JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 97, 266–276 (1983)

Fenchel–Rockafellar Type Duality for a Non-Convex Non-Differential Optimization Problem*

P. KANNIAPPAN

School of Mathematical Sciences, Madurai Kamaraj University, Madurai-625 021, Tamil Nadu, India, and
Department of Mathematics, Gandhigram Rural University, Gandhigram-624302, Madurai District, India

Submitted by George Leitmann

A Fenchel–Rockafellar type duality theorem is obtained for a non-convex and non-differentiable maximization problem by embedding the original problem in a family of perturbed problems. The recent results of Ivan Singer are developed in this more general framework. A relationship is also established between the solutions and optimal values of the primal and dual problems using the theory of subdifferential calculus.

1. INTRODUCTION

In the case of convex minimization problems, Rockafellar [3, 4] has shown that, by embedding the problem in a family of perturbed problems and using the theory of conjugate convex functions, a dual problem can be associated with the primal problem. The duality theory deals with the relationship between the primal and the dual problems. For complete details about Rockafellar's duality theory, refer to [2, Chap. 3].

For any optimization problem, convex or not, it is natural to ask whether there is a dual problem associated with it. In a recent paper by Ivan Singer [6], a notion of duality for non-convex optimization problems was discussed. That is, by generalizing the methods of his previous paper [7], he has obtained in [6] a Fenchel–Rockafellar type [5] duality theorem for maximization. Many such problems do in fact have applications in approximation theory. It is significant to note that the duality principle which is proved in [6] is not arrived at by embedding the original problem in a family of perturbed problems.

The aim of the present paper is to introduce a duality theory for extremal problems when no assumptions like convexity and differentiability are made on the functional to be maximized. In Section 3, the dual problem will be

* The research was supported by the University Grants Commission of India.

0022-247X/83 $3.00

defined by embedding the original problem in a family of perturbed problems and calculating the Lagrangian functional corresponding to this embedding. Further we show how the Lagrangian can be used to define an extremal problem which is dual to the original problem. In Section 4, the entire duality theory of [6] is seen to be a special case of this perturbational duality principle. We shall also investigate the relationship between the solutions and the optimal values of the primal and the dual problems.

2. Preliminaries

In this paper X and X^* shall be a pair of real vector spaces in duality, with their respective weak topologies $\sigma(X, X^*)$ and $\sigma(X^*, X)$. Thus X and X^* will be locally convex spaces. We denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear form of the dualities between the spaces X and X^*. Thus, all statements concerning continuity, lower-semi-continuity, upper-semi-continuity, convergence, etc., will refer to continuity, lower-semi-continuity, upper-semi-continuity, convergence, etc., in these topologies.

The functional $f: X \to \bar{R} = [-\infty, \infty]$ is said to be *proper*, if $f(X) \subset [-\infty, +\infty]$ and f is not identically $+\infty$.

The functional $f: X \to \bar{R}$ is called *convex* if the inequality

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

holds for every $t \in [0, 1]$ and for all $x, y \in X$.

The functional $f: X \to \bar{R}$ is called *lower-semi-continuous* if for each $x \in X$ and the sequence $x_n \to x$, we have

$$\lim_{n \to \infty} \inf f(x_n) \geqslant f(x)$$

and f is called *upper-semi-continuous* if

$$\lim_{n \to \infty} \sup f(x_n) \leqslant f(x).$$

Consider any functional $f: X \to \bar{R}$. The functional $f^*: X^* \to \bar{R}$ defined by

$$f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x): x \in X\}, \qquad x^* \in X^*,$$

is called the *conjugate function* of f. The conjugate of f^*, that is, the functional $f^{**}: X \to \bar{R}$ defined by

$$f^{**}(x) = \sup\{\langle x, x^* \rangle - f^*(x^*): x^* \in X^*\}, \qquad x \in X, \tag{1}$$

is called the *biconjugate* of f. Clearly, for any functional $f: X \to \bar{R}$, f^* and

f^{**} are convex and lower-semi-continuous. In fact f^{**} is the largest convex and lower-semi-continuous function which is less than f. Hence in general, $f^{**}(x) \leqslant f(x)$ for all $x \in X$ [1, p. 84, Proposition 1.8].

