Linear Algebra and its Applications 433 (2010) 606–617

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

A matrix completion problem over integral domains: the case with $2n-3$ prescribed entries ☆

Alberto Borobia *, Roberto Canogar, Helena Šmigoc

Dpto. Matemáticas, UNED, 28040 Madrid, Spain
School of Mathematical Sciences, University College Dublin, Ireland

ARTICLE INFO

Article history:
Received 21 January 2009
Accepted 19 March 2010
Available online 17 April 2010

Submitted by R.A. Brualdi

Keywords:
Matrix completions
Inverse eigenvalue problems
Matrices over integral domains

ABSTRACT

Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, $n \geqslant 2$, be a given multiset of elements in an integral domain $\mathfrak{R}$ and let P be a matrix of order n with at most $2n-3$ prescribed entries that belong to $\mathfrak{R}$. Under the assumption that each row, each column and the diagonal of P have at least one unprescribed entry, we prove that P can be completed over $\mathfrak{R}$ to obtain a matrix A with spectrum Λ. We describe an algorithm to construct A. This result is an extension to integral domains of a classical completion result by Herskowitz for fields.

1. Introduction

An inverse eigenvalue problem consists of the construction of a matrix with prescribed structural and spectral constraints. This is a two level problem: on a theoretical level the target is to determine if there exists a solution matrix with the given constraints; and on a practical level the target is an effective construction of a solution matrix when the problem is solvable. Inverse eigenvalue problems are classified into different types according to the specific constraints. For interested readers, we refer to the book by Chu and Golub [3] where an account of inverse eigenvalue problems with applications and exhaustive bibliography can be found.

A particular class of inverse eigenvalue problems are completion problems: given a matrix P with some of its entries specified, we would like to decide if and how we can choose unspecified entries of P in such a way that the completed matrix satisfies certain spectral properties. A survey on these type of problems is given by Ikramov and Chugunov in [5], where they are specially interested in

☆ Supported by the Spanish Ministerio de Ciencia y Tecnología Grants MTM2009-09180 and MTM2006-05361.

* Corresponding author at: Dpto. Matemáticas, UNED, 28040 Madrid, Spain.
E-mail addresses: aborobia@mat.uned.es (A. Borobia), rcanogar@mat.uned.es (R. Canogar), helena.smigoc@ucd.ie (H. Šmigoc).

0024-3795/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2010.03.025

the development of finite rational algorithms to construct a solution matrix. A different approach to the problem is given by Chu, Diele and Sgura in [2], where they consider gradient flow methods. An extensive list of results in completion problems is given by Borobia in [1].

Our work was motivated by an interesting result of Hershkowitz [4]. He considered the case of a matrix of order n with prescribed spectrum, with at most $2n-3$ prescribed entries in arbitrary positions, and with the prescribed entries of the matrix and the prescribed eigenvalues lying in the same field. He showed that, except in some very special cases, such a matrix can always be completed with elements of the field to a matrix with the given spectrum.

When presented with a partially prescribed matrix P of order n there are some situations in which we can immediately see that the completion to a matrix with a given spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ is not possible. For example, let P have a line (row or column) with all its elements prescribed, with all the off-diagonal entries in that line equal to 0 and the diagonal entry not in Λ. If such a line in a matrix P does not exist we will say that the lines of P are *consistent with* Λ. Another example, where the completion is clearly impossible, is when we have all the diagonal elements of P prescribed and the sum of the diagonal elements is different to the sum of the elements in Λ. If this is not the case, we will say that the diagonal of P is *consistent with* Λ. Hershkowitz proved that the two situations mentioned above are the only ones that we need to exclude if we want to find a completion of a matrix P with at most $2n-3$ prescribed entries to a matrix with prescribed spectrum Λ.

Theorem 1.1 (Herskowitz [4]). *For $n \geqslant 2$ let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a given multiset of elements in a field $\mathfrak{F}$. Let P be a matrix of order n with at most $2n-3$ prescribed entries that belong to $\mathfrak{F}$, and such that the lines and the diagonal of P are consistent with Λ. Then P can be completed with elements of $\mathfrak{F}$ to obtain a matrix with spectrum Λ.*

While matrix completion problems over fields have been extensively studied, little is known about completion problems over rings. Except in the case where there exists a line with all its elements prescribed, we will be able to extend Theorem 1.1 to arbitrary integral domains. The following theorem is the main result of this work.

Theorem 1.2. *For $n \geqslant 2$ let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a given multiset of elements in an integral domain $\mathfrak{R}$. Let P be a matrix of order n with at most $2n-3$ prescribed entries that belong to $\mathfrak{R}$, and such that the diagonal and each line of P have at least one element unprescribed. Then P can be completed with elements of $\mathfrak{R}$ to obtain a matrix with spectrum Λ.*

Note that the completion problem for a matrix P with the diagonal fully prescribed and consistent with Λ is equivalent to a completion problem for a matrix obtained from P by changing one of the prescribed entries on the diagonal to unprescribed. Therefore Theorem 1.2 can be trivially extended to matrices that have the diagonal fully prescribed and consistent with Λ.

