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n-hypergroups and binary relations

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Abstract

In this paper, we introduce and study the notion of a partial n-hypergroupoid, associated with a binary relation. Some important results concerning Rosenberg partial hypergroupoids, induced by relations, are generalized to the case of n-hypergroupoids. Then, n-hypergroups associated with union, intersection, products of relations and also mutually associative n-hypergroupoids are analyzed. Finally, n-hypergroupoids associated with relations are used in order to study the transitivity of the relation β in n-semihypergroups.

1. Introduction

Since 1934, when Marty introduced for the first time the notion of a hypergroup, the hyperstructure theory had applications to several domains, for instance non-Euclidean geometry, graphs and hypergraphs, binary relations, lattices, automata, cryptography, codes, artificial intelligence, probabilities etc. (see [7]). 70 years later, a nice generalization of a hypergroup, called an n-hypergroup has been introduced and studied by Davvaz and Vougiouklis [9]. Nowadays, hyperstructure theory has arrived at its maturity, being studied extensively in Europe, Asia, North America and Australia. There are thousands of papers and some books written on this topic. Meanwhile, n-hyperstructures are just at the beginning of their study.

In this paper, we study a connection between n-hypergroups and binary relations. This generalizes the first connection between hyperstructures and binary relations, which was considered by Rosenberg [13], in the most general meaning. There are many types of connections between hyperstructures and binary relations. Particular hyperstructures have already been

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associated with binary relations by Corsini (with hypergraphs [2]), and by Nieminen and Rosenberg (with graphs [12,14]); also see [4,8,11]. The Rosenberg hypergroup was studied by Corsini and then by Corsini and Leoreanu, who analyzed hypergroups associated with union, intersection, product, Cartesian product, direct limit of relations. Later, Spartalis [15], De Salvo and Lo Faro [10] have obtained new results on hyperstructures associated with binary relations.

After introducing the n-hypergroupoid induced by a binary relation, we study here some of its properties. Several results are obtained, one for which mutual associativity for n-hyperstructures plays a part. Among the examples, we mention the n-hypergroupoid associated with a hypergraph. Finally, by using the above mentioned n-hypergroupoid, we analyse the transitivity of the relation β in n-semihypergroups, by characterizing the class of n-semihypergroups for which the relation β is transitive. The basic definitions concerning hypergroup theory can be found in [1,6,16].

2. The partial n-hypergroupoid (H, f_R)

Let H be a nonempty set and $f : H^n \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H. Then f is called an *n-ary hyperoperation* on H and the pair (H, f) is called an *n-hypergroupoid*.

If $A_1, A_2, \dots, A_n$ are subsets of H, then we define

$$f(A_1, A_2, \dots, A_n) = \cup\{f(a_1, \dots, a_n) \mid a_i \in A_i, i \in \{1, 2, \dots, n\}\}.$$

In the following, we shall denote the sequence $a_i, a_{i+1}, \dots, a_j$ by a_i^j. For $j < i$, a_i^j is the empty set.

The n-hypergroupoid (H, f) is called an *n-semihypergroup* if for any $i, j \in \{1, 2, \dots, n\}$ and a_1^{2n-1} of H, we have

$$f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1}) = f(a_1^{j-1}, f(a_j^{n+j-1}), a_{n+j}^{2n-1}).$$

An n-semihypergroup (H, f) for which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has the solution $x_i \in H$, for any $a_1^{i-1}, x_i, a_{i+1}^n$ of H and $1 \le i \le n$, is called an *n-hypergroup* [9].

Let R be a binary relation on a nonempty set H. We define a partial n-hypergroupoid (H, f_R) as follows:

$$\forall a \in H, \quad f_R(\underbrace{a, \dots, a}_{n}) = \{y \mid (a, y) \in R\}$$

and $\forall a_1, a_2, \dots, a_n \in H$,

$$f_R(a_1, a_2, \dots, a_n) = f_R(\underbrace{a_1, \dots, a_1}_{n}) \cup f_R(\underbrace{a_2, \dots, a_2}_{n}) \cup \dots \cup f_R(\underbrace{a_n, \dots, a_n}_{n}).$$

Notice that (H, f_R) is commutative, since the value of $f_R(a_1, \dots, a_n)$ does not depend on the permutation of $a_1, \dots, a_n$.

The partial n-hypergroupoid (H, f_R) is a generalization of the Rosenberg partial hypergroupoid H_R (see [13]).

Define $f_R(a_1, a_2, \dots, a_n)$ by $f_R(a_1^n)$.

Theorem 2.1. *The relation R is transitive if and only if for any $a \in H$, we have*

$$f_R(f_R(\underbrace{a,\ldots,a}_{n}), \underbrace{a,\ldots,a}_{n-1}) = f_R(\underbrace{a,\ldots,a}_{n}).$$

Proof. We have $f_R(\underbrace{a,\ldots,a}_{n}) = \{b \mid (a,b) \in R\}$ and

$$\begin{aligned} f_R(f_R(\underbrace{a,\ldots,a}_{n}), \underbrace{a,\ldots,a}_{n-1}) &= \bigcup_{(a,b)\in R} f_R(b, \underbrace{a,\ldots,a}_{n-1}) \\ &= \bigcup_{(a,b)\in R} [f_R(\underbrace{b,\ldots,b}_{n}) \cup f_R(\underbrace{a,\ldots,a}_{n})] \\ &= f_R(\underbrace{a,\ldots,a}_{n}) \cup \bigcup_{(a,b)\in R} f_R(\underbrace{b,\ldots,b}_{n}) \\ &= f_R(\underbrace{a,\ldots,a}_{n}) \cup \bigcup_{(a,b)\in R} \{c \mid (b,c) \in R\}. \end{aligned}$$

