_\nu(G^{(3)}\Psi^{\mu\nu}) \\
& + 3G^{(3)}\square\phi\nabla_\rho h^{\rho\mu} - 3h^{\rho\mu}\nabla_\rho(G^{(3)}\square\phi) - 3G^{(3)}\square\phi\nabla^\mu h \\
& + 3h\nabla^\mu(G^{(3)}\square\phi) + 6G^{(3)}h_{\rho\sigma}G^{\mu\rho}\nabla^\sigma\phi \\
& - 3G^{(3)}h_{\rho\sigma}G^{\rho\sigma}\nabla^\mu\phi + 6\delta\phi G^{\mu\nu}\nabla_\nu G^{(3)} \\
& - 6G^{(3)}G^{\mu\nu}\nabla_\nu\delta\phi + 6\delta\phi G_{,X}^{(3)}G^{\rho\sigma}\Psi_{\rho\sigma}\nabla^\mu\phi. \tag{A.11}
\end{aligned}$$

In the works [3,4,21], the equations of motion for the Horndeski theory were also presented but the surface terms were absent. To compare the field equations in this work with the ones in [4], one can find that all the equations of motion for the matter field  $\phi$  coincide with each other and the relationship between the expressions of the equations of motion with respect to the gravitational field  $g_{\mu\nu}$  is  $-2\mathcal{T}_{\mu\nu}^{(i)} = T_{\mu\nu}^{(i)}$ , where  $T_{\mu\nu}^{(i)}$  is the notation in [4].

## Appendix B. The variation for the off-shell Noether potentials $K_{(i)}^{\mu\nu}$ and $K_{(s)}^{\mu\nu}$

In the present appendix, we give a derivation of the variation for the off-shell Noether potentials  $K_{(i)}^{\mu\nu}$  and  $K_{(s)}^{\mu\nu}$  under the condition that  $\delta\zeta^\mu = 0$ . Varying the off-shell Noether potentials  $K_{(1)}^{\mu\nu}$  and  $K_{(2)}^{\mu\nu}$  in Eq. (10), we have

$$\begin{aligned}
\delta K_{(1)}^{\mu\nu} = & 2\delta G^{(1)}\zeta^{[\mu}\nabla^{\nu]}\phi + 2G^{(1)}(\zeta^{[\mu}\nabla^{\nu]}\delta\phi - \zeta^{[\mu}h^{\nu]\sigma}\nabla_\sigma\phi), \\
\delta K_{(2)}^{\mu\nu} = & 4\delta G_{,X}^{(2)}(\square\phi\zeta^{[\mu}\nabla^{\nu]}\phi - \zeta_\sigma\Psi^{\sigma[\mu}\nabla^{\nu]}\phi) \\
& + 4G_{,X}^{(2)}[\delta(\square\phi)\zeta^{[\mu}\nabla^{\nu]}\phi \\
& + \square\phi(\zeta^{[\mu}\nabla^{\nu]}\delta\phi - \zeta^{[\mu}h^{\nu]\sigma}\nabla_\sigma\phi) \\
& - h_{\rho\sigma}\zeta^\rho\Psi^{\sigma[\mu}\nabla^{\nu]}\phi - \zeta_\sigma\delta(\Psi^{\sigma[\mu}\nabla^{\nu]}\phi)] \\
& - 4\zeta^{[\mu}h^{\nu]\sigma}\nabla_\sigma G^{(2)} + 4\zeta^{[\mu}\nabla^{\nu]}\delta G^{(2)} \\
& + 2G^{(2)}(\zeta_\sigma\nabla^{[\mu}h^{\nu]\sigma} - h^{\sigma[\mu}\nabla_\sigma\zeta^{\nu]}) \\
& + 2\delta G^{(2)}\nabla^{[\mu}\zeta^{\nu]}, \tag{B.1}
\end{aligned}$$

where

$$\begin{aligned}
\delta X = & \frac{1}{2}h_{\rho\sigma}\Phi^{\rho\sigma} - (\nabla^\sigma\phi)(\nabla_\sigma\delta\phi), \\
\delta\Psi_{\mu\nu} = & \nabla_\mu\nabla_\nu\delta\phi - \frac{1}{2}(2\nabla_{(\mu}h_{\nu)\lambda} - \nabla_\lambda h_{\mu\nu})\nabla^\lambda\phi, \\
\delta G^{(i)} = & G_{,\phi}^{(i)}\delta\phi + G_{,X}^{(i)}\delta X, \quad \delta G_{,X}^{(i)} = G_{,X\phi}^{(i)}\delta\phi + G_{,XX}^{(i)}\delta X, \\
\delta\Psi^{\mu\nu} = & -2h^{\sigma(\mu}\Psi^{\nu)}{}_\sigma + g^{\mu\rho}g^{\nu\sigma}\delta\Psi_{\rho\sigma}, \\
\delta\square\phi = & h_{\rho\sigma}\Psi^{\rho\sigma} + g_{\rho\sigma}\delta\Psi^{\rho\sigma}, \tag{B.2}
\end{aligned}$$