PROPOSITION 1 [1, p. 86]. *$f(x) = f^{**}(x)$ for all $x \in X$, if and only if f is proper, lower-semi-continuous, and convex.*

An element $x^* \in X^*$ is said to be a subgradient of $f: X \to R$ at a point $x_0 \in X$ if

$$\langle x - x_0, x^* \rangle \leqslant f(x) - f(x_0)$$

for every $x \in X$. The set of all subgradients of f at x_0 is called the subdifferential of f at x_0 and is denoted by $\partial f(x_0)$.

PROPOSITION 2 [1, p. 91]. *Let $f: X \to \bar{R}$ be any functional and let $f^*: X^* \to \bar{R}$ be its conjugate. Then $x^* \in \partial f(x)$ if and only if*

$$f(x) + f^*(x^*) = \langle x, x^* \rangle.$$

3. The Duality Theory

Let X and X^* be a pair of real vector spaces in duality and let $\langle \cdot, \cdot \rangle: X \times X^* \to R$ denote the corresponding bilinear form which is compatible with the topologies on X and X^*. Let $J: X \to \bar{R}$ be any functional. Then by *primal problem* (P) we shall mean the problem

$$(P) \sup_{x \in X} J(x)$$

and we shall call an element $\bar{x}$ of X a solution of (P) if

$$-\infty < \sup_{x \in X} J(x) = J(\bar{x}) < +\infty.$$

Now let Y and Y^* be another pair of real vector spaces in duality, and without any ambiguity we shall use $\langle \cdot, \cdot \rangle$ to denote the bilinear form which is compatible with the topologies on Y and Y^*. Thus, the spaces Y and Y^* are locally convex spaces.

Let $\phi: X \times Y \to R$ be a map satisfying

$$\phi(x, 0) = J(x) \tag{2}$$

for all $x \in X$. For each $x \in X$, let $\phi_x: Y \to \bar{R}$ be defined by

$$\phi_x(p) = \phi(x, p). \tag{3}$$

Then the Lagrangian functional $\mathscr{L}$ is defined on $X \times Y^*$ by

$$\begin{aligned}\mathscr{L}(x, p^*) &= \inf_{p \in Y} \{\phi(x, p) - \langle p, p^* \rangle\} \\ &= -\sup_{p \in Y} \{\langle p, p^* \rangle - \phi(x, p)\} \\ &= -\phi_x^*(p^*).\end{aligned} \tag{4}$$

Now for each $p^* \in Y^*$, let

$$L(p^*) = \sup_{x \in X} \mathscr{L}(x, p^*). \tag{5}$$

Then the problem

$$(P^*) \quad \sup_{p^* \in Y^*} L(p^*)$$

is termed the *dual problem* of (P).

We prove the following theorems between the primal and the dual problems (P) and (P^*).

THEOREM 1. $\sup_{x \in X} J(x) \geqslant \sup_{p^* \in Y^*} L(p^*)$.

Proof.

$$\begin{aligned}\sup_{x \in X} J(x) &= \sup_{x \in X} \phi(x, 0) \\ &= \sup_{x \in X} \phi_x(0), && \text{by (2) and (3),} \\ &\geqslant \sup_{x \in X} \phi_x^{**}(0), && \text{since } \phi_x(0) \geqslant \phi_x^{**}(0), \\ &= \sup_{x \in X} \{\sup_{p^* \in Y^*} [\langle 0, p^* \rangle - \phi_x^*(p^*)]\}, && \text{by (1),} \\ &= \sup_{p^* \in Y^*} \{\sup_{x \in X} -\phi_x^*(p^*)\} \\ &= \sup_{p^* \in Y^*} \{\sup_{x \in X} \mathscr{L}(x, p^*)\}, && \text{by (4),} \\ &= \sup_{p^* \in Y^*} L(p^*), && \text{by (5).}\end{aligned}$$

Hence the theorem.