Denote $K = \bigcup_{(a,b)\in R}\{c \mid (b,c) \in R\}$. We have

$$K = \{c \mid (a,b) \in R, (b,c) \in R\} = \{c \mid (a,c) \in R^2\}.$$

"$\Longrightarrow$" Suppose R is transitive. Then $K \subset \{c \mid (a,c) \in R\} = f_R(\underbrace{a,\ldots,a}_{n})$, whence we get the thesis.

"$\Longleftarrow$" We have $K \subset f_R(\underbrace{a,\ldots,a}_{n})$, for any $a \in H$, that is

$$\{c \mid (a,c) \in R^2\} \subset \{b \mid (a,b) \in R\},$$

whence $R^2 \subset R$. ■

Remark 2.1. (H, f_R) is an n-hypergroupoid if the domain of R is H.

Recall now what an *outer element* of R is (see [13]). This notion is useful in order to characterize the n-hypergroup (H, f_R).

Definition 2.1. An element $z \in H$ is called an *outer element* of R if there exists $y \in H$ such that $(y,z) \notin R^2$.

Theorem 2.2. *Let R be a binary relation on H, with full domain. The n-hypergroupoid (H, f_R) is an n-semihypergroup if and only if $R \subset R^2$ and for any outer element z of R, the following implication holds:*

$$(x,z) \in R^2 \Longrightarrow (x,z) \in R. \qquad (*)$$

Proof. Recall that (H, f_R) is an n-ary semihypergroup if for any $i, j \in \{1,2,\ldots,n\}$ and $a_1, a_2, \ldots, a_{2n-1} \in H$, we have

$$f_R(a_1^{i-1}, f_R(a_i^{n+i-1}), a_{n+i}^{2n-1}) = f_R(a_1^{j-1}, f_R(a_j^{n+j-1}), a_{n+j}^{2n-1}).$$

Denote by T_1 the left side and by T_2 the right side of the above equality.

We have

$$\begin{aligned} T_1 &= f_R(a_1,\ldots,a_1)\cup\cdots\cup f_R(a_{i-1},\ldots,a_{i-1})\cup f_R(f_R(a_i^{n+i-1}),\ldots,f_R(a_i^{n+i-1}))\\ &\quad \cup f_R(a_{n+i},\ldots,a_{n+i})\cup\cdots\cup f_R(a_{2n-1},\ldots,a_{2n-1})\\ &= \{z \mid (a_1,z)\in R \text{ or } \ldots \text{ or } (a_{i-1},z)\in R \text{ or } (a_{n+i},z)\\ &\quad \in R \text{ or } \ldots \text{ or } (a_{2n-1},z)\in R\}\cup A, \end{aligned}$$

where $A = f_R(f_R(a_i^{n+i-1}),\ldots,f_R(a_i^{n+i-1})) = \{z \mid (y,z)\in R,$ where y is such that $(a_i,y)\in R$ or $\ldots$ or $(a_{n+i-1},y)\in R\} = \{z \mid (a_i,z)\in R^2$ or $\ldots$ or $(a_{n+i-1},z)\in R^2\}$. Hence $T_1 = \{z \mid (a_1,z)\in R$ or $\ldots$ or $(a_{i-1},z)\in R$ or $(a_i,z)\in R^2$ or $\ldots$ or $(a_{n+i-1},z)\in R^2$ or $(a_{n+i},z)\in R$ or $\ldots$ or $(a_{2n-1},z)\in R\}$.

Similarly, we get that $T_2 = \{z \mid (a_1,z)\in R$ or $\ldots$ or $(a_{j-1},z)\in R$ or $(a_j,z)\in R^2$ or $\ldots$ or $(a_{n+j-1},z)\in R^2$ or $(a_{n+j},z)\in R$ or $\ldots$ or $(a_{2n-1},z)\in R\}$.

"$\Longrightarrow$" For any $i,j\in\{1,2,\ldots,n\}$, we have $T_1 = T_2$. Set $i=n$ and $j=1$.

We get $T_1' = \{z \mid (a_1,z)\in R$ or $\ldots$ or $(a_{n-1},z)\in R$ or $\ldots$ or $(a_n,z)\in R^2$ or $\ldots$ or $(a_{2n-1},z)\in R^2\}$ and $T_2' = \{z \mid (a_1,z)\in R^2$ or $\ldots$ or $(a_n,z)\in R^2$ or $\ldots$ or $(a_{n+1},z)\in R$ or $\ldots$ or $(a_{2n-1},z)\in R\}$.

Suppose $R\not\subset R^2$; that means there exists $(a,b)\in R-R^2$.

Now, set $a_1=\cdots=a_{n-1}=b$, $a_n=\cdots=a_{2n-1}=a$ and $z=b$. Then, it follows that $b\in T_2'$ and so we get $b\in T_1'$. On the other hand $(a,b)\notin R^2$; hence $(b,b)\in R$. Since $(a,b)\in R$, we get $(a,b)\in R^2$, a contradiction. Therefore $R\subset R^2$.