and

$$\delta(\Psi^{\sigma[\mu}\nabla^{\nu]}\phi) = \delta\Psi^{\sigma[\mu}\nabla^{\nu]}\phi - \Psi^{\sigma[\mu}h^{\nu]\rho}\nabla_\rho\phi + \Psi^{\sigma[\mu}\nabla^{\nu]}\delta\phi. \tag{B.3}$$

The perturbation of  $K_{(3)}^{\mu\nu}$  in Eq. (11) takes the form

$$\delta K_{(3)}^{\mu\nu} = \delta K_{(31)}^{\mu\nu} + \delta K_{(32)}^{\mu\nu}, \tag{B.4}$$

where

$$\begin{aligned}
\delta K_{(31)}^{\mu\nu} = & 6\delta G_{,X}^{(3)}[(\square\phi)^2 - \Psi_{\alpha\beta}\Psi^{\alpha\beta}]\zeta^{[\mu}\nabla^{\nu]}\phi \\
& + 2\Psi^{\sigma[\mu}\nabla^{\nu]}\phi(\zeta^\rho\Psi_{\rho\sigma} - \square\phi\zeta_\sigma) \\
& + 6G_{,X}^{(3)}[(2\square\phi\delta\square\phi - \delta\Psi_{\alpha\beta}\Psi^{\alpha\beta} - \Psi_{\alpha\beta}\delta\Psi^{\alpha\beta})\zeta^{[\mu}\nabla^{\nu]}\phi \\
& + ((\square\phi)^2 - \Psi_{\alpha\beta}\Psi^{\alpha\beta})(\zeta^{[\mu}\nabla^{\nu]}\delta\phi - \zeta^{[\mu}h^{\nu]\sigma}\nabla_\sigma\phi) \\
& + 2\delta(\Psi^{\sigma[\mu}\nabla^{\nu]}\phi)(\zeta^\rho\Psi_{\rho\sigma} - \square\phi\zeta_\sigma) \\
& + 2(\zeta^\rho\delta\Psi_{\rho\sigma} - \zeta_\sigma\delta\square\phi - h_{\sigma\lambda}\zeta^\lambda\square\phi)\Psi^{\sigma[\mu}\nabla^{\nu]}\phi] \tag{B.5}
\end{aligned}$$

and

$$\begin{aligned}
\delta K_{(32)}^{\mu\nu} = & -6[2\zeta^{[\mu}\delta Y_{\sigma}{}^{\nu]\sigma} - 2h_{\rho\sigma}\zeta^{\rho}Y^{[\mu\nu]\sigma} + 2\zeta_{\sigma}h^{\rho[\mu}Y_{\rho}{}^{\nu]\sigma} \\
& - 2\zeta_{\sigma}g^{\rho[\mu}\delta Y_{\rho}{}^{\nu]\sigma} - 2h_{\rho\sigma}\zeta^{[\mu}Y^{\nu]\rho\sigma} \\
& + 2g_{\rho\sigma}\zeta^{[\mu}h^{\nu]\lambda}Y_{\lambda}{}^{\rho\sigma} - 2g_{\rho\sigma}\zeta^{[\mu}g^{\nu]\lambda}\delta Y_{\lambda}{}^{\rho\sigma} \\
& + 2(G^{(3)}h_{\rho\sigma}\zeta^{\rho} + \delta G^{(3)}\zeta_{\sigma})G^{\sigma[\mu}\nabla^{\nu]}\phi \\
& + 2G^{(3)}\zeta_{\sigma}(\delta G^{\sigma[\mu}\nabla^{\nu]}\phi - G^{\sigma[\mu}h^{\nu]\rho}\nabla_{\rho}\phi + G^{\sigma[\mu}\nabla^{\nu]}\delta\phi) \\
& + 2\delta G^{(3)}\Psi^{\sigma[\mu}\nabla_{\sigma}\zeta^{\nu]} \\
& + G^{(3)}\zeta^{\rho}(\Psi^{\sigma[\mu}\nabla_{\sigma}h^{\nu]}\rho + \Psi_{\sigma}{}^{[\mu}\nabla_{\rho}h^{\nu]\sigma} - \Psi^{\sigma[\mu}\nabla^{\nu]}h_{\rho\sigma}) \\
& + 2G^{(3)}\delta\Psi^{\sigma[\mu}\nabla_{\sigma}\zeta^{\nu]} - \delta(G^{(3)}\square\phi)\nabla^{[\mu}\zeta^{\nu]} \\
& - G^{(3)}\square\phi(\zeta_{\sigma}\nabla^{[\mu}h^{\nu]\sigma} - h^{\sigma[\mu}\nabla_{\sigma}\zeta^{\nu]})]. \quad (B.6)
\end{aligned}$$