Corollary 1.1. *For $n \geqslant 2$ let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a given multiset of elements in an integral domain $\mathfrak{R}$. Let P be a matrix of order n with at most $2n-3$ prescribed entries that belong to $\mathfrak{R}$, such that each line of P has at least one element unprescribed, and such that the diagonal of P is consistent with Λ. Then P can be completed over $\mathfrak{R}$ to obtain a matrix with spectrum Λ.*

Remark 1.1. We can not extend Theorem 1.2 to matrices that contain a line that has all its elements prescribed. Let P have a line with all its elements prescribed and contained in a proper ideal $\mathfrak{I}$ of the integral domain $\mathfrak{R}$. The determinant of any completion of P is an element of $\mathfrak{I}$. Therefore no completion of P can achieve the spectrum $\Lambda = \{1, \ldots, 1\}$, where 1 denotes the unity of $\mathfrak{R}$, since the product of the elements of Λ is 1, and 1 is not an element of $\mathfrak{I}$.

2. Notation

For convenience of the reader we provide a comprehensive list of the notation used at this point, although the motivation for some of the notation will not become apparent until later in the manuscript.

- $\mathfrak{R}$ will denote an arbitrary integral domain.
- $\mathcal{M}_n$ will denote the set of $n \times n$ matrices with entries in $\mathfrak{R}$.
- $\overline{\mathfrak{R}}$ will denote the set $\mathfrak{R} \cup \{\Box\}$, where $\Box$ will denote an entry that is unprescribed.
- $\overline{\mathcal{M}_n}$ will denote the set of $n \times n$ matrices with entries in $\overline{\mathfrak{R}}$ (in other words, $\overline{\mathcal{M}_n}$ denotes the set of matrices that can have some unprescribed entries).
- $\widehat{M}_n$ will denote the set of all matrices $P \in \overline{\mathcal{M}_n}$ such that for all $i = 1, \ldots, n$ we have $p_{i1} = \Box$ or $p_{i2} = \Box$ (observe that $\widehat{M}_n$ denotes a class of matrices that have a restriction on the prescribed-unprescribed pattern in the first two columns).
- Let $\tau \in S_n$ where S_n is the symmetric group on n elements, and let $P \in \overline{\mathcal{M}_n}$, we define
$$\tau(P) = \left(p_{\tau(i)\,\tau(j)}\right)_{i,j=1}^{n}.$$
If $P \in \mathcal{M}_n$ then $\tau(P) = TPT^{-1}$, where T is the permutation matrix corresponding to τ.
- For $P, Q \in \overline{\mathcal{M}_n}$ we say that they are related if $Q = \tau(P)$ or $Q = \tau(P^T)$ for some $\tau \in S_n$. It is clear that this defines an equivalence relation. We will denote the equivalence class of P by $\mathcal{E}(P)$. That is,
$$\mathcal{E}(P) = \{\tau(P) : \tau \in S_n\} \cup \{\tau(P^T) : \tau \in S_n\}.$$
Notice that if $P \in \mathcal{M}_n$ then all matrices in $\mathcal{E}(P)$ have the same spectrum.
- $P_{(i)}$ will denote the i-th row of P,
- $P^{(j)}$ will denote the j-th column of P,
- $\#P$ will denote the number of prescribed entries in P,
- $\#P_{(i)}$ will denote the number of prescribed entries in $P_{(i)}$,
- $\#P^{(j)}$ will denote the number of prescribed entries in $P^{(j)}$.

3. Previous results

3.1. Reductions

In this section we will see how to construct, from a given matrix $P \in \widehat{M}_n$ and a given $\lambda \in \mathfrak{R}$, a *reduced* matrix $\Gamma_\lambda(P) \in \overline{\mathcal{M}_{n-1}}$. The interesting point is that if $\Gamma_\lambda(P)$ can be completed with elements of $\mathfrak{R}$ to obtain a matrix with spectrum $\{\lambda_1, \ldots, \lambda_{n-1}\}$ then P can be completed with elements of $\mathfrak{R}$ to obtain a matrix with spectrum $\{\lambda, \lambda_1, \ldots, \lambda_{n-1}\}$.

Our method is based on a result obtained by Šmigoc in [6]. Here we extend to integral domains a simplified version of that result.

Lemma 3.1. *Let $a \in \mathfrak{R}$; $\mathbf{b}, \mathbf{c} \in \mathfrak{R}^{n-2}$ and $D \in \mathcal{M}_{n-2}$. Define the matrix:*
$$B = \left(\begin{array}{c|c} a & \mathbf{b}^T \\ \hline \mathbf{c} & D \end{array}\right) \in \mathcal{M}_{n-1}. \tag{1}$$
For any $x, \lambda \in \mathfrak{R}$ and any $\mathbf{y} \in \mathfrak{R}^{n-2}$ define the matrix:
$$A = \left(\begin{array}{cc|c} x+\lambda & x & \mathbf{y}^T \\ a-x-\lambda & a-x & \mathbf{b}^T - \mathbf{y}^T \\ \hline \mathbf{c} & \mathbf{c} & D \end{array}\right) \in \mathcal{M}_n \tag{2}$$
Then the spectrum of A consists of all the eigenvalues of B with λ adjoined.

