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Construction of biorthogonal wavelet vectors

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ABSTRACT

The construction of all possible biorthogonal wavelet vectors corresponding to a given biorthogonal scaling vector may not be easy as that of biorthogonal uniwavelets. In this paper, we give some theorems about the construction of biorthogonal wavelet vectors, which is followed by simple computations for constructing all parametrized biorthogonal wavelet vectors supported in $[-1, 1]$. This approach is also suitable for the case of compactly supported orthogonal uniwavelet. Moreover, we give examples parametrizing all biorthogonal wavelet vectors corresponding to well known biorthogonal scaling vectors.

1. Introduction

Most orthonormal wavelets can be constructed using the multiresolution analysis formalism introduced by Mallat in [1]. In this setting there is a single scaling function ϕ whose integer translates form an orthonormal basis of a central approximation space V_0. In many applications it is desirable for ϕ to be compactly supported. Daubechies constructed such scaling functions in [2]. The conditions of orthonormality and compact support together are restrictive and it is known that certain other desirable properties such as symmetry and continuity cannot also be simultaneously achieved. Recently, two generalizations of orthonormal wavelets, namely biorthogonal wavelets and multiwavelets, have been introduced which have all desirable properties, see [4,6–8].

Biorthogonal wavelets are constructed using two dual multiresolution analyses generated by dual scaling functions ϕ and $\tilde{\phi}$ satisfying

$$\left\langle \phi(\cdot - n), \tilde{\phi}(\cdot - m) \right\rangle = \delta_{n,m}.$$

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In [3], Cohen, Daubechies and Feauveau give a method for finding a compactly supported dual scaling function $\tilde{\phi}$ given a compactly supported scaling function ϕ. Once the biorthogonal scaling functions are found there is a simple formula giving the biorthogonal wavelets. Also, if ϕ is symmetric then $\tilde{\phi}$ can be chosen to be symmetric as well. This allows great freedom in the choice of ϕ.

Orthonormal multiwavelets are constructed from $n(>1)$ scaling functions whose integer translates form an orthonormal basis of a central approximation space V_0. In [4,5], symmetric, compactly supported, continuous and orthogonal scaling functions (and associated wavelets) are constructed using $n = 2$ scaling functions. If the scaling functions have compact supports then by reindexing the multiresolution analysis the scaling functions can all be assumed to be supported in $[-1, 1]$. In [6] a general theory was developed for constructing orthonormal scaling vectors in $[-1, 1]$.

In [7], Hardin and Marasovich give a procedure for constructing biorthogonal wavelet vectors associated with a given pair of biorthogonal scaling vectors, the components of which have supports in $[-1, 1]$. And they provide necessary and sufficient conditions for the existence of biorthogonal multiwavelets supported in $[-1, 1]$. Further, they give biorthogonal scaling vectors and corresponding wavelet vectors of multiplicity 2.

In [8], Yang et al. give a procedure for constructing easily compactly supported biorthogonal wavelet vector associated with a given biorthogonal scaling vectors. But we note that it gives a way to obtain only one wavelet vector. Those construction procedures could not completely characterize the corresponding biorthogonal wavelet vectors.

In this paper, we give some theorems about the construction of biorthogonal wavelet vectors containing the maximal number of wavelet functions supported in $[0, 1]$ and generalize results in [7,9]. By using the existence theorem, we give the procedure for generalized construction of all possible biorthogonal wavelet vectors corresponding to biorthogonal scaling vectors.

2. Preliminaries

2.1. Multiwavelets

A single scaling function ϕ that generates a multiresolution analysis of $L^2(R)$ cannot be compactly supported, orthonormal, have any degree of regularity, and also be symmetric.

Recently, multiwavelets have been studied as a means of overcoming this obstacle. In the multiwavelet setting, $n(>1)$ scaling functions are used to generate a multiresolution analysis of $L^2(R)$. In [4,6,9] two scaling functions ϕ_1 and ϕ_2 are constructed that are compactly supported, orthonormal, continuous, and symmetric. These scaling functions generated symmetric and antisymmetric wavelets ψ_1 and ψ_2 that were also compactly supported, orthonormal, and continuous.

