Two sharp upper bounds for the Laplacian eigenvalues

Xiao-Dong Zhang

Department of Mathematics, Shanghai Jiao Tong University, 1954 Huashan road, Shanghai 200030,
PR China

Received 18 November 2002; accepted 27 June 2003

Submitted by R.A. Brualdi

Abstract

In this paper, we first obtain a sharp upper bound for the eigenvalues of the adjacency matrix of the line graph of a graph. Then this result is used to present a sharp upper bound for the Laplacian eigenvalues. Another sharp upper bound is presented also. Moreover, we determine all extreme graphs which attain these upper bounds. In last, two examples illustrate that our results are, in some sense, best.

© 2003 Elsevier Inc. All rights reserved.

AMS classification: 05C50; 15A42

Keywords: Laplacian eigenvalue; Eigenvalue of graph; Line graph

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. Denote by $d(v_i)$ the degree of vertex $v_i$. If $D(G) = \text{diag}(d_u, u \in V)$ is the diagonal matrix of vertex degrees of $G$ and $A(G)$ is the $(0, 1)$ adjacency matrix of $G$, then the matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of a graph $G$. Moreover the eigenvalues of $L(G)$ are called Laplacian eigenvalues of $G$. The largest eigenvalue of $L(G)$ is denoted by $\lambda(G)$. Moreover, if $X$ is a symmetric...
matrix, the largest eigenvalue of \( X \) is denoted by \( \mu(X) \). The line graph \( H \) of a graph \( G \) is defined by \( V(H) = E(G) \), where any two vertices in \( H \) are adjacent if and only if they are adjacent as edges of \( G \). Many researchers have investigated upper bounds for \( \lambda(G) \). Let us recall some known results.

In 1985, Anderson and Morley [1] showed that
\[
\lambda(G) \leq \max\{d(u) + d(v) | (u, v) \in E(G)\}. \tag{1}
\]

\[
\lambda(G) \leq 2 + \sqrt{(r - 2)(s - 2)}, \tag{2}
\]
where \( r = \max\{d(u) + d(v) | (u, v) \in E(G)\} \) and \( s = \max\{d(u) + d(v) | (u, v) \in E(G) - (x, y)\} \) with \((x, y) \in E(G)\) such that \( d(x) + d(y) = r \).

\[
\lambda(G) \leq \max\{d(v) + m(v) | v \in V(G)\}, \tag{3}
\]
where \( m(v) \) is the average of the degrees of the vertices adjacent to \( v \), which is called average 2-degree of vertex \( v \).

In 2000, Rojo et al. [10] obtained an always nontrivial bound as follows:
\[
\lambda(G) \leq \max\{d(u) + d(v) - |N(u) \cap N(v)| | u \neq v\}, \tag{4}
\]
where \( N(u) \) is the set of neighbor of \( u \).

In 2001, Li and Pan [6] showed that
\[
\lambda(G) \leq \max \{\sqrt{2d(u)(d(v) + m(v))} | v \in V(G)\}. \tag{5}
\]

In 2002, Shu et al. [11] gave an upper bound in terms of degree sequences. Assume that the degree sequence of \( G \) is \( d_1 \geq d_2 \geq \cdots \geq d_n \). Then
\[
\lambda(G) \leq d_n + \frac{1}{2} + \sqrt{\left( d_n - \frac{1}{2} \right)^2 + \sum_{i=1}^{n} d_i (d_i - d_n)}. \tag{6}
\]

In this paper, we present two sharp bounds in terms of degree sequence and average 2-degree as follows:

**Theorem 1.1.** Let \( G \) be a simple connected graph. Then
\[
\lambda(G) \leq \max \left\{ 2 + \sqrt{d(u)(d(u) + m(u) - 4) + d(v)(d(v) + m(v) - 4) + 4} \right\}, \tag{7}
\]
where the maximum is taken over all pairs \((u, v) \in E(G)\). Moreover, equality in (7) holds if and only if \( G \) is bipartite regular or semi-regular (i.e., there exists a bipartite partition of \( G \) such that the degrees of all vertices in each partitions are the same), or a path of order 4.
Theorem 1.2. Let $G$ be a simple connected graph. Then
\[ \lambda(G) \leq \max \left\{ d(v) + \sqrt{d(v)m(v)} \mid v \in V(G) \right\}, \tag{8} \]
with equality if and only if $G$ is bipartite regular or semi-regular.

The terminology and notations not defined may be found in [2,4,5]. The rest of this paper is organized as follows. In Section 2, we present a sharp upper bound for the largest eigenvalue of the adjacency matrix of the line graph of a graph in terms of degree and average 2-degree. This result is used, in Section 3, to provide proofs of Theorems 1.1 and 1.2. In Section 4, we compare our bounds with known bounds and two examples illustrate that Theorems 1.1 and 1.2 are, in some sense, best.

2. Eigenvalues of line graphs

In order to obtain a sharp upper bound for the largest eigenvalue of the adjacency matrix of the line graph of a graph, we need some lemmas.