Now, let z be an outer element of H and suppose there exists $x\in H$ such that $(x,z)\in R^2-R$.

Set $a_1=\cdots=a_{n-1}=x$ and $a_n=\cdots=a_{2n-1}=y$, where $(y,z)\notin R^2$. Since $(x,z)\in R^2$, we get that $z\in T_2'=T_1'$. On the other hand $(x,z)\notin R$, so we must have $(y,z)\in R^2$, a contradiction.

Therefore, the implication (∗) holds for any outer element of R.

"$\Longleftarrow$" If z is an outer element of R, then the implication (∗) holds and since $R\subset R^2$, we get $(x,z)\in R^2$ if and only if $(x,z)\in R$, whence $z\in T_1$ if and only if $z\in T_2$.

If z is not an outer element of R, that is for any $x\in H$, we have $(x,z)\in R^2$, then $z\in T_1$ and $z\in T_2$.

Hence $T_1=T_2$. ■

Remark 2.2. The n-semihypergroup (H, f_R) is an n-hypergroup if and only if R has a full range.

Therefore, we get:

Theorem 2.3. *Let R be a binary relation with full domain. The n-hypergroupoid (H, f_R) is an n-hypergroup if and only if the following conditions hold:*

1°. *R has a full range;*
2°. *$R\subset R^2$;*
3°. *$(x,z)\in R^2 \Longrightarrow (x,z)\in R$ for any outer element z of R.*

Remark 2.3. If $R\subset R^2$, then x is an outer element of R if and only if $x\notin f_R(f_R(\underbrace{a,\ldots,a}_{n}),\underbrace{a,\ldots,a}_{n-1})$ for some $a\in H$.

Proof. Indeed, we have $f_R(f_R(\underbrace{a,\ldots,a}_{n}),\underbrace{a,\ldots,a}_{n-1}) = \{x \mid (a,x) \in R \cup R^2\} = \{x \mid (a,x) \in R^2\}$, whence we get the thesis. ■

Remark 2.4. If $R \subset R^2$, then there are no outer elements of R if and only if for any $a \in H$, we have

$$f_R(f_R(\underbrace{a,\ldots,a}_{n}),\underbrace{a,\ldots,a}_{n-1}) = H.$$

Examples 2.1. 1°. Let $H = \{1,2,\ldots,m\}$ and $R = I \cup \{(x,1) \mid x \in H\} \cup \{(1,2)\}$ where $I = \{(x,x) \mid x \in H\}$. Then $R^2 = R \cup \{(x,2) \mid x \in H\}$.

Notice that the relation R satisfies all the conditions of Theorem 2.3, so (H, f_R) is an n-hypergroup. We have $f_R(\underbrace{1,\ldots,1}_{n}) = \{1,2\}$ and $f_R(\underbrace{k,\ldots,k}_{n}) = \{1,k\}$, for any $k \in H - \{1\}$.

Hence $f_R(a_1,\ldots,a_n) = \{1,a_1,\ldots,a_n\}$, for any $(a_1,\ldots,a_n)$ such that $1 \notin \{a_1,\ldots,a_n\}$ and $f_R(a_1,\ldots,a_n) = \{1,2,a_1,\ldots,a_n\}$ if $1 \in \{a_1,\ldots,a_n\}$.

2°. Let R be an equivalence relation on a nonempty set H. According to Theorem 2.3, it follows that (H, f_R) is an n-hypergroup. For any $a \in H$, we have that $f_R(\underbrace{a,\ldots,a}_{n})$ is the equivalence class $\hat{a}$ of a.

Theorem 2.4. *Let R be a reflexive and symmetric relation on H. Then (H, f_R) is an n-hypergroup if and only if for any $a, c \in H$, we have*

$$f_R(f_R(\underbrace{c,\ldots,c}_{n}),\underbrace{c,\ldots,c}_{n-1}) - f_R(\underbrace{c,\ldots,c}_{n}) \subset f_R(f_R(\underbrace{a,\ldots,a}_{n}),\underbrace{a,\ldots,a}_{n-1}). \qquad (\tau)$$

Proof. Since $R \subset R^2$, it follows that for any $c \in H$, we have

$$f_R(f_R(\underbrace{c,\ldots,c}_{n}),\underbrace{c,\ldots,c}_{n-1}) = \{y \mid (c,y) \in R^2\}.$$

"$\Longrightarrow$" Suppose the implication (τ) holds, for any $a, c \in H$. ■

Let $y \in f_R(f_R(\underbrace{c,\ldots,c}_{n}),\underbrace{c,\ldots,c}_{n-1}) - f_R(\underbrace{c,\ldots,c}_{n})$, that is $(c,y) \in R^2 - R$.

Since (H, f_R) is an n-hypergroup and according to Theorem 2.3, we deduce that y is not an outer element of R. In other words, for any $a \in H$, we have $(a,y) \in R^2$, that is $y \in f_R(f_R(\underbrace{a,\ldots,a}_{n}),\underbrace{a,\ldots,a}_{n-1})$.