Similar to the concept of a multiresolution analysis generated from a single scaling function ϕ is the idea of a multiresolution analysis of multiplicity $n > 1$ for multiwavelets. Let $\Phi = (\phi^1, \ldots, \phi^n)^T$ be a column vector of length n whose elements are in $L^2(R)$. Let $\tau(\Phi)$ denote the set of all integer translates of components of Φ, that is $\tau(\Phi) = \{\phi^i(\cdot - j) | i = 1, \ldots, n\,;\ j \in Z\}$ and let $\sigma(\Phi)$ denote the L^2 closed linear span of $\tau(\Phi)$. A space V is called a *finitely-generated shift invariant* (FSI) space if $V = \sigma(\Phi)$ for some finite-length vector Φ. In this case Φ is called a *generating vector* for V.

We will be most interested in FSI spaces that arise from a multiresolution analysis: A *multiresolution analysis of multiplicity n* is a sequence of closed linear subspaces $(V_p)_{p\in Z}$ in $L^2(R)$ satisfying the following:

(1) $\cdots V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$,
(2) $\overline{\bigcup_{p\in Z} V_p} = L^2(R)$ and $\bigcap_{p\in Z} V_p = \{0\}$,
(3) $f \in V_p$ if and only if $f(2^{-p}\cdot) \in V_0,\ \forall p \in Z$,
(4) V_0 is an FSI space generated by some n vector $\Phi = (\phi^1, \ldots, \phi^n)^T$ such that $\tau(\Phi)$ is a Riesz basis of V_0.

We call Φ a *scaling vector* for $(V_p)_{p\in Z}$ and the components of Φ are called *scaling functions*.

Without loss of generality, let us restrict our attention to multiresolution analyses with scaling functions supported in $[-1, 1]$ such that their support meets $(0, 1)$ [6,11].

Let $\chi_{[0,1]}$ denote the characteristic function of $[0, 1]$ and let Φ have support $[-1, 1]$ and be such that none of the components have support contained in $[-1, 0]$. If Φ is such that the set of nonzero restrictions of the components of Φ and their integer shifts restricted to $[0, 1]$, i.e.,

$$\{\phi_{\chi_{[0,1]}} | \phi \in \tau(\Phi), \phi\chi_{[0,1]} \neq 0\}$$

is linearly independent, then we say that Φ is *minimally supported in* $[-1, 1]$ (or just *minimally supported* for short). By the following lemma [6], we may assume that any multiresolution analysis is generated by a scaling vector minimally supported in $[-1, 1]$.

Lemma 2.1. *Suppose* $(V_p)_{p\in Z}$ *is multiresolution analysis generated by compactly supported scaling functions.Then there are some n and some set of scaling functions minimally supported in* $[-1, 1]$ *that generate the multiresolution analysis* $(V'_p)_{p\in Z}$ *given by*

$$V'_p = V_{p+n}.$$

If Φ is minimally supported we say $V = \sigma(\Phi)$ is *minimally generated* and we denote the number of generators supported in $[-1, 1]$ but not supported in $[0, 1]$ by $k = k(\Phi)$. If $\widetilde{\Phi}$ is minimally supported and generates the same FSI space V as Φ, i.e., $\sigma(\widetilde{\Phi}) = V$, then it follows that $k(\Phi) = k(\widetilde{\Phi})$. Thus we can define $k(V) := k(\Phi)$ independent of the choice of minimally supported generating vector. Also any component ϕ_i of Φ that is supported in $[-1, 0]$ may be replaced by its shift $\phi^i(\cdot - 1)$ and so we will assume that supports of all of the components of Φ meet $(0, 1)$. Furthermore, we order the component of Φ minimally supported in $[-1, 1]$ such that the components $\phi^{k(V)+1}, \ldots, \phi^n$ are supported in $[0, 1]$ while the supports of the components $\phi^1, \ldots, \phi^{k(V)}$ meet $(-1, 0)$ and $(0, 1)$. In the sequel we will assume that the components of any minimally supported generating vector are so ordered.