Proof
$$\det(\mu I - A) = \det\left(\begin{array}{cc|c} \mu-x-\lambda & -x & -\mathbf{y}^T \\ -a+x+\lambda & \mu-a+x & -\mathbf{b}^T + \mathbf{y}^T \\ \hline -\mathbf{c} & -\mathbf{c} & \mu I - D \end{array}\right) \overrightarrow{R_1 + R_2 \to R_1}$$

$$= \det\left(\begin{array}{cc|c} \mu - a & \mu - a & -\mathbf{b}^T \\ -a + x + \lambda & \mu - a + x & -\mathbf{b}^T + \mathbf{y}^T \\ \hline -\mathbf{c} & -\mathbf{c} & \mu I - D \end{array}\right) \overrightarrow{C_2 - C_1 \to C_2}$$

$$= \det\left(\begin{array}{cc|c} \mu - a & 0 & -\mathbf{b}^T \\ -a + x + \lambda & \mu - \lambda & -\mathbf{b}^T + \mathbf{y}^T \\ \hline -\mathbf{c} & \mathbf{0} & \mu I - D \end{array}\right)$$

$$= (\mu - \lambda) \det\left(\begin{array}{c|c} \mu - a & -\mathbf{b}^T \\ \hline -\mathbf{c} & \mu I - D \end{array}\right)$$

$$= (\mu - \lambda) \det(\mu I - B)$$

Then the spectrum of A consists of all eigenvalues of B with λ adjoined. $\square$

We will use Lemma 3.1 to prove Theorem 1.2 by induction on the size of the partially prescribed matrix, since it permits to reduce a completion problem for a matrix in $\widehat{\mathcal{M}}_n$ to a completion problem for a matrix in $\overline{\mathcal{M}}_{n-1}$. The following three definitions give an explicit formulation of the reduction.

Definition 3.1. We introduce the following two operations between elements in $\overline{\mathfrak{R}}$:

1. Given $r_1, r_2 \in \overline{\mathfrak{R}}$ we define

$$r_1 \oplus r_2 = \begin{cases} r_1 + r_2 & \text{if } r_1, r_2 \in \mathfrak{R} \\ \square & \text{otherwise} \end{cases}$$

2. Given $s_1, s_2 \in \overline{\mathfrak{R}}$ with at least one of the elements equal to $\square$, we define

$$s_1 \odot s_2 = \begin{cases} \square & \text{if } s_1 = s_2 = \square \\ s_i & \text{if } s_i \in \mathfrak{R} \text{ for some } i \end{cases}$$

(Operation $\odot$ is not defined if both s_1 and s_2 belong to $\mathfrak{R}$.)

Definition 3.2. Given a matrix

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \in \widehat{\mathcal{M}}_2$$

and given $\lambda \in \mathfrak{R}$, we define *the λ-reduction* of Q in the following way:

$$\Gamma_\lambda(Q) = \begin{cases} q_{11} + q_{21} & \text{if } q_{11}, q_{21} \in \mathfrak{R} \\ q_{12} + q_{22} & \text{if } q_{12}, q_{22} \in \mathfrak{R} \\ q_{11} + q_{22} - \lambda & \text{if } q_{11}, q_{22} \in \mathfrak{R} \\ q_{12} + q_{21} + \lambda & \text{if } q_{12}, q_{21} \in \mathfrak{R} \\ \square & \text{otherwise} \end{cases}$$

Definition 3.3. For $n \geqslant 3$, given a matrix

$$Q = \left(\begin{array}{cc|ccc} q_{11} & q_{12} & q_{13} & \cdots & q_{1n} \\ q_{21} & q_{22} & q_{23} & \cdots & q_{2n} \\ \hline q_{31} & q_{32} & q_{33} & \cdots & q_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & q_{n3} & \cdots & q_{nn} \end{array}\right) \in \widehat{\mathcal{M}}_n \tag{3}$$

and given $\lambda \in \mathfrak{R}$, we define *the λ-reduction* of Q as the matrix

$$\Gamma_\lambda(Q)=\left(\begin{array}{c|ccc} \Gamma_\lambda\begin{pmatrix} q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix} & q_{13}\oplus q_{23} & \cdots & q_{1n}\oplus q_{2n}\\ \hline q_{31}\odot q_{32} & q_{33} & \cdots & q_{3n}\\ \vdots & \vdots & \ddots & \vdots \\ q_{n1}\odot q_{n2} & q_{n3} & \cdots & q_{nn}\end{array}\right)\in \overline{\mathcal{M}}_{n-1}. \tag{4}$$

Example 3.1. We provide some examples:

1. $Q=\left(\begin{array}{cc|ccc} a & \square & b & \square & \square\\ \square & c & d & \square & e\\ \hline \square & f & \square & g & h\\ \square & \square & i & \square & \square\\ j & \square & k & \square & l\end{array}\right)\in\widehat{\mathcal{M}}_5$. Then $\Gamma_\lambda(Q)=\left(\begin{array}{c|ccc} a+c-\lambda & b+d & \square & \square\\ \hline f & \square & g & h\\ \square & i & \square & \square\\ j & k & \square & l\end{array}\right)$.
2. $Q=\left(\begin{array}{cc|cc} 2 & \square & 8 & 3\\ 4 & \square & 1 & \square\\ \hline \square & 7 & \square & 1\\ \square & \square & 2 & \square\end{array}\right)\in\widehat{\mathcal{M}}_4$. Then $\Gamma_\lambda(Q)=\left(\begin{array}{c|cc} 6 & 9 & \square\\ \hline 7 & \square & 1\\ \square & 2 & \square\end{array}\right)$.
3. $Q=\left(\begin{array}{cc|c} \square & \square & 5\\ 4 & \square & 4\\ \hline 3 & \square & \square\end{array}\right)\in\widehat{\mathcal{M}}_3$. Then $\Gamma_\lambda(Q)=\left(\begin{array}{c|c} \square & 9\\ \hline 3 & \square\end{array}\right)$.