"$\Longleftarrow$" Conversely, it is sufficient to check only the condition (3°) of Theorem 2.3. Let y be an outer element of R, that is there is $a \in H$ such that $(a,y) \notin R^2$. Then, according to (τ), it follows that for any $c \in H$, we have $y \notin f_R(f_R(\underbrace{c,\ldots,c}_{n}),\underbrace{c,\ldots,c}_{n-1}) - f_R(\underbrace{c,\ldots,c}_{n})$. Hence, for any $c \in H$, if $(c,y) \in R^2$, then $(c,y) \in R$, that is the condition (3°) is satisfied.

Remark 2.5. If (H, f_R) is an n-hypergroupoid and R is symmetric, then R^2 is an equivalence.

Indeed, it follows from Theorem 2.3, exactly as for hypergroups induced by relations (see [5]).

Remark 2.6. If (H, f_R) is an n-hypergroup and R is reflexive, symmetric, but it is not transitive, then $R^2 = H^2$.

Indeed, if R is not transitive, then $R \neq R^2$ and so, there exists $c, y \in H$ such that $(c, y) \in R^2 - R$. According to the above theorem, it follows that $(a, y) \in R^2$, for any $a \in H$; that means any two elements of H are in the relation R^2. In other words, $R^2 = H^2$.

Corollary 2.1. *Let R be a reflexive, symmetric, but not transitive relation on H. Then (H, f_R) is an n-hypergroup if and only if $R^2 = H^2$.*

Example 2.2. Let R_Γ be the binary relation associated with a hypergraph $\Gamma = (H, \{A_i\}_{i\in I})$, i.e. for any $i \in I$, $\emptyset \neq A_i \subset H$ and $\bigcup_{i\in I} A_i = H$.

In other words,

$$a\, R_\Gamma\, b \Longleftrightarrow \{a, b\} \subset A_i, \quad \text{for some } i \in I.$$

Clearly, the relation R_Γ is reflexive and symmetric.

For any $a \in H$, we have

$$f_{R_\Gamma}(\underbrace{a, \ldots, a}_{n}) = \{y \mid (a, y) \in R_\Gamma\} = \bigcup_{a\in A_i} A_i.$$

Hence, we get a natural generalization of the hypergroupoid induced by a hypergraph, considered by Corsini in [2].

By the above theorem, (H, f_{R_Γ}) is an n-hypergroup if and only if the corresponding inclusion (τ) holds. Moreover, according to the above corollary, if R is not transitive, then (H, f_{R_Γ}) is an n-hypergroup if and only if we have $R_\Gamma^2 = H^2$; that means for any $a, c \in H$, there is $b \in H$ such that $\{a, c\} \subset \bigcup_{b\in A_i} A_i$.

In the same way as for binary hyperoperations, associated with binary relations (see [7]), from Theorem 2.3, there follows the next result, that we present here without proof.

Theorem 2.5. *Let R and S be binary relations on H.*

1°. *If (H, f_R) is an n-hypergroup, then for any $k \in \mathbb{N}^*$, (H, f_{R^k}) is an n-hypergroup, too.*
2°. *If (H, f_R) is an n-hypergroup and $R \subset S \subset S^2 \subset C_t(R)$, where $C_t(R)$ is the transitive closure of R, then (H, f_S) is an n-hypergroup, too.*
3°. *If (H, f_R) is an n-hypergroup and $S \subset S^2 \subset C_t(R)$, then for any $s, k \in \mathbb{N}^*$, $(H, f_{R^s\cup S^k})$ is an n-hypergroup, too.*
4°. *If $(H, f_{R\cap S})$ is an n-hypergroup and $R \subset R^2 \subset C_t(R\cap S)$, then (H, f_R) is an n-hypergroup, too.*
5°. *If (H, f_R) is an n-hypergroup and $R \subset S \subset C_t(R)$, then for any $k_1, k_2 \in \mathbb{N}^*$, $(H, f_{R^{k_1}S^{k_2}})$ and $(H, f_{S^{k_2}R^{k_1}})$ are n-hypergroups, too.*
6°. *If $(H, f_{R\cup S})$ is an n-hypergroup, R, S are reflexive and $S \subset C_t(R)$, then for any $k_1, k_2 \in \mathbb{N}^*$, $(H, f_{R^{k_1}S^{k_2}})$ and $(H, f_{S^{k_2}R^{k_1}})$ are n-hypergroups, too.*

3. Mutually associative (H, f_R) n-hypergroups

We generalize here the concept of mutually associative hypergroupoids, introduced by Corsini [3].

Definition 3.1. We say that two partial n-hypergroupoids (H, f_1) and (H, f_2) are *mutually associative* (m.a.) if for any $a_1, \ldots, a_{2n-1} \in H$, the following equalities hold:

(i$_1$) $f_2(f_1(a_1^n), a_{n+1}^{2n-1}) = f_1(a_1^{n-1}, f_2(a_n^{2n-1}))$;
(i$_2$) $f_2(a_1, f_1(a_2^{n+1}), a_{n+2}^{2n-1}) = f_1(a_1^{n-2}, f_2(a_{n-1}^{2n-2}), a_{2n-1})$;
(i$_3$) $f_2(a_1, a_2, f_1(a_3^{n+2}), a_{n+3}^{2n-1}) = f_1(a_1^{n-3}, f_2(a_{n-2}^{2n-3}), a_{2n-2}, a_{2n-1})$;
...
(i$_{n-1}$) $f_2(a_1^{n-2}, f_1(a_{n-1}^{2n-2}), a_{2n-1}) = f_1(a_1, f_2(a_2^{n+1}), a_{n+2}^{2n-1})$;
(i$_n$) $f_2(a_1^{n-1}, f_1(a_n^{2n-1})) = f_1(f_2(a_1^n), a_{n+1}^{2n-1})$.