2.2. Biorthogonal multiwavelets

Let multiresolution analyses $(V_p)_{p\in Z}$ and $(\widetilde{V}_p)_{p\in Z}$ be generated by scaling vectors $\Phi = (\phi^1, \ldots, \phi^n)^T$ and $\widetilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^n)^T$. Then we say that $(V_p)_{p\in Z}$ and $(\widetilde{V}_p)_{p\in Z}$ are biorthogonal with respect to $[-1, 1]$ (or just biorthogonal for short) [11] if $k(V_0) = k(\widetilde{V}_0)$ and Φ and $\widetilde{\Phi}$ are minimally supported generating vectors for V_0 and $\widetilde{V}_0$, respectively, such that

$$\left\langle \phi^i(\cdot), \tilde{\phi}^j(\cdot - k) \right\rangle = \delta_{ij}\delta_{0k} \quad \text{for} \quad i, j = 1, 2, \ldots, n \text{ and } k \in Z.$$

These vector functions Φ and $\widetilde{\Phi}$ are called biorthogonal scaling vectors. And the vector functions $\Psi = (\psi^1, \ldots, \psi^n)^T$ and $\widetilde{\Psi} = (\tilde{\psi}^1, \ldots, \tilde{\psi}^n)^T$ are called biorthogonal wavelet vectors if $\tau(\Phi) \cup \tau(\Psi)$ and $\tau(\widetilde{\Phi}) \cup \tau(\widetilde{\Psi})$ are Riesz bases of V_1 and $\widetilde{V}_1$, respectively, and

$$2^{\frac{l}{2}} 2^{\frac{m}{2}} \left\langle \psi^i(2^l \cdot), \tilde{\psi}^j(2^m \cdot - k) \right\rangle = \delta_{ij}\delta_{lm}\delta_{0k} \quad \text{for} \quad i, j = 1, 2, \ldots, n \text{ and } k, l, m \in Z.$$

In the paper [7], Hardin and Marsovich developed a theory of constructing compactly supported biorthogonal multiwavelets, corresponding to given compactly supported biorthogonal scaling vectors Φ and $\widetilde{\Phi}$ generating multiresolution analyses $(V_p)_{p\in Z}$ and $(\widetilde{V}_p)_{p\in Z}$, respectively. Let us assume that compactly supported biorthogonal scaling vectors Φ and $\widetilde{\Phi}$ can be found that generate multiresolution analyses $(V_p)_{p\in Z}$ and $(\widetilde{V}_p)_{p\in Z}$, respectively. It is desired that our scaling vectors Φ and $\widetilde{\Phi}$ and the associated wavelet vectors Ψ and $\widetilde{\Psi}$ generate a biorthogonal system.

Since our scaling and wavelet functions are compactly supported in $[-1, 1]$, only a finite number of matrix dilation coefficients will be nonzero, i.e.,

$$\begin{aligned} \Phi(x) &= \sum_{i=-2}^{1} C_i \Phi(2x - i), \\ \widetilde{\Phi}(x) &= \sum_{i=-2}^{1} \widetilde{C}_i \widetilde{\Phi}(2x - i). \end{aligned} \tag{1}$$

And the wavelet generating vectors Ψ and $\widetilde{\Psi}$ can be written in terms of the scaling functions at the next finest scale

$$\begin{aligned} \Psi(x) &= \sum_{i=-2}^{1} D_i \Phi(2x-i), \\ \widetilde{\Psi}(x) &= \sum_{i=-2}^{1} \widetilde{D}_i \widetilde{\Phi}(2x-i). \end{aligned} \tag{2}$$

Let $l_m = [C_{2m}\ C_{2m+1}]$ and $\widetilde{l}_m = [\widetilde{C}_{2m}\ \widetilde{C}_{2m+1}]$ for $m \in Z$. Then the following theorem gives the necessary and sufficient conditions for given biorthogonal scaling vectors Φ and $\widetilde{\Phi}$ to have the associated biorthogonal wavelet vectors Ψ and $\widetilde{\Psi}$, respectively.

Theorem 1 [7]. *Let multiresolution analyses $(V_p)_{p\in Z}$ and $(\widetilde{V}_p)_{p\in Z}$ be biorthogonal multiresolution analyses generated by scaling vectors $\Phi = (\phi^1, \ldots, \phi^n)^T$ and $\widetilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^n)^T$. Then there exist biorthogonal wavelet vectors Ψ and $\widetilde{\Psi}$ if and only if*

$$\text{range}\, l_0^T \cap \text{range}\, l_{-1}^T = \{0\},$$
$$\text{range}\, \widetilde{l}_0^T \cap \text{range}\, \widetilde{l}_{-1}^T = \{0\}.$$

3. Construction of biorthogonal multiwavelets

If a compactly supported scaling function is found that generates a multiresolution analysis of $L^2(R)$, then a construction of the compactly supported wavelet is given easily by Daubechies [2]. But for scaling vector generating a multiresolution analysis of multiplicity $n > 1$, it is not easy to find the corresponding wavelet vector. Theorem 1 gives the necessary and sufficient conditions that the biorthogonal scaling vectors Φ and $\widetilde{\Phi}$ in multiresolution analysis of multiplicity n have the corresponding wavelet vectors Ψ and $\widetilde{\Psi}$.