3.2. Lemmas

Let us assume that $P\in\overline{\mathcal{M}}_n$ satisfies conditions of Theorem 1.2, that is, P is a matrix of order n with at most $2n-3$ prescribed entries that belong to $\mathfrak{R}$, and such that the diagonal and each line of P have at least one element unprescribed. It is convenient to present these conditions in the following form:

S1 First row of P is not fully prescribed: $\#P_{(1)}<n$.
S2 Rows $2,\ldots,n$ of P are not fully prescribed: $\#P_{(2)},\ldots,\#P_{(n)}<n$.
S3 First column of P is not fully prescribed: $\#P^{(1)}<n$.
S4 Columns $2,\ldots,n$ of P are not fully prescribed: $\#P^{(2)},\ldots,\#P^{(n)}<n$.
S5 The diagonal of P is not fully prescribed: $\#\mathrm{diag}(P)<n$.
S6 P has at most $2n-3$ prescribed elements: $\#P\leqslant 2n-3$.

Notice that if P satisfies conditions **S1**–**S6**, then so does every matrix in $\mathcal{E}(P)$. Let us assume that we have found in $\mathcal{E}(P)$ some $Q\in\widehat{\mathcal{M}}_n$. Next we list some initial observations that we can make for $\Gamma_\lambda(Q)$.

Lemma 3.2. *Let $Q\in\widehat{\mathcal{M}}_n$ satisfy conditions* **S1**–**S6** *with $n\geqslant 3$. Then*

1. *$\Gamma_\lambda(Q)$ satisfies* **S1**.
2. *$\Gamma_\lambda(Q)$ satisfies* **S2** *if $\#Q_{(3)},\ldots,\#Q_{(n)}\leqslant n-2$. In particular, this happens in the following situations:*
   (a) Q has no empty rows.
   (b) $\#Q_{(2)}\geqslant\#Q_{(3)}\geqslant\ldots\geqslant\#Q_{(n)}$.
3. *$\Gamma_\lambda(Q)$ satisfies* **S3** *if $\#Q^{(1)}+\#Q^{(2)}<n$.*
4. *$\Gamma_\lambda(Q)$ satisfies* **S4**.
5. *$\Gamma_\lambda(Q)$ satisfies* **S5** *unless $q_{33},\ldots,q_{nn}$ are prescribed and exactly two of the elements $q_{11},q_{12},q_{21},q_{22}$ are prescribed.*
6. *$\Gamma_\lambda(Q)$ satisfies* **S6** *in the following cases:*

(a) Prescribed entries in rows $Q_{(1)}$ and $Q_{(2)}$ lay in at least two different columns, and at least one prescribed entry in rows $Q_{(1)}$ and $Q_{(2)}$ lays in columns $Q^{(3)}, \ldots, Q^{(n)}$. In particular, this happens if $\#Q_{(1)} \geqslant 2$ or $\#Q_{(2)} \geqslant 2$.
(b) $\#Q_{(1)} = 0$ and $\#Q_{(2)} \geqslant \#Q_{(3)} \geqslant \ldots \geqslant \#Q_{(n)}$.
(c) $\#Q_{(n)} = 0$ and $\#Q_{(2)} \geqslant \#Q_{(3)} \geqslant \ldots \geqslant \#Q_{(n)}$.

Proof

1. As $\#Q_{(1)} + \#Q_{(2)} \leqslant \#Q \leqslant 2n-3$ we have
$$\#\Gamma_\lambda(Q)_{(1)} \leqslant \min\{\#Q_{(1)}, \#Q_{(2)}\} < n-1.$$
2. For $i = 2, \ldots, n-1$ we have $\#\Gamma_\lambda(Q)_{(i)} = \#Q_{(i+1)} \leqslant n-2$.
3. Assumption $\#Q^{(1)} + \#Q^{(2)} < n$ tells us that either at most one of the elements $q_{11}, q_{12}, q_{21}, q_{22}$ is prescribed or there exists $i \in \{3, \ldots, n\}$ so that $q_{i1} = q_{i2} = \Box$. Both situations give us $\#\Gamma_\lambda(Q)^{(1)} < n-1$.
4. If $\#\Gamma_\lambda(Q)^{(j)} = n-1$ for some $j = 2, \ldots, n-1$, then $\#Q^{(j+1)} = n$, which contradicts the assumption that Q satisfies **S4**.
5. Clear.
6.
   (a) In this case we have $\#Q_{(1)} + \#Q_{(2)} \geqslant \#(\Gamma_\lambda(Q))_{(1)} - 2$, which implies
   $$\#\Gamma_\lambda(Q) \leqslant \#Q - 2 \leqslant 2(n-1) - 3.$$
   (b) Item (a) allows us to assume that $\#Q_{(2)} \leqslant 1$. Then
   $$\#\Gamma_\lambda(Q) = \#Q_{(3)} + \ldots + \#Q_{(n)} \leqslant n-2 \leqslant 2(n-1) - 3.$$
   (c) Item (a) allows us to assume $\#Q_{(2)} \leqslant 1$. Then
   $$\#\Gamma_\lambda(Q) \leqslant \#Q_{(2)} + \ldots + \#Q_{(n)} \leqslant n-2 \leqslant 2(n-1) - 3. \qquad \Box$$

In our next result we will see that if P satisfies all conditions of Theorem 1.2 then we can find in the equivalence class $\mathcal{E}(P)$ a matrix $Q \in \widehat{\mathcal{M}}_n$ such that $\Gamma_\lambda(Q)$ satisfies all conditions of Theorem 1.2.