THEOREM 2. *If* $\phi_x : Y \to \bar{R}$ *is such that* $\phi_x(0) = \phi_x^{**}(0)$ *for every* $x \in X$, *then*

$$\sup_{x \in X} J(x) = \sup_{p^* \in Y^*} L(p^*).$$

Proof. Since $\phi_x(0) = \phi^{**}(0)$, the theorem follows from the proof of Theorem 1.

Remark. If $\partial\phi_x(0)$ is non-empty, then also $\phi_x(0) = \phi_x^{**}(0)$ (see [2, p. 21]). Hence, Theorem 2 follows, if we assume that ϕ_x is subdifferentiable at the origin.

By Proposition 1, Theorem 2 also follows by assuming ϕ_x is proper, lower-semi-continuous, and convex.

The following theorem establishes a relationship between the solutions and a relationship between the optimal values of the problems (P) and (P^*).

THEOREM 3. *If $\bar{x} \in X$ solves (P) and $\bar{p}^* \in \partial\phi_x(0)$, then $\bar{p}^*$ solves the problem (P^*). Furthermore, we have the following extremality conditions*

$$J(\bar{x}) - L(\bar{p}^*) = 0$$

$$J(\bar{x}) + \phi_{\bar{x}}^*(\bar{p}^*) = 0.$$

Proof. Since $\bar{x} \in X$ is a solution of (P), we have

$$J(\bar{x}) = \sup_{x \in X} J(x) = \alpha \in R.$$

Since $\bar{p}^* \in \partial\phi_{\bar{x}}(0)$, we have

$$\langle p, \bar{p}^* \rangle \leqslant \phi_{\bar{x}}(p) - \phi_{\bar{x}}(0), \qquad \text{for all } p \in Y.$$

Hence

$$\begin{aligned} \phi(\bar{x}, p) = \phi_{\bar{x}}(p) &\geqslant \phi_{\bar{x}}(0) + \langle p, \bar{p}^* \rangle, && \text{for all } p \in Y, \\ &= \alpha + \langle p, \bar{p}^* \rangle, && \text{by (2) and (3).} \end{aligned}$$

that is, $\phi(\bar{x}, p) - \langle p, \bar{p}^* \rangle \geqslant \alpha$, for all $p \in Y$. Therefore, $\inf_{p \in Y}\{\phi(\bar{x}, p) - \langle p, \bar{p}^* \rangle\} \geqslant \alpha$. That is $\mathscr{L}(\bar{x}, \bar{p}^*) \geqslant \alpha$. Hence, $L(\bar{p}^*) = \sup_{x \in X} \mathscr{L}(x, \bar{p}^*) \geqslant \alpha$. From Theorem 1, $\alpha \geqslant L(\bar{p}^*)$. Hence by Theorem 2 and the remark, we have

$$L(\bar{p}^*) = \alpha = J(\bar{x}) = \sup_{x \in X} J(x) = \sup_{p^* \in Y^*} L(p^*). \tag{6}$$

This implies that $\bar{p}^*$ solves (P^*). By (6), we have

$$J(\bar{x}) - L(\bar{p}^*) = 0$$

and from $\bar{p}^* \in \partial\phi_{\bar{x}}(0)$ and by Proposition 2, we have

$$J(\bar{x}) + \phi_{\bar{x}}^*(\bar{p}^*) = 0.$$

Hence the theorem.

THEOREM 4. *Assume that* $\{x_n\}$ *is a maximizing sequence of problem* (P) *such that* $x_n \to x_0$ *and* $\phi(\cdot, p^*): X \to \bar{R}$ *is upper-semi-continuous for every* $p^* \in Y^*$. *Then* x_0 *is an optimal solution of problem* (P).