Notice that if f_1 and f_2 are binary hyperoperations, we get two mutually associative partial hypergroupoids.

Now, if R is a binary relation on H and $A \subset H$, we define

$$R(A) = \{b \mid (a, b) \in R, \text{ for some } a \in A\}.$$

If $A = \{a_1, a_2, \ldots, a_k\}$, we write $R(a_1^k)$ for $R(A)$.

If R and S are binary relations on H, then we denote by SR the relation $\{(a, c) \in H^2 \mid (a, b) \in R \text{ and } (b, c) \in S, \text{ for some } b \in H\}$.

Theorem 3.1. *Let R and S be relations of H with full domain. Then the n-hypergroupoids (H, f_R) and (H, f_S) are mutually associative if and only if for any $a_1, a_2, \ldots, a_{2n-1} \in H$, we have*

$$SR(a_1^n) \cup S(a_{n+1}^{2n-1}) = RS(a_n^{2n-1}) \cup R(a_1^{n-1}). \quad (*)$$

Proof. We have $f_S(f_R(a_1^n), a_{n+1}^{2n-1}) = \{c \mid (b, c) \in S \text{ and } [(a_1, b) \in R \text{ or } \ldots \text{ or } (a_n, b) \in R]\} \cup \{c \mid (a_{n+1}, c) \in S \text{ or } \ldots \text{ or } (a_{2n-1}, c) \in S\} = SR(a_1^n) \cup S(a_{n+1}^{2n-1})$ and $f_R(a_1^{n-1}, f_S(a_n^{2n-1})) = \{c \mid (b, c) \in R \text{ and } [(a_n, b) \in S \text{ or } \ldots \text{ or } (a_{2n-1}, b) \in S]\} \cup \{c \mid (a_1, c) \in R \text{ or } \ldots \text{ or } (a_{n-1}, c) \in R\} = RS(a_n^{2n-1}) \cup R(a_1^{n-1})$. Hence the equality $(*)$ coincides with the equality (i$_1$).

Since both (H, f_R) and (H, f_S) are commutative, the equalities (i$_2$), ..., (i$_n$) coincide with the equality (i$_1$). ■

Theorem 3.2. *If R and S are relations on H such that (H, f_R) and (H, f_S) are mutually associative n-hypergroups, then $(H, f_{R\cup S})$ is an n-hypergroup, too.*

Proof. It is sufficient to check the condition (3°) of Theorem 2.3. The other conditions are clearly satisfied.

Let z be an outer element of $R \cup S$, that is $(y, z) \notin (R \cup S)^2 = R^2 \cup S^2 \cup RS \cup SR$ for some $y \in H$. Hence z is an outer element for both R and S. Let $(x, z) \in (R \cup S)^2$. If $(x, z) \in R^2 \cup S^2$, then $(x, z) \in R \cup S$, since (H, f_R) and (H, f_S) are n-hypergroups.

If $(x, z) \in SR$, then we consider $a_1 = a_2 = \cdots = a_{n-1} = x$ and $a_n = \cdots = a_{2n-1} = y$ in the equality $(*)$. We get $SR(x, y) \cup S(y) = RS(y) \cup R(x)$. We have $z \in SR(x)$, whence $z \in RS(y) \cup R(x)$.

Since $(y, z) \notin RS$, it follows that $(x, z) \in R \subset R \cup S$. Similarly, if $(x, z) \in RS$ we get $(x, z) \in R \cup S$; that means the condition (3°) of Theorem 2.3 holds. ■

Theorem 3.3. *Let R and S be relations on H, such that $R \subset SR$. If (H, f_R) is an n-hypergroup, (H, f_R) and (H, f_S) are mutually associative and one of the following two conditions is satisfied:*

(i) $RS \cap \{(x, x) \mid x \in H\} = \emptyset$;
(ii) *the domain $\mathbb{D}(RS)$ of RS is different from H,*
then (H, f_{SR}) is an n-hypergroup, too.

Proof. Also for this proof, it is sufficient to check the condition (3°) of Theorem 2.3.

Let z be an outer element of SR. Then $(y, z) \notin (SR)^2$ for some $y \in H$. Since $R \subset SR$, it follows that z is an outer element of R, too.

Now, let $(x, z) \in (SR)^2$. Then there is $t \in H$ such that $(x, t) \in SR \ni (t, z)$.

Let us analyze the two situations:

(i) We have $t \in f_S(f_R(\underbrace{x, \ldots, x}_{n-1}, t), \underbrace{t, \ldots, t}_{n-1}) = \{c \mid (t, c) \in S\} \cup \{c \mid (u, c) \in S$ and $[(x, u) \in R$ or $(t, u) \in R]$ for some $u \in H\} = \{c \mid (t, c) \in S$ or $(x, c) \in SR$ or $(t, c) \in SR\}$. Since (H, f_R) and (H, f_S) are mutually associative, it follows that $t \in (f_R(\underbrace{x, \ldots, x}_{n-1}), f_S(\underbrace{t, \ldots, t}_{n})) = \{c \mid (x, c) \in R\} \cup \{c \mid (v, c) \in R$ and $(t, v) \in S$, for some $v \in H\} = \{c \mid (x, c) \in R$ or $(t, c) \in RS\}$ and according to the hypothesis, we get $(x, t) \in R$.