Lemma 2.1. Let $G$ be a simple connected graph. Then (i) The line graph $H$ of $G$ is regular if and only if $G$ is regular or semi-regular. (ii) The line graph $H$ of $G$ is semi-regular and not regular if and only if $G$ is a path of order 4.

Proof. It follows from Lemmas 3.4 and 3.5 in [12].

A matrix $X$ is row-regular (column-regular) if all row (column) sums $r_i(X)$ are equal. For each $i$, let $R_i(X) = \sum_{j=1}^{n} x_{ij} r_j$ and $R(X) = \max \{ R_i(X) \mid 1 \leq i \leq n \}$. Cao in [3] obtained the following result.

Lemma 2.2. Let $X$ be an $n \times n$ nonnegative irreducible symmetric matrix. Then the largest eigenvalue $\mu(X)$ of $X$ satisfies
\[ \mu(X) \leq \sqrt{R(X)} \tag{9} \]
with equality if and only if $X$ is row-regular or there exists a permutation matrix $P$ such that $PAP^T$ is in the form \( \begin{pmatrix} 0 & Y \\ Y^T & 0 \end{pmatrix} \), where $Y$ is row-regular and column-regular.

We are ready to present a sharp bound for the largest eigenvalue of the adjacency matrix of the line graph of a graph $G$.

Theorem 2.3. Let $G$ be a simple connected graph and $H$ be the line graph of $G$. Then the largest eigenvalue of the adjacency matrix $A(H)$ of $H$ satisfies
\[ \mu(A(H)) \leq \max \left\{ \sqrt{d(u)(d(u) + m(u) - 4)} + d(v)(d(v) + m(v) - 4) + 4 \right\}. \tag{10} \]
where the maximum is taken over all pair \((u, v) \in E(G)\). Moreover, equality in (10) holds if and only if \(G\) is regular, or semi-regular, or a path of order 4.

**Proof.** It is easy to see that the \(e_{uv}\)th row sum in \(A(H) = (a_{e_{pq},e_{uv}})\) is equal to \(r(e_{uv}) = d(u) + d(v) - 2\), where \(e_{uv} = (u, v) \in E(G)\) is an edge of \(G\). Then

\[
R_{uv}(A(H)) = \sum_{(e_{uv}, e_{pq}) \in E(H)} a_{e_{uv}, e_{pq}} r(e_{pq})
\]

\[
= \sum_{(u,q) \in E(G)} r(e_{uq}) + \sum_{(p,v) \in E(G)} r(e_{pv}) - 2r(e_{uv})
\]

\[
= d(u)(d(u) + m(u) - 4) + d(v)(d(v) + m(v) - 4) + 4.
\]

Hence it follows from Lemma 2.2 that (10) holds. Moreover, equality in (10) holds if and only if \(H\) is regular or semi-regular by Lemma 2.2. Therefore, by Lemma 2.1, equality in (10) holds if and only if \(G\) is regular, or semi-regular, or a path of order 4. □

**Remark 1.** If we apply Lemma 2.2 to the nonnegative irreducible symmetric matrix \(A(H) + 2I\), where \(I\) is the identity matrix of order \(|E(G)|\), it is easy to obtain the following result by a similar argument.

**Theorem 2.4.** Let \(G\) be a simple connected graph and \(H\) be the line graph of \(G\). Then the largest eigenvalue of the adjacency matrix \(A(H)\) of \(H\) satisfies

\[
\mu(A(H)) \leq \max \{ \sqrt{d(u)(d(u) + m(u))} + d(v)(d(v) + m(v)) - 2 \},
\]

(11)

where the maximum is taken over all pair \((u, v) \in E(G)\). Moreover, equality in (11) holds if and only if \(G\) is regular, or semi-regular.

3. Laplacian eigenvalues of a graph

In order to obtain sharp upper bounds for the Laplacian eigenvalues, we need the following lemma from [11].

**Lemma 3.1.** Let \(G\) be a simple connected graph and \(\mu(A(H))\) be the largest eigenvalue of the adjacency matrix \(A(H)\) of \(H\). Then

\[
\lambda(G) \leq 2 + \mu(A(H)),
\]

(12)

with equality holding if and only if \(G\) is bipartite.

Now we are ready to provide proofs of Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** It follows directly from Theorem 2.3 and Lemma 3.1. □
Similarly, we may get another bound by Theorem 2.4 and Lemma 3.1.

**Theorem 3.2.** Let $G$ be a simple graph of order $n$. Then

$$\lambda(G) \leq \max \left\{ \sqrt{d(u)(d(u) + m(u))} + d(v)(d(v) + m(v)) \right\},$$

(13)

where the maximum is taken over all pair $(u, v) \in E(G)$. Moreover equality in (13) holds if and only if $G$ is bipartite regular or semi-regular.