In this section, we give several lemmas about biorthogonality. And by using these lemmas, we develop two theorems with proofs that will illustrate the simple computational algorithm constructing complete form of biorthogonal wavelet vectors corresponding to given scaling vectors of biorthogonal multiresolution analyses.

If not mentioned otherwise, all subspaces are assumed to be of finite dimension in $L^2(R)$.

We need the following lemma to provide a necessary and sufficient condition on the subspaces U and $\widetilde{U}$ that enables us to construct biorthogonal bases.

Lemma 3.1. *Let U and $\widetilde{U}$ be subspaces of the same finite dimension. Then $U \cap \widetilde{U}^\perp = \{0\}$ is equivalent to $U^\perp \cap \widetilde{U} = \{0\}$.*

Proof. Let $u = (u_1, \ldots, u_n)$ and $\tilde{u} = (\tilde{u_1}, \ldots, \tilde{u_n})$ be bases of U and $\widetilde{U}$, respectively. Then $U \cap \widetilde{U}^\perp \neq \{0\}$ if and only if there exists a nonzero vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in R^n$ such that $v = \sum_{i=1}^n \alpha_i u_i$ and for every i,

$$\langle \tilde{u}_i, v\rangle = \sum_{j=1}^{n} \langle \tilde{u}_i, u_j\rangle \alpha_j = 0.$$

Putting $G = (g_{ij})$ by $g_{ij} = \langle \tilde{u}_i, u_j\rangle$, it implies that $U \cap \widetilde{U}^\perp = \{0\}$ if and only if $n \times n$ matrix G is nonsingular. Since the nonsingularity of G is equivalent to that of G^T, $U \cap \widetilde{U}^\perp = \{0\}$ must be equivalent to $U^\perp \cap \widetilde{U} = \{0\}$. □

By the above equivalence of $U \cap \widetilde{U}^\perp = \{0\}$ and $U^\perp \cap \widetilde{U} = \{0\}$ we have the following necessary and sufficient condition for constructing biorthogonal bases.

Lemma 3.2. *Let U and $\widetilde{U}$ be subspaces of the same finite dimension. If $u = (u_1, \ldots, u_n)$ is a basis for U, then there exists a basis $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)$ for $\widetilde{U}$ which is biorthogonal to u if and only if $U \cap \widetilde{U}^\perp = \{0\}$.*

Proof. Let $u = (u_1, \ldots, u_n)$ be a basis for U. Suppose $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)$ is the dual basis for $\widetilde{U}$. Assume $v \in U \cap \widetilde{U}^{\perp}$. Since $v \in U$ there exist $\alpha_i \in R$, $i = 1, \ldots, n$ such that $v = \sum_{i=1}^{n} \alpha_i u_i$. In addition since $v \in \widetilde{U}^{\perp}$, for each j

$$\langle v, \tilde{u}_j \rangle = \sum_{i=1}^{n} \alpha_i \langle u_i, \tilde{u}_j \rangle = \alpha_j = 0.$$

This implies $v = 0$, hence $U \cap \widetilde{U}^{\perp} = \{0\}$. Conversely assume that $U \cap \widetilde{U}^{\perp} = \{0\}$ and $\{\tilde{v}_1, \ldots, \tilde{v}_n\}$ is a basis for $\widetilde{U}$. Define $n \times n$ matrix $G = (g_{ij})$ by $g_{ij} = \langle u_i, \tilde{v}_j \rangle$. In the proof of Lemma 3.1 $G = (g_{ij})$ is shown to be nonsingular. If we set $G^{-1} = (h_{ij})_{n \times n}$, then

$$\sum_k \langle u_i, \tilde{v}_k \rangle \, h_{kj} = \left\langle u_i, \sum_k h_{kj} \tilde{v}_k \right\rangle = \delta_{ij}.$$

Let $\tilde{u}_j = \sum_k h_{kj} \tilde{v}_k$, then $\langle u_i, \tilde{u}_j \rangle = \delta_{ij}$. Hence $\tilde{u} = \{\tilde{u}_1, \ldots, \tilde{u}_n\}$ is the basis for $\widetilde{U}$, such that u and $\tilde{u}$ are biorthogonal. □

By the Lemma 3.2, we define biorthogonal subspaces as follows.