On the other hand, $z \in f_S(f_R(\underbrace{t, \ldots, t}_{n-1}, z), \underbrace{z, \ldots, z}_{n-1}) = \{c \mid (z, c) \in S$ or $(t, c) \in SR$ or $(z, c) \in SR\}$.

Again, since (H, f_R) and (H, f_S) are mutually associative, it follows that $z \in f_R(\underbrace{t, \ldots, t}_{n-1}, f_S(\underbrace{z, \ldots, z}_{n})) = \{c \mid (t, c) \in R$ or $(z, c) \in RS\}$.

According to the hypothesis, it follows that $(t, z) \in R$.

(ii) Let $h \in H - \mathbb{D}(RS)$. We have $t \in f_S(f_R(\underbrace{x, \ldots, x}_{n-1}, h), \underbrace{h, \ldots, h}_{n-1}) = \{c \mid (h, c) \in S$ or $(x, c) \in SR$ or $(h, c) \in SR\}$. Since (H, f_R) and (H, f_S) are mutually associative, it follows that $t \in f_R(\underbrace{x, \ldots, x}_{n-1}, f_S(\underbrace{h, \ldots, h}_{n})) = \{c \mid (x, c) \in R$ or $(h, c) \in RS\}$. According to the hypothesis, we get $(x, t) \in R$.

On the other hand, $z \in f_S(f_R(\underbrace{t, \ldots, t}_{n-1}, h), \underbrace{h, \ldots, h}_{n-1}) = \{c \mid (h, c) \in S$ or $(t, c) \in SR$ or $(h, c) \in SR\}$. Since (H, f_R) and (H, f_S) are mutually associative, we get $z \in f_R(\underbrace{t, \ldots, t}_{n-1}, f_S(\underbrace{h, \ldots, h}_{n})) = \{c \mid (t, c) \in R$ or $(h, c) \in RS\}$.

According to the hypothesis, we get $(t, z) \in R$.

In both situations (i) and (ii), we get $(x, z) \in R^2$, whence $(x, z) \in R \subset SR$, since z is an outer element of R.

Therefore, the condition (3°) of Theorem 2.3 is satisfied and we get that (H, f_{SR}) is an n-hypergroup. ■

4. The transitivity of the relation β in n-semihypergroups

Let (H, f) be an n-semihypergroup. We define

$$\begin{aligned} f_{(1)} &= \{f(a_1^n) \mid a_i \in H, \forall i \in \{1,2,\dots,n\}\},\\ f_{(2)} &= \{f(f(x_1^n), a_2^n) \mid x_i \in H, a_j \in H, \forall i \in \{1,\dots,n\}, \forall j \in \{2,\dots,n\}\},\\ f_{(3)} &= \{f(f(f(z_1^n), x_2^n), a_2^n) \mid z_i \in H, x_j \in H, a_j \in H,\\ &\quad \forall i \in \{1,2,\dots,n\},\ \forall j \in \{2,\dots,n\}\}, \end{aligned}$$

and so on. Define $\mathcal{U} = \bigcup_{k\in\mathbb{N}^*} f_{(k)}$. Now, we can define the relation β, which is an important binary relation on an n-semihypergroup (H, f). We have $\beta = \bigcup\limits_{k\geq 1} \beta_k$ and for x, y of H, we define

$$a\beta_k y \Leftrightarrow \exists u \in f_{(k)}, \quad \text{such that } \{x, y\} \subseteq u.$$

Clearly, β is reflexive and symmetric. Let β^* be the transitive closure of β. Then β^* is the smallest equivalence relation such that the quotient $(H/\beta^*, f\,|_{\beta^*})$ is an n-semigroup, where H/β^* is the quotient set and $f\,|_{\beta^*}(\beta^*(a_1),\dots,\beta^*(a_n)) = \beta^*(a)$, for any $a \in f(a_1,\dots,a_n)$.

In this paragraph, we shall prove that if (H, f) is an n-hypergroup, then β is transitive. Moreover, we find necessary and sufficient conditions such that β is transitive in an n-semihypergroup.

For any $a \in H$, we have

$$f_\beta(a,a,\dots,a) = \{y \mid (a,y) \in \beta\} = \{y \mid \exists u \in \mathcal{U}, \text{ such that } \{a,y\} \subset u\} = \bigcup_{\substack{a\in u\\ u\in\mathcal{U}}} u.$$

Define $\bigcup_{\substack{a\in u\\ u\in\mathcal{U}}} u$ by $\mathcal{C}_1(a)$; that means

$$\mathcal{C}_1(a) = \{x \mid \exists u \in \mathcal{U} : a \in u, x \in u\}.$$

For any $n \in \mathbb{N}^*$, define

$$\mathcal{C}_{n+1}(a) = \{x \mid \exists u \in \mathcal{U} : \mathcal{C}_n(a) \cap u \neq \emptyset, x \in u\}.$$

Definition 4.1. We say that A is a *complete part* of (H, f) if the following implication holds, for any $u \in \mathcal{U}$:

$$A \cap u = \emptyset \Longrightarrow u \subset A.$$

Let $\mathcal{C}(a)$ be the complete closure of a. Like for semihypergroups, we get that

Remark 4.1. $\mathcal{C}(a) = \bigcup_{i\in\mathbb{N}^*} \mathcal{C}_i(a)$, for any $a \in H$.