Proof. By the definition of upper-semi-continuity, we have

$$\lim_{n\to\infty} \sup \phi(x_n, p^*) \leqslant \phi(x_0, p^*)$$

for every $p^* \in Y^*$. In particular, we have

$$\lim_{n\to\infty} \sup \phi(x_n, 0) \leqslant \phi(x_0, 0).$$

That is, $\lim_{n\to\infty} \sup J(x_n) \leqslant J(x_0)$, which implies $\alpha \leqslant J(x_0)$, where α is the optimal value of (P). Hence $\alpha = J(x_0)$. Hence the theorem.

Now, we have the following application of Theorem 3.

THEOREM 5. *We assume that* $\partial\phi_x(0)$ *is non-empty, for every* $x \in X$ *and that* $p^* \in Y^*$, $\{x_n\}$ *is a sequence in* X *satisfying*

$$\lim_{n\to\infty} \mathscr{L}(x_n, p^*) = L(p^*). \tag{7}$$

Then p^* *is a solution of the dual problem* (P^*) *if and only if* $\{x_n\}$ *is a maximizing sequence for problem* (P) *and* $J(x_n) - \mathscr{L}(x_n, p^*)$ *converges to zero.*

Proof. Suppose that p^* is an optimal solution of problem (P^*). Let $L(P^*) = \alpha$. Then by (7), for any $\eta > 0$, there exists an N such that

$$\alpha - \eta \leqslant \mathscr{L}(x_n, p^*) \leqslant \alpha.$$

That is,

$$\alpha - \eta \leqslant -\sup_{p \in Y} \{\langle p, p^*\rangle - \phi(x_n, p^*)\} \leqslant \alpha$$

for all $n \geqslant N$. Considering the left hand side inequality, we have

$$\sup_{p\in Y} \{\langle p, p^*\rangle - \phi(x_n, p^*)\} \leqslant -\alpha + \eta$$

or all $n \geqslant N$. Hence $-\phi(x_n, 0) \leqslant -\alpha + \eta$, for all $n \geqslant N$. That is

$$J(x_n) \geqslant \alpha - \eta \tag{8}$$

for all $n \geqslant N$.

Since $\partial\phi_x(0)$ is non-empty for all $x \in X$, we have by Theorem 2 and the remark,

$$\sup_{x\in X} J(x) = \sup_{q^*\in Y^*} L(q^*) = L(p^*) = \alpha,$$

which implies

$$J(x_n) \leqslant \alpha, \tag{9}$$

for all n. By (8) and (9), we have

$$\alpha - \eta \leqslant J(x_n) \leqslant \alpha$$

for all $n \geqslant N$. Hence $\{x_n\}$ is a maximizing sequence for problem (P). That is, $J(x_n)$ converges to α. By hypothesis, $\mathscr{L}(x_n, p^*)$ converges to α. Hence $J(x_n) - \mathscr{L}(x_n, p^*)$ converges to zero.

Conversely, suppose $\{x_n\}$ is a maximizing sequence of (P) and $J(x_n) - \mathscr{L}(x_n, p^*)$ converges to zero. By hypothesis, $\mathscr{L}(x_n, p^*)$ converges to $L(p^*)$. Hence $J(x_n)$ converges to $L(p^*)$. That is,

$$\sup_{x\in X} J(x) = L(p^*).$$

Further by Theorem 2, we have

$$\sup_{x\in X} J(x) = \sup_{q^*\in Y^*} L(q^*) = L(p^*).$$

Hence p^* is a solution of problem (P^*).

THEOREM 6. *Assume that ϕ_x is subdifferentiable at the origin for every $x \in X$ and that $\{x_n\}$ is a maximizing sequence for (P). If $p_n^* \in \partial\phi_{x_n}(0)$, then $\{p_n^*\}$ is a maximizing sequence for (P^*).*

Proof. Suppose that $\sup_{x\in X} J(x) = \alpha \in R$. Since $\{x_n\}$ is a maximizing sequence for (P), we have

$$\lim_{n\to\infty} J(x_n) = \lim_{n\to\infty} \phi(x_n, 0) = \sup_{x\in X} J(x) = \alpha.$$