Definition 3.1. Let U and $\widetilde{U}$ be subspaces of the same dimension. If $U \cap \widetilde{U}^{\perp} = \{0\}$, then U is said to be biorthogonal to $\widetilde{U}$.

If V is a subspace that is orthogonal to $\widetilde{U}$, then V is a subset of $\widetilde{U}^{\perp}$, which means that $U \cap V \subset U \cap \widetilde{U}^{\perp}$. Hence we can get the following lemma obviously.

Lemma 3.3. *Let U and $\widetilde{U}$ be subspaces such that $U \cap \widetilde{U}^{\perp} = \{0\}$. If a subspace V is orthogonal to $\widetilde{U}$, then $U \cap V = \{0\}$.*

To prove our main results we need the following lemmas.

Lemma 3.4. *Let W_0 and $\widetilde{W}_0$ be subspaces such that $W_0 \cap \widetilde{W}_0^{\perp} = \{0\}$. If V_0 is orthogonal to $\widetilde{W}_0$, then $(V_0 \oplus W_0) \cap \widetilde{W}_0^{\perp} = V_0$.*

Proof. Assume that $W_0 \cap \widetilde{W}_0^{\perp} = \{0\}$. If V_0 is orthogonal to $\widetilde{W}_0$, then by Lemma 3.3 $V_0 \cap W_0 = \{0\}$ and $V_0 \subset \widetilde{W}_0^{\perp}$. Hence $V_0 \subset (V_0 \oplus W_0) \cap \widetilde{W}_0^{\perp}$. To show $(V_0 \oplus W_0) \cap \widetilde{W}_0^{\perp} \subset V_0$, let $u \in (V_0 \oplus W_0) \cap \widetilde{W}_0^{\perp}$. Then $u = v + w$ for $v \in V_0$ and $w \in W_0$, and $u \in \widetilde{W}_0^{\perp}$. Since $V_0 \subset \widetilde{W}_0^{\perp}$, $w = u - v \in \widetilde{W}_0^{\perp}$, which implies $w \in W_0 \cap \widetilde{W}_0^{\perp} = \{0\}$. Hence $u = v \in V_0$. □

Lemma 3.5. *Let U and V be orthogonal to $\widetilde{V}$ and $\widetilde{U}$, respectively. If $(U + V) \cap (\widetilde{U} + \widetilde{V})^{\perp} = \{0\}$, then $U \cap V = \{0\}$.*

Proof. Suppose that $(U + V) \cap (\widetilde{U} + \widetilde{V})^{\perp} = \{0\}$. To show that $U \bigcap V = \{0\}$, assume that there exists a nonzero $w \in U \cap V$. Since $U \perp \widetilde{V}$ and $V \perp \widetilde{U}$, $U \subset \widetilde{V}^{\perp}$ and $V \subset \widetilde{U}^{\perp}$. Hence $w \in U \cap V \subset \widetilde{V}^{\perp} \cap \widetilde{U}^{\perp}$, and also $w \in U + V$. From the fact that $(\widetilde{U} + \widetilde{V})^{\perp} = \widetilde{U}^{\perp} \cap \widetilde{V}^{\perp}$,

$$(U + V) \cap (\widetilde{U} + \widetilde{V})^{\perp} = (U + V) \cap (\widetilde{U}^{\perp} \cap \widetilde{V}^{\perp}).$$

This implies that nonzero vector w is in $(U+V) \cap (\widetilde{U}+\widetilde{V})^{\perp}$, which contradicts to $(U+V) \cap (\widetilde{U}+\widetilde{V})^{\perp} = \{0\}$. Therefore $U \bigcap V = \{0\}$. □

For subspaces A, B of U and $\widetilde{A}$, $\widetilde{B}$ of $\widetilde{U}$, we have the following two lemmas.