Lemma 3.3. *For $n \geqslant 3$ let $P \in \overline{\mathcal{M}}_n$ be a matrix that satisfies conditions* **S1**–**S6**, *and let $\lambda \in \mathfrak{R}$. Then there exists in the equivalence class $\mathcal{E}(P)$ a matrix $Q \in \widehat{\mathcal{M}}_n$ such that $\Gamma_\lambda(Q)$ satisfies conditions* **S1**–**S6**.

Proof. In the proof we will often assume that elements of a matrix in some specific positions are unprescribed, elements of a matrix in some specific positions are prescribed, and we won't assume anything for the rest of positions. To emphasize that the later positions can be any element from $\overline{\mathfrak{R}}$ we will denote such entries by ?. The main difficulty in the proof is to decide on the appropriate cases to look at, so the proof is divided into many different cases. However, most of the cases are not difficult to resolve.

(**I**) ***P* has a line with no prescribed elements**.

(I.A) P has a line with no prescribed elements and $\#\mathrm{diag}\,(P) = n-1$. Then there exists $Q \in \mathcal{E}(P)$ of the following form:
$$Q = \begin{pmatrix} \Box & \Box & \ldots & \Box & \mathbf{b}_{i+1} & \ldots & \mathbf{b}_n \\ \Box & \mathbf{a}_2 & \ldots & ? & ? & \cdots & ? \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \Box & ? & \ldots & \mathbf{a}_i & ? & \ldots & ? \\ \Box & ? & \ldots & ? & \mathbf{a}_{i+1} & \ldots & ? \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ \Box & ? & \ldots & ? & ? & \ldots & \mathbf{a_n} \end{pmatrix} \in \widehat{\mathcal{M}}_n$$

with $\#Q_{(2)} \geqslant \ldots \geqslant \#Q_{(i)}$ for some $2 \leqslant i \leqslant n$, and $\mathbf{b}_{i+1}, \ldots, \mathbf{b}_n, \mathbf{a}_2, \ldots, \mathbf{a}_n$ prescribed. Note that $i = 1$ would imply $\#Q \geqslant 2n - 2$.

- $\Gamma_\lambda(Q)$ satisfies **S1**, **S3**, **S4**, and **S5** by Lemma 3.2.
- $\Gamma_\lambda(Q)$ satisfies **S2**:
  - If $\#Q_{(1)} \geqslant 1$ then Q has no empty rows. Apply Lemma 3.2, item 2(a).
  - If $\#Q_{(1)} = 0$ then $\#Q_{(2)} \geqslant \#Q_{(3)} \geqslant \ldots \geqslant \#Q_{(n)}$. Apply Lemma 3.2, item 2(b).
- $\Gamma_\lambda(Q)$ satisfies **S6**:
  - If $\#Q_{(1)} \geqslant 1$ then $q_{1n} = \mathbf{b_n}$ and $q_{22} = \mathbf{a_2}$ are prescribed. Apply Lemma 3.2, item 6(a).
  - If $\#Q_{(1)} = 0$ then $\#Q_{(2)} \geqslant \cdots \geqslant \#Q_{(n)}$. Apply Lemma 3.2, item 6(b).

(I.B) P has an empty line and $\#\mathrm{diag}\,(P) \leqslant n - 2$. Then there exists $Q \in \mathcal{E}(P)$ of the form:

$$Q = \begin{pmatrix} \square & ? & \cdots & ? \\ \square & ? & \cdots & ? \\ \vdots & \vdots & & \vdots \\ \square & ? & \cdots & ? \end{pmatrix} \in \widehat{\mathcal{M}}_n$$

with $\#Q_{(2)} \geqslant \#Q_{(3)} \geqslant \ldots \geqslant \#Q_{(n)}$.

- $\Gamma_\lambda(Q)$ satisfies **S1**, **S2**, **S3**, **S4**, and **S5** by Lemma 3.2.
- $\Gamma_\lambda(Q)$ satisfies property **S6**:
  - Lemma 3.2, item 6(a), resolves this case if $\#Q_{(1)} \geqslant 2$, if $\#Q_{(2)} \geqslant 2$, or if $\#Q_{(1)} = \cdots = \#Q_{(n)} = 1$. In the last subcase we can assume that the prescribed entry in the first row and the prescribed entry in the second row are in different columns, as Q has no full line.
  - If $\#Q_{(1)} = 0$, then apply Lemma 3.2, item 6(b).
  - If $\#Q_{(1)} = 1$, $\#Q_{(2)} \leqslant 1$ and $\#Q_{(j)} \neq 1$ for some $j = 2, 3, \ldots, n$, then $\#Q_{(n)} = 0$ and we can apply Lemma 3.2, item 6(c).

(**II**) ***P* has no empty lines.** Notice, that in this case no line of P can have more than $n - 2$ entries.

(II.A) There exists a line with its only prescribed entry on the diagonal. Then $\mathcal{E}(P)$ contains a matrix

$$Q = \begin{pmatrix} \mathbf{a}_1 & \square & \ldots & \square & \mathbf{a}_{i+1} & \ldots & \mathbf{a}_n \\ \square & ? & \ldots & ? & ? & \cdots & ? \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \square & ? & \ldots & ? & ? & \ldots & ? \\ \square & ? & \ldots & ? & ? & \ldots & ? \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ \square & ? & \ldots & ? & ? & \ldots & ? \end{pmatrix} \in \widehat{\mathcal{M}}_n$$

with $\#Q_{(2)} \geqslant \ldots \geqslant \#Q_{(i)}$ for some $3 \leqslant i \leqslant n$, where $\mathbf{a}_1, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n \in \Re$.