Hence, given any $\eta > 0$, there exists an N such that

$$-\eta \leqslant \phi(x_n, 0) - \alpha \leqslant 0$$

for all $n \geqslant N$. Since $p_n^* \in \partial\phi_{x_n}(0)$, we have

$$\begin{aligned} \phi(x_n, p) &\geqslant \phi(x_n, 0) + \langle p, p_n^*\rangle, && \text{for every } p \in Y, \\ &\geqslant \alpha - \eta + \langle p, p_n^*\rangle, && \text{for every } p \in Y. \end{aligned}$$

Hence, $\sup_{p \in Y}\{\langle p, p_n^* \rangle - \phi(x_n, p)\} \leqslant -\alpha + \eta$. Then, we have by (4) $\mathscr{L}(x_n, p_n^*) \geqslant \alpha - \eta$. Hence $L(p_n^*) = \sup_{x \in X} \mathscr{L}(x, p_n^*) \geqslant \alpha - \eta$. By Theorem 2, $\alpha \geqslant L(p_n^*)$. Therefore, $\{p_n^*\}$ is a maximizing sequence for (P^*).

If $\sup_{x \in X} J(x) = \infty$, then the proof is immediate from the above argument.

4. Special Case

In this section we shall consider problems (P) of the form

$$(P) \sup_{x \in X} F(x) - G(x)$$

where F is a proper, lower-semi-continuous and convex functional on X and G is an arbitrary functional on X. The purpose of this section is to show how the results in [6] are consequences of the duality theory of the last section of this paper.

Let $J(x) = F(x) - G(x)$ for all $x \in X$. Define $\phi: X \times X \to \bar{R}$ by

$$\phi(x, p) = F(x + p) - G(x).$$

Since F is a proper, lower-semi-continuous, and convex functional, we have $\phi_x^{**}(0) = \phi_x(0) = \phi(x, 0) = F(x) - G(x)$, for all $x \in X$. This is so because, once x is fixed, ϕ_x is a function of p alone, which is proper, lower-semi-continuous, and convex. Hence, the duality Theorem 2 holds in this case,

$$\sup_{x \in X} F(x) - G(x) = \sup_{p^* \in Y^*} L(p^*).$$

Let us now calculate $L(p^*)$ in this case. Consider,

$$\begin{aligned}
\mathscr{L}(x, p^*) &= -\sup_{p \in X} \{\langle p, p^* \rangle - \phi(x, p)\} \\
&= -\sup_{p \in X} \{\langle p, p^* \rangle - F(x, p) + G(x)\} \\
&= -\sup_{p \in X} \{\langle x + p, p^* \rangle - F(x + p)\} - G(x) + \langle x, p^* \rangle \\
&= -\sup_{x + p \in X} \{\langle x + p, p^* \rangle - F(x + p)\} - G(x) + \langle x, p^* \rangle \\
&\qquad \text{since } X \text{ is the whole space} \\
&= -F^*(p^*) - G(x) + \langle x, p^* \rangle.
\end{aligned}$$

Consequently,

$$\begin{aligned} L(p^*) &= \sup_{p \in X} \mathscr{L}(x, p^*) \\ &= \sup_{p \in X} \{-F^*(p^*) - G(x) + \langle x, p^* \rangle\} \\ &= \sup_{p \in X} \{\langle x, p^* \rangle - G(x)\} - F^*(p^*) \\ &= G^*(p^*) - F^*(p^*). \end{aligned}$$

Thus, we have proved the following theorem.

THEOREM 7. *If F is a proper, lower-semi-continuous, and convex functional on X and G is an arbitrary functional on X, then*

$$\sup_{p \in X} F(x) - G(x) = \sup_{p^* \in X^*} G^*(p^*) - F^*(p^*). \tag{10}$$

This is precisely the duality theorem of Fenchel–Rockafellar type for maximization proved by Ivan Singer in [6].

In this context, Theorem 3 takes the following form.