**Proof.** It follows from Theorem 2.4 and Lemma 3.1. □

**Proof of Theorem 1.2.** Let $x = (x(v), v \in V(G))^T$ be an eigenvector with $||x||^2 = 1$ corresponding to $\lambda(G)$. Thus $L(G)x = \lambda(G)x$. Hence for any $u \in V(G)$, $\lambda(G)x_u = d(u)x_u = \sum_{v \in V(G)} a_{uv}x_v = \sum_{(u,v) \in E(G)} (x_u - x_v).$ By the Cauchy–Schwarz inequality, we have

$$\lambda(G)^2 x_u^2 \leq \left( \sum_{(u,v) \in E(G)} 1^2 \right) \left( \sum_{(u,v) \in E(G)} (x_u - x_v)^2 \right)$$

$$\quad = d(u)^2 x_u^2 + 2d(u)x_u^2(\lambda - d(u)) + d(u) \sum_{(u,v) \in E(G)} x_v^2.$$

Hence

$$\sum_{u \in V(G)} \lambda(G)^2 x_u^2 \leq \sum_{u \in V(G)} (2d(u)\lambda - d(u)^2)x_u^2 + \sum_{u \in V(G)} d(u) \sum_{(u,v) \in E(G)} x_v^2$$

$$\quad = \sum_{u \in V(G)} (2d(u)\lambda - d(u)^2)x_u^2 + \sum_{u \in V(G)} d(u)m(u)x_u^2.$$

Therefore, we have

$$\sum_{u \in V(G)} (\lambda(G)^2 - 2d(u)\lambda(G) + d(u)^2 - d(u)m(u))x_u^2 \leq 0.$$

Then there must exist a vertex $u$ such that

$$\lambda(G)^2 - 2d(u)\lambda(G) + d(u)^2 - d(u)m(u) \leq 0,$$

which implies $\lambda(G) \leq d(u) + \sqrt{d(u)m(u)}$. it follows that (8) holds.

If $G$ is bipartite regular or semi-regular, it is easy to see that equality in (8) holds by a simply calculation.

Conversely, if equality in (8) holds, it follows from the above proof that for each $u \in V(G)$, $(u, v) \in E(G)$, $(u, v) \in E(G)$, we have $x_u - x_v = x_u - x_w$, which implies that all $x_v$ are equal for all vertices adjacent to vertex $u$. Fixed a vertex $w \in V(G)$, we may define that $V_1(G) = \{v \in V(G)\}$ the distance between $v$ and $w$ is
even] and \( V_2(G) = \{ v \in V(G) \mid \text{the distance between } v \text{ and } w \text{ is odd} \} \). Clearly, \( V_1 \) and \( V_2 \) are a partition of \( V(G) \). Since \( G \) is connected, it is not difficult to see that all \( x_v \) are equal for any \( v \in V_1 \) and denoted by \( a \), and that all \( x_v \) are equal for any \( v \in V_2 \) and denoted by \( b \). We claim that \( G \) is bipartite. In fact, if there exists an edge \((u_1, u_2) \in E(G)\), where \( u_1, u_2 \in V_1 \) or \( u_1, u_2 \in V_2 \), then \( a = b \). Hence \( \lambda(G)x_w = \sum_{(v, w) \in E(G)}(x_w - x_v) = 0 \) which implies \( x_w = 0 \). Therefore \( x = 0 \) and it is a contradiction. For any \( u \in V_1 \), we have \( \lambda(G)x_u = \sum_{(v, u) \in E(G)}(x_u - x_v) = (a - b)d(u) \), which result in \( d(u) = \frac{\lambda(G)}{a - b} \) for any \( u \in V_1 \). Similarly, \( d(u) = -\frac{b\lambda(G)}{a - b} \) for any \( u \in V_2 \). Hence we conclude that \( G \) is regular or semi-regular. □

4. Remark and example

**Remark 2.** First, the main results in this paper may be extended to mixed graphs (the reader may refer to [12]). In fact, we may only modify slightly the proofs of Theorems 1.1 and 1.2 and utilize the properties of mixed graphs in [12]. Second, since \( d(u) + \sqrt{d(u)m(u)} \leq \sqrt{2d(u)(d(u) + m(u))} \) for any \( u \in V(G) \), we have that (8) is always better than (5). Moreover, it is easy to see that (2) is always better than (1). However, the rest upper bounds, in general, are not comparable. Let us present two examples to illustrate that (7) and (8) are, in some cases, best.

**Example 4.1.** Let \( G_1 \) and \( G_2 \) be trees of order 9 and 18 respectively in Fig. 1.

We summarize all known upper bounds for the largest eigenvalue of the Laplacian matrix of a graph as follows:

\[
\begin{array}{cccccccccc}
\lambda_1(L(G)) & (1) & (2) & (3) & (4) & (5) & (6) & (7) & (8) & (13) \\
\end{array}
\]

Hence bound (7) is the best in all known upper bounds for \( G_1 \), while bound (8) is the best in all known upper bounds for \( G_2 \).
Acknowledgements

The author would like to thank the referee very much for valuable suggestions, comments and corrections which results in an improvement of the original manuscript.

References