Theorem 4.1. *Let (H, f) be an n-semihypergroup. The relation β is transitive if and only if $\mathcal{C}(a) = \mathcal{C}_1(a)$, for any $a \in H$.*

Proof. According to Theorem 2.1, β is transitive if and only if

$$f_\beta(f_\beta(\underbrace{a,\dots,a}_{n}), \underbrace{a,\dots,a}_{n-1}) = f_\beta(\underbrace{a,\dots,a}_{n}),$$

since $\beta \subset \beta^2$. We have

$$f_\beta(f_\beta(\underbrace{a, \ldots, a}_{n}), \underbrace{a, \ldots, a}_{n-1}) = f_\beta(\underbrace{a, \ldots, a}_{n}) \cup K,$$

where $K = \{c \mid \exists b : (b, c) \in \beta, (a, b) \in \beta\}$.

We have

$$\begin{aligned} K &= \{c \mid \exists b \in H, \exists u, v \in \mathcal{U} : \{b, c\} \subset u, \{a, b\} \subset v\} \\ &= \{c \mid \exists u, v \in \mathcal{U} : u \cap v \neq \emptyset, a \in v, c \in u\} \\ &= \left\{ c \mid \exists u \in \mathcal{U} : \bigcup_{\substack{a \in v \\ v \in \mathcal{U}}} v \cap u \neq \emptyset, c \in u \right\} \\ &= \{c \mid \exists u \in \mathcal{U} : \mathcal{C}_1(a) \cap u \neq \emptyset, c \in u\} = \mathcal{C}_2(a). \end{aligned}$$

Hence $f_\beta(f_\beta(\underbrace{a, \ldots, a}_{n}), \underbrace{a, \ldots, a}_{n-1}) = \mathcal{C}_1(a) \cup \mathcal{C}_2(a)$. Therefore, β is transitive if and only if $\mathcal{C}_1(a) \cup \mathcal{C}_2(a) = \mathcal{C}_1(a)$, that is $\mathcal{C}_2(a) \subset \mathcal{C}_1(a)$. Then $\forall n \in \mathbb{N}^*$, we have $\mathcal{C}_{n+1} \subset \mathcal{C}_n(a)$. Indeed, if we suppose $\mathcal{C}_k(a) \subset \mathcal{C}_{k-1}(a)$, where $k > 2$, then $\mathcal{C}_{k+1} = \{c \mid \exists u \in \mathcal{U} : \mathcal{C}_k(a) \cap u \neq \emptyset, c \in u\} \subset \{c \mid \exists u \in \mathcal{U} : \mathcal{C}_{k-1}(a) \cap u \neq \emptyset, c \in u\} = \mathcal{C}_k(a)$. Hence β is transitive if and only if $\mathcal{C}(a) = \mathcal{C}_1(a)$, for any $a \in H$. ■

Theorem 4.2. *If (H, f) is an n-hypergroup, then β is transitive.*

Proof. We check that $\mathcal{C}_2(a) \subset \mathcal{C}_1(a)$, that is if $\mathcal{C}_1(a) \cap u \neq \emptyset$, then $u \subset \mathcal{C}_1(a)$, since $\mathcal{C}_2(a) = \{x \mid \exists u \in \mathcal{U} : \mathcal{C}_1(a) \cap u \neq \emptyset,\ x \in u\}$.

In other words, let us see that $\mathcal{C}_1(a)$ is a complete part of (H, f).

Let $z \in \mathcal{C}_1(a) \cap u$, whence there exists $v \in \mathcal{U}$ such that $z \in v \cap u$ and $a \in v$. Suppose $u \in f_{(k)}$, more exactly

$$u = (f(\ldots f(f(y_1^n), z_2^n), \ldots), \ell_2^n).$$

There are $\alpha, \beta \in H$ such that

$$y_1 \in f(a, t_2^{n-1}, \alpha) \quad \text{and} \quad a \in f(\beta, s_2^{n-1}, z).$$

We have

$$\begin{aligned} u &\subset (f(\ldots f(f(f(a, t_2^{n-1}, \alpha), y_2^n), z_2^n), \ldots), \ell_2^n) \\ &\subset (f(\ldots f(f(f(f(\beta, s_2^{n-1}, z), t_2^{n-1}, \alpha), y_2^n), z_2^n), \ldots), \ell_2^n). \end{aligned}$$

Since $z \in v$, we get $u \subset w$, where

$$w = (f(\ldots f(f(f(f(\beta, s_2^{n-1}, v), t_2^{n-1}, \alpha), y_2^n), z_2^n), \ldots), \ell_2^n).$$

We have $w \in \mathcal{U}$.