- $\Gamma_\lambda(Q)$ satisfies **S1**, **S2**, **S3**, **S4**, and **S5** by Lemma 3.2.
- $\Gamma_\lambda(Q)$ satisfies **S6**:
  - If $\#Q_{(1)} \geqslant 2$ or $\#Q_{(2)} \geqslant 2$ then apply Lemma 3.2, item 6(a).
  - If $\#Q_{(1)} = \#Q_{(2)} = 1$ then $\#Q_{(2)} = \ldots = \#Q_{(n)} = 1$. As Q has at least one unprescribed entry on the diagonal, we can assume that $q_{22} = \square$. Apply Lemma 3.2, item 6(a).

(II.B) All lines with exactly one prescribed entry have this entry out of the diagonal. Suppose that there exist t columns and s rows each of which have exactly one prescribed entry. Note that $t, s \geqslant 3$ since $\#P \leqslant 2n - 3$ and P has no empty lines. As we are looking for $Q \in \mathcal{E}(P)$, we can assume that $t \geqslant s \geqslant 3$ and the first t columns of P have exactly one prescribed entry. Let $p_{i_1 1}, \ldots, p_{i_t t}$ where $i_k \neq k$ for $k = 1, \ldots, t$, be the prescribed entries in the first t columns of P. We have the following possibilities:

(II.B.1) Two of the prescribed entries in the first t columns are p_{ij} and p_{jk} with $i \neq k$. Then $\mathcal{E}(P)$ contains a matrix

$$Q = \begin{pmatrix} \square & \mathbf{b} & ? & \cdots & ? \\ \square & \square & ? & \cdots & ? \\ \mathbf{a} & \square & ? & \cdots & ? \\ \square & \square & ? & \cdots & ? \\ \vdots & \vdots & \vdots & & \vdots \\ \square & \square & ? & \cdots & ? \end{pmatrix} \in \widehat{\mathcal{M}}_n,$$

where $\mathbf{a}, \mathbf{b} \in \Re$.

- $\Gamma_\lambda(Q)$ satisfies **S1**, **S2**, **S3**, **S4**, and **S5** by Lemma 3.2.
- As Q has no empty line, the second row has a prescribed entry. Then $\Gamma_\lambda(Q)$ satisfies **S6** by item 6(a) of Lemma 3.2.

(II.B.2) No two prescribed entries in the first t columns are of the form p_{ij} and p_{jk} with $i \neq k$, and there exist two prescribed entries in the first t columns of the form p_{ij} and p_{ji}. Then $\mathcal{E}(P)$ contains a matrix

$$Q = \begin{pmatrix} \square & \square & \mathbf{c} & ? & \cdots & ? \\ \square & \square & \square & ? & \cdots & ? \\ \mathbf{a} & \square & \square & ? & \cdots & ? \\ \square & \mathbf{b} & \square & ? & \cdots & ? \\ \square & \square & \square & ? & \cdots & ? \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \square & \square & \square & ? & \cdots & ? \end{pmatrix} \in \widehat{\mathcal{M}}_n,$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \Re$.

- $\Gamma_\lambda(Q)$ satisfies **S1**, **S2**, **S3**, **S4**, and **S5** by Lemma 3.2.
- As Q has no empty line, the second row has a prescribed entry, and $\Gamma_\lambda(Q)$ satisfies **S6** by item 6(a) of Lemma 3.2.

(II.B.3) We have $i_1, \ldots, i_t > t$.
Recall that $\#P \leqslant 2n - 3$, that every line of P has at least one prescribed entry, and that P has exactly t columns and exactly s rows with one prescribed entry. Those assumptions give us:

$$\max\{\#P^{(1)},\ldots,\#P^{(n)}\}\leqslant t-1 \quad\text{and}\quad \max\{\#P_{(1)},\ldots,\#P_{(n)}\}\leqslant s-1.$$

In particular, this means that the first t rows of P can not have the same pattern of prescribed entries, and that the first $s \leqslant t$ columns of P can not all have their prescribed entry in the same position. We may assume that the first and the second rows of P have different patterns of prescribed entries.

If $i_1 \neq i_2$, then we consider the matrix $Q = P$. If $i_1 = i_2$ then we chose some $k \leqslant t$ such that $i_1 \neq i_k$. Then either rows 1 and k have different pattern of prescribed entries or rows 2 and k have different pattern of prescribed entries. Without loss of generality we may assume that rows 1 and k have different pattern of prescribed entries. Then we consider the matrix $Q = \tau(P)$ where $\tau \in S_n$ is the transposition of 2 and k.

We have obtained $Q \in \mathcal{E}(P)$ of the form

$$Q=\left(\begin{array}{cc|ccccc}
\square & \square & \cdots & \mathbf{c} & \cdots & ? & \cdots\\
\square & \square & \cdots & ? & \cdots & \mathbf{d} & \cdots\\
\hline
\vdots & \vdots & ? & & \cdots & & ?\\
\mathbf{a} & \square & & & & & \\
\vdots & \vdots & \vdots & & \ddots & & \vdots\\
\square & \mathbf{b} & & & & & \\
\vdots & \vdots & ? & & \cdots & & ?
\end{array}\right)\in\widehat{\mathcal{M}}_n$$

with $\#Q^{(1)} = \#Q^{(2)} = 1$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \Re$, where $\mathbf{a}$ and $\mathbf{b}$ are in different rows, and $\mathbf{c}$ and $\mathbf{d}$ are in different columns.