In the above equation,

$$\begin{aligned}
Y_{\rho}{}^{\mu\nu} &= \nabla_{\rho}(G^{(3)}\Psi^{\mu\nu}), \\
\delta Y_{\rho}{}^{\mu\nu} &= \nabla_{\rho}(\delta G^{(3)}\Psi^{\mu\nu} + G^{(3)}\delta\Psi^{\mu\nu}) \\
&\quad + G^{(3)}g^{\alpha(\mu}\Psi^{\nu)\beta}(2\nabla_{(\rho}h_{\beta)\alpha} - \nabla_{\alpha}h_{\rho\beta}), \\
\delta G^{\sigma\mu} &= \frac{1}{2}[2\nabla_{\lambda}\nabla^{(\sigma}h^{\mu)\lambda} - 4h_{\lambda}{}^{(\sigma}R^{\mu)\lambda} - \square h^{\sigma\mu} - \nabla^{\sigma}\nabla^{\mu}h \\
&\quad + Rh^{\sigma\mu} - g^{\sigma\mu}(\nabla^{\alpha}\nabla^{\beta}h_{\alpha\beta} - h^{\alpha\beta}R_{\alpha\beta} - \square h)]. \quad (B.7)
\end{aligned}$$

Besides, the variation of the total off-shell Noether potential  $K^{\mu\nu}$  can be expressed as  $\delta K^{\mu\nu} = \sum_{i=1}^3 \delta K_{(i)}^{\mu\nu}$ .

Finally, the variation of the off-shell Noether potential  $K_{(s)}^{\mu\nu}$  in Eq. (20) is given by

$$\begin{aligned}
\delta K_{(s)}^{\mu\nu} &= 2(\lambda + \beta X)(\zeta_{\sigma}\nabla^{[\mu}h^{\nu]\sigma} - h^{\sigma[\mu}\nabla_{\sigma}\zeta^{\nu]}) + 2\beta(\delta X)\nabla^{[\mu}\zeta^{\nu]} \\
&\quad + \beta[4\zeta^{[\mu}\nabla^{\nu]}\delta X - 4\zeta^{[\mu}h^{\nu]\sigma}\nabla_{\sigma}X + 2\zeta^{[\mu}\delta(\nabla_{\sigma}\Phi^{\nu]\sigma}) \\
&\quad - 2\zeta_{\sigma}g^{\rho[\mu}\delta(\nabla_{\rho}\Phi^{\nu]\sigma}) - 2h_{\rho\sigma}\zeta^{\rho}\nabla^{[\mu}\Phi^{\nu]\sigma} \\
&\quad + 2\zeta_{\sigma}h^{\rho[\mu}\nabla_{\rho}\Phi^{\nu]\sigma} + 2(\delta\Phi^{\sigma[\mu}\nabla_{\sigma}\zeta^{\nu]}) + \zeta_{\rho}\Phi^{\sigma[\mu}\nabla_{\sigma}h^{\nu]\rho} \\
&\quad + \zeta^{\rho}\Phi_{\sigma}{}^{[\mu}\nabla_{\rho}h^{\nu]\sigma} - \zeta^{\rho}\Phi^{\sigma[\mu}\nabla^{\nu]}h_{\rho\sigma}], \quad (B.8)
\end{aligned}$$

where

$$\begin{aligned}
\delta\Phi^{\mu\nu} &= -2h^{\sigma(\mu}\Phi^{\nu)\sigma} + 2(\nabla^{(\mu}\phi)(\nabla^{\nu)}\delta\phi), \\
\delta(\nabla_{\rho}\Phi^{\nu\sigma}) &= \nabla_{\rho}\delta\Phi^{\nu\sigma} + g^{\alpha(\nu}\Phi^{\sigma)\beta}(2\nabla_{(\rho}h_{\beta)\alpha} - \nabla_{\alpha}h_{\rho\beta}). \quad (B.9)
\end{aligned}$$

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