THEOREM 3. *If $\bar{x} \in X$ is an optimal solution of problem (P) and $\bar{p}^* \in \partial F(\bar{x})$, then $\bar{p}^*$ is a solution of problem (P^*). Furthermore,*

$$F(\bar{x}) + F^*(\bar{p}^*) = \langle \bar{x}, \bar{p}^* \rangle \tag{11}$$

$$G(\bar{x}) + G^*(\bar{p}^*) = \langle \bar{x}, \bar{p}^* \rangle. \tag{12}$$

Note. Conditions (11) and (12) are called extremality conditions for problems (P) and (P^*) expressed in terms of conjugate functions.

5. APPLICATIONS

(1) In the particular case when $G = \delta_B$, the indicator function of a bounded subset B of X (that is $\delta_B(x) = 0$ if $x \in B$ and $\delta_B(x) = +\infty$ if $x \in X \backslash B$), we get a formula for supremum of a proper, lower-semi-continuous, and convex functional F on a bounded subset, as an application of the result (10). That is, when $G = \delta_B$, then its conjugate function is just the support function of the set B. That is,

$$G^*(x^*) = \sup_{x \in B} \langle x, x^* \rangle.$$

Now, by taking $G=\delta_B$ in (10), we have

$$\begin{aligned}\sup_{x\in B} F(x) &= \sup_{p^*\in X^*} G^*(p^*)-F^*(p^*)\\ &= \sup_{p^*\in X^*} \{\sup_{x\in B}\langle x,p^*\rangle - \sup_{p\in X}[\langle p,p^*\rangle - F(p)]\}\\ &= \sup_{p^*\in X^*}\{\inf_{p\in X}[F(p)-\langle p,p^*\rangle] + \sup_{x\in B}\langle x,p^*\rangle\}.\end{aligned}$$

Hence,

$$\begin{aligned}\sup F(B) = \sup_{x\in B} F(x) &= \sup_{p^*\in X^*}\{\inf_{p\in X}[F(p)-\langle p,p^*\rangle] + \sup_{x\in B}\langle x,p^*\rangle\}\\ &= \sup_{p^*\in X^*}\{\inf_{p\in X}[F(p)+\langle p,p^*\rangle] - \inf_{x\in B}\langle x,p^*\rangle\} \qquad (13)\end{aligned}$$

(see [6], for details).

(2) When X is a normed linear space and F is a continuous convex functional on X defined by

$$F(y)=\|x-y\|, \qquad y\in X,$$

where x is any fixed element of X, we have from formula (13) the following new formula for the deviation [7],

$$\delta(B,x)=\sup_{b\in B}\|b-x\|$$

of a bounded subset B from x:

$$\delta(B,x)=\sup_{p^*\in X^*}\{\inf_{y\in X}[\|x-y\|-\langle y,p^*\rangle] + \sup_{b\in B}\langle b,p^*\rangle\}.$$

Acknowledgment

The author thanks Dr. Sundaram M. A. Sastry for his helpful suggestions.

References

1. V. Barbu and Th. Precupanu, "Convexity and Optimization in Banach Spaces," Sijthoff & Noordhoff Intern. Pub., 1978.
2. I. Ekeland and R. Temam, Convex analysis and variational problems *in* "Studies in Mathematics and Its Applications," Vol. 1, North-Holland, Amsterdam, Oxford; American Elsevier, New York, 1976.
3. R. T. Rockafellar, "Convex Analysis," Princeton Mathematics Series, Princeton Univ. Press, Princeton, New Jersey, 1970.

4. R. T. Rockafellar, Duality and stability in extremum problems involving convex functions, *Pacific J. Math.* **21** (1967), 167–187.
5. R. T. Rockafellar, An extension of Fenchel's duality theorem for Convex functions, *Duke Math. J.* **33** (1966), 81–89.
6. I. Singer, A Fenchel–Rockafellar type duality theorem for maximization, *Bull. Austral. Math. Soc.* **20** (1979), 193–198.
7. I. Singer, "Three papers on optimization," Research Report No. 16, Department of Mathematics, National Institute for Scientific and Technical Creation, Bucharest, 1979