- $\Gamma_\lambda(Q)$ satisfies **S1**, **S2**, **S3**, **S4**, **S5** and **S6** by Lemma 3.2. $\square$

3.3. Completions

Let $n \geqslant 3$, $Q \in \widehat{\mathcal{M}}_n$ and $\lambda \in \Re$. Then $\Gamma_\lambda(Q)$ is well defined. In this section we show for every completion B of $\Gamma_\lambda(Q)$ how to construct a completion A of Q with spectrum the spectrum of B with λ adjoined.

Let $Q \in \widehat{\mathcal{M}}_n$ and $\Gamma_\lambda(Q) \in \overline{\mathcal{M}}_{n-1}$ be given as in (3) and (4), and let

$$B=\left(\begin{array}{c|ccc}
b_{11} & b_{12} & \cdots & b_{1,n-1}\\
\hline
b_{21} & b_{22} & \cdots & b_{2,n-1}\\
\vdots & \vdots & \ddots & \vdots\\
b_{n-1,1} & b_{n-1,2} & \cdots & b_{n-1,n-1}
\end{array}\right)\in\mathcal{M}_{n-1}$$

be a completion of $\Gamma_\lambda(Q)$. The tables below show how matrix

$$A=\left(\begin{array}{cc|ccc}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n}\\
\hline
a_{31} & a_{32} & a_{33} & \cdots & a_{3n}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{array}\right)\in\mathcal{M}_n$$

can be constructed:

1. The entries in $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ depend on $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, λ and b_{11} as follows:

$\begin{matrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{matrix}$	$\Gamma_\lambda \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$	b_{11}	$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}$
$\begin{matrix} \square & \square \\ \square & \square \end{matrix}$	$\square$	a	$\begin{matrix} a & a-\lambda \\ 0 & \lambda \end{matrix}$
$\begin{matrix} r & \square \\ \square & \square \end{matrix}$	$\square$	a	$\begin{matrix} r & r-\lambda \\ a-r & a-r+\lambda \end{matrix}$
$\begin{matrix} \square & r \\ \square & \square \end{matrix}$	$\square$	a	$\begin{matrix} r+\lambda & r \\ a-r-\lambda & a-r \end{matrix}$
$\begin{matrix} \square & \square \\ r & \square \end{matrix}$	$\square$	a	$\begin{matrix} a-r & a-r-\lambda \\ r & r+\lambda \end{matrix}$
$\begin{matrix} \square & \square \\ \square & r \end{matrix}$	$\square$	a	$\begin{matrix} a-r+\lambda & a-r \\ r-\lambda & r \end{matrix}$
$\begin{matrix} r & \square \\ s & \square \end{matrix}$	$r+s$	$r+s$	$\begin{matrix} r & r-\lambda \\ s & s+\lambda \end{matrix}$
$\begin{matrix} \square & r \\ \square & s \end{matrix}$	$r+s$	$r+s$	$\begin{matrix} r+\lambda & r \\ s-\lambda & s \end{matrix}$
$\begin{matrix} r & \square \\ \square & s \end{matrix}$	$r+s-\lambda$	$r+s-\lambda$	$\begin{matrix} r & r-\lambda \\ s-\lambda & s \end{matrix}$
$\begin{matrix} \square & r \\ s & \square \end{matrix}$	$r+s+\lambda$	$r+s+\lambda$	$\begin{matrix} r+\lambda & r \\ s & s+\lambda \end{matrix}$

.

Note that $Q \in \widehat{\mathcal{M}}_n$ implies $\#(q_{i1}\ q_{i2}) \leqslant 1$ for $i = 1, 2$ which explains why the first column in the previous table considers all possibilities.

2. For $j = 3, \ldots, n$ the entries in $\begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix}$ depend on $\begin{pmatrix} q_{1j} \\ q_{2j} \end{pmatrix}$ and $b_{1,j-1}$ as follows:

$\begin{matrix} q_{1j} \\ q_{2j} \end{matrix}$	$q_{1j} \oplus q_{2j}$	$b_{1,j-1}$	$\begin{matrix} a_{1j} \\ a_{2j} \end{matrix}$
$\begin{matrix} \square \\ \square \end{matrix}$	$\square$	a	$\begin{matrix} 0 \\ a \end{matrix}$
$\begin{matrix} r \\ \square \end{matrix}$	$\square$	a	$\begin{matrix} r \\ a-r \end{matrix}$
$\begin{matrix} \square \\ r \end{matrix}$	$\square$	a	$\begin{matrix} a-r \\ r \end{matrix}$
$\begin{matrix} r \\ s \end{matrix}$	$r+s$	$r+s$	$\begin{matrix} r \\ s \end{matrix}$

.

3. For $i = 3, \ldots, n$ the entries in $(a_{i1}\ a_{i2})$ depend on $b_{i-1,1}$ as follows:

$q_{i1}\quad q_{i2}$	$q_{i1} \odot q_{i2}$	$b_{i-1,1}$	$a_{i1}\quad a_{i2}$
$\square\quad \square$	$\square$	a	$a\quad a$
$r\quad \square$	r	r	$r\quad r$
$\square\quad r$	r	r	$r\quad r$

.

Note that $Q \in \widehat{\mathcal{M}}_n$ implies $\#(q_{i1}\ q_{i2}) \leqslant 1$ for $i = 3, \ldots, n$ which explains why the first column in the previous table considers all possibilities.