On the other hand, $a \in v$, so

$$w \supset (f(\ldots f(f(f(f(\beta, s_2^{n-1}, a), t_2^{n-1}, \alpha), y_2^n), z_2^n), \ldots), \ell_2^n)$$

and since

$$f(f(\beta, s_2^{n-1}, a), t_2^{n-1}, \alpha) = f(\beta, s_2^{n-1}, f(a, t_2^{n-1}, \alpha)) \supset f(\beta, s_2^{n-1}, y_1),$$

it follows that

$$w \supset (f(\ldots f(f(f(\beta, s_2^{n-1}, y_1), y_2^n), z_2^n), \ldots), \ell_2^n).$$

Now, we have

$$\begin{aligned} f(f(f(\beta, s_2^{n-1}, y_1), y_2^n), z_2^n) &= f(f(\beta, s_2^{n-1}, f(y_1^n)), z_2^n) \\ &= f(\beta, s_2^{n-1}, f(f(y_1^n), z_2^n)), \end{aligned}$$

whence we get

$$\begin{aligned} w &\supset f(\beta, s_2^{n-1}, f(f(\ldots f(f(y_1^n), z_2^n), \ldots), \ell_2^n)) \\ &= f(\beta, s_2^{n-1}, u) \supset f(\beta, s_2^{n-1}, z) \ni a. \end{aligned}$$

Hence we have $u \in w$, $w \in \mathcal{U}$ and $a \in w$; that means $u \in \bigcup_{\substack{u' \in \mathcal{U} \\ a \in u'}} u' = \mathcal{C}_1(a)$.

Therefore $\mathcal{C}_2(a) \subset \mathcal{C}_1(a)$, whence we get $\mathcal{C}(a) = \mathcal{C}_1(a)$ and, according to the above theorem, we get that β is transitive. ■

Example 4.1. Example of an n-semihypergroup, for which β is not transitive.

Let $|H| \geq 4$ and $f : H^3 \longrightarrow \mathcal{P}^*(H)$ (where $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H), defined as follows:

$$\begin{aligned} &f(x_0, x_0, x_0) = H - \{x_0, x_1\} \\ &f(x, y, z) = H - \{x_0, x_2\}, \quad \forall (x, y, z) \neq (x_0, x_0, x_0) \end{aligned}$$

and $x_0 \neq x_1 \neq x_2 \neq x_0$.

(H, f) is an n-semihypergroup, since $\forall \alpha, \beta, \gamma, \delta, \mu \in H$, we have

$$\begin{aligned} f(\alpha, \beta, f(\gamma, \delta, \mu)) &= f(\alpha, f(\beta, \gamma, \delta), \mu) \\ &= f(f(\alpha, \beta, \gamma), \delta, \mu) = H - \{x_0, x_2\}, \end{aligned}$$

because $x_0 \notin f(\alpha, \beta, \gamma)$.

(H, f) is not an n-hypergroup, since

$$x_0 \notin f(\alpha, \beta, \gamma) \quad \text{for any } \alpha, \beta, \gamma \in H.$$

For any $x_3 \in H - \{x_0, x_1, x_2\}$, we have $x_3 \, \beta \, x_2$ and $x_3 \, \beta \, x_1$ so $x_1 \, \beta^* \, x_2$, but $x_1 \not\beta \, x_2$.

By β^* we have denoted the transitive closure of β.

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References

[1] P. Corsini, Aviani (Eds.), Prolegomena of Hypergroup Theory, second ed., 1993.
[2] P. Corsini, Hypergraphs and hypergroups, Algebra Universalis 35 (1996) 548–555.
[3] P. Corsini, Mutually associative hypergroupoids, Algebraic Hyperstructures and Applications, Prague, 1996, Democritus Univ. Thrace, Alexandroupolis, 1997, pp. 25–33.
[4] P. Corsini, Binary relations and hypergroupoids, Ital. J. Pure Appl. Math. 7 (2000) 11–18.
[5] P. Corsini, On the hypergroups associated with a binary relation, Mult. Valued Log. 5 (2000) 407–419.
[6] P. Corsini, V. Leoreanu, Applications of hyperstructure theory, in: Advanced in Mathematics, Kluwer Academic Publisher, 2003.

[7] P. Corsini, V. Leoreanu, Hypergroups and binary relations, Algebra Universalis 43 (2000) 321–330.

[8] B. Davvaz, M. Karimian, On the γ_n^*-complete hypergroups, European J. Combin. 28 (2007) 86–93.

[9] B. Davvaz, T. Vougiouklis, n-ary hypergroups, Iran. J. Sci. Technol. 30 (A2) (2006) 165–174.

[10] M. De Salvo, G. Lo Faro, Hypergroups and binary relations, Mult. Valued Log. 8 (2002) 645–657.

[11] M. Karimian, B. Davvaz, On the γ-cyclic hypergroups, Commun. Algebra 34 (12) (2006) 4579–4589.

[12] J. Nieminen, Join Space graphs, J. Geom. 33 (1988) 99–103.

[13] I.G. Rosenberg, Hypergroups and join spaces determined by relations, Ital. J. Pure Appl. Math. (4) (1998) 93–101.

[14] I.G. Rosenberg, Hypergroupes induced by paths of a direct graph, Ital. J. Pure Appl. Math. 4 (1998) 133–142.

[15] S. Spartalis, C. Mamaloukas, On the hyperstructures associated with binary relations, Comput. Math. Appl. 51 (2006) 41–50.

[16] T. Vougiouklis, Hyperstructures and their Representations, Hadronic Press, Inc., Palm Harber, USA, 1994, pp. 115.