4. For $i, j = 3, \ldots, n$ the entry a_{ij} depends on $b_{i-1,j-1}$ as follows:

q_{ij}	$b_{i-1,j-1}$	a_{ij}
$\square$	a	a
r	r	r

.

It is clear that A is a completion of Q. Note that matrices B and A are of the form (1) and (2) respectively, so Lemma 3.1 can be applied to conclude that the spectrum of A consists of all the eigenvalues of B with λ adjoined.

4. Main result

At this point we will rewrite our main result, Theorem 1.2, and give a proof that implicitly contains the algorithm to construct a solution matrix.

Theorem 4.1. *For $n \geqslant 2$ let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a given multiset of elements in an integral domain $\mathfrak{R}$. Let P be a matrix of order n with at most $2n-3$ prescribed entries that belong to $\mathfrak{R}$, and such that the diagonal and each line of P have at least one element unprescribed. Then P can be completed with elements of $\mathfrak{R}$ to obtain a matrix with spectrum Λ.*

Proof. We begin by showing that the theorem holds for $n = 2$. Assume $P \in \overline{\mathcal{M}}_2$ is a matrix that satisfies the conditions of Theorem 4.1, i.e. P has at most one prescribed entry. Then P is of one of the types in the table below, where $r \in \mathfrak{R}$. The second column of the table shows the desired completions of P with spectrum $\{\lambda_1, \lambda_2\}$.

P	completion of P
$\begin{matrix} \square & \square \\ \square & \square \end{matrix}$	$\begin{matrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{matrix}$
$\begin{matrix} r & \square \\ \square & \square \end{matrix}$	$\begin{matrix} r & r-\lambda_2 \\ \lambda_1 - r & \lambda_1+\lambda_2-r \end{matrix}$
$\begin{matrix} \square & \square \\ \square & r \end{matrix}$	$\begin{matrix} \lambda_1+\lambda_2-r & \lambda_1 - r \\ r-\lambda_2 & r \end{matrix}$
$\begin{matrix} \square & r \\ \square & \square \end{matrix}$	$\begin{matrix} \lambda_1 & r \\ 0 & \lambda_2 \end{matrix}$
$\begin{matrix} \square & \square \\ r & \square \end{matrix}$	$\begin{matrix} \lambda_1 & 0 \\ r & \lambda_2 \end{matrix}$

We proceed by induction on n. Let $P \in \overline{\mathcal{M}}_n$ satisfy conditions of Theorem 4.1, and let $\{\lambda_1, \ldots, \lambda_n\}$ be a given multiset of elements that belong to $\mathfrak{R}$. In Lemma 3.3 we showed how to find in the equivalence class $\mathcal{E}(P)$ a matrix $Q \in \widehat{\mathcal{M}}_n$ such that $\Gamma_{\lambda_n}(Q) \in \overline{\mathcal{M}}_{n-1}$ is a matrix that satisfies conditions of Theorem 4.1. By induction hypothesis, $\Gamma_{\lambda_n}(Q)$ can be completed to a matrix $B \in \mathcal{M}_{n-1}$ with spectrum $\{\lambda_1, \ldots, \lambda_{n-1}\}$. In Section 3.3 we showed how to construct a matrix $A \in \mathcal{M}_n$ with the spectrum equal to the spectrum of B with λ_n adjoined and such that A is a completion of Q.

Any matrix in the equivalence class $\mathcal{E}(A)$ has spectrum $\{\lambda_1, \ldots, \lambda_n\}$. Since $Q \in \mathcal{E}(P)$ then there exists some $\tau \in S_n$ such that $Q = \tau(P)$ or $Q = \tau(P^T)$, therefore we conclude that $\tau^{-1}(A)$ or $\tau^{-1}(A^T)$ is a desired completion of matrix P. $\square$

Remark 4.1. This paper concentrates on completion problems over integral domains. Using similar methods and working over fields, we were able to develop a rational algorithm that completely covers Theorem 1.1, including the case where we want to complete a matrix that contains a line with all its elements prescribed. This algorithm will be presented in a forthcoming paper.

Acknowledgment

We would like to thank the referee for his comments, specially for the proof of Lemma 3.1 that he provided.

References

[1] Alberto Borobia, Inverse eigenvalue problems, in: Leslie Hogben (Ed.), Handbook of Linear Algebra, Discrete Mathematics and its Applications, Chapman and Hall/CRC, Boca Raton, FL, 2007, Chapter 20.
[2] Moody T. Chu, Fasma Diele, Ivonne Sgura, Gradient flow methods for matrix completion with prescribed eigenvalues, in: Tenth Conference of the International Linear Algebra Society, Linear Algebra Appl. 379 (2004) 85–112.
[3] Moody T. Chu, Gene H. Golub, Inverse eigenvalue problems: theory, algorithms, and applications, Numerical Mathematics and Scientific Computation, Oxford University Press, New York, 2005.
[4] D. Hershkowitz, Existence of matrices with prescribed eigenvalues and entries, Linear and Multilinear Algebra 14 (4) (1983) 315–342.
[5] Kh.D. Ikramov, V.N. Chugunov, Inverse matrix eigenvalue problems, J. Math. Sci. (New York) 98 (1) (2000) 51–136.
[6] Helena Šmigoc, The inverse eigenvalue problem for nonnegative matrices, Linear Algebra Appl. 393 (2004) 365–374.