Lemma 3.6. *Let $U = A + B$ and $\widetilde{U} = \widetilde{A} + \widetilde{B}$. If A and B are orthogonal to $\widetilde{B}$ and $\widetilde{A}$, respectively, then $U \cap \widetilde{U}^{\perp} = \{0\}$ if and only if $A \cap \widetilde{A}^{\perp} = \{0\}$ and $B \cap \widetilde{B}^{\perp} = \{0\}$.*

Proof. Suppose $U \cap \widetilde{U}^{\perp} = \{0\}$, and let B be orthogonal to $\widetilde{A}$. Then $B \subset \widetilde{A}^{\perp}$. But

$$\begin{aligned} U \cap \widetilde{U}^{\perp} &= (A+B) \cap (\widetilde{A}+\widetilde{B})^{\perp} \\ &= (A+B) \cap (\widetilde{A}^{\perp} \cap \widetilde{B}^{\perp}) \\ &\supset (A+B) \cap (B \cap \widetilde{B}^{\perp}) \\ &= ((A+B) \cap B) \cap \widetilde{B}^{\perp} = B \cap \widetilde{B}^{\perp}, \end{aligned}$$

which implies that $B \cap \widetilde{B}^{\perp} = \{0\}$. If A is orthogonal to $\widetilde{B}$, then we can easily get $A \cap \widetilde{A}^{\perp} = \{0\}$. Conversely, let $A \cap \widetilde{A}^{\perp} = \{0\}$ and $B \cap \widetilde{B}^{\perp} = \{0\}$. And assume that B is orthogonal to $\widetilde{A}$. Then

$$(A+B) \cap (\widetilde{A}+\widetilde{B})^{\perp} = (A+B) \cap \widetilde{A}^{\perp} \cap \widetilde{B}^{\perp}.$$

Since $A \cap \widetilde{A}^{\perp} = \{0\}$, by Lemma 3.4 $(A+B) \cap \widetilde{A}^{\perp} = B$. And since $B \cap \widetilde{B}^{\perp} = \{0\}$,

$$\begin{aligned} U \cap \widetilde{U}^{\perp} &= (A+B) \cap (\widetilde{A}+\widetilde{B})^{\perp} \\ &= (A+B) \cap \widetilde{A}^{\perp} \cap \widetilde{B}^{\perp} \\ &= B \cap \widetilde{B}^{\perp} \\ &= \{0\}. \end{aligned}$$

Hence $U \cap \widetilde{U}^{\perp} = \{0\}$. □

Lemma 3.7. *Let $U = A + B$ and $\widetilde{U} = \widetilde{A} + \widetilde{B}$. If A and B are orthogonal to $\widetilde{B}$ and $\widetilde{A}$, respectively, then U and $\widetilde{U}$ are biorthogonal if and only if A and B are biorthogonal to $\widetilde{A}$ and $\widetilde{B}$, respectively.*

Proof. Let A and B are orthogonal to $\widetilde{B}$ and $\widetilde{A}$, respectively, and suppose $U = A + B$ is biorthogonal to $\widetilde{U} = \widetilde{A} + \widetilde{B}$. Then by Lemma 3.6, $A \cap \widetilde{A}^{\perp} = \{0\}$ and $B \cap \widetilde{B}^{\perp} = \{0\}$. And by Lemma 3.5, $A \cap B = \{0\}$ and $\widetilde{A} \cap \widetilde{B} = \{0\}$. Hence $U = A \oplus B$ and $\widetilde{U} = \widetilde{A} \oplus \widetilde{B}$. Now to show that $\dim A = \dim \widetilde{A}$ and $\dim B = \dim \widetilde{B}$, let c and $\tilde{c}$ be coordinate functions for given biorthogonal bases of U and $\widetilde{U}$. Since B is orthogonal to $\widetilde{A}$, $c(B)$ is orthogonal to $\tilde{c}(\widetilde{A})$ in $R^{\dim U}$, and $\dim c(B) + \dim c(A) = \dim U$. Hence $\dim \widetilde{A} = \dim \tilde{c}(\widetilde{A}) \leqslant \dim c(A) = \dim A$. Since A is also orthogonal to $\widetilde{B}$, $\dim A \leqslant \dim \widetilde{A}$. That is, $\dim A = \dim \widetilde{A}$. And similarly we have also $\dim B = \dim \widetilde{B}$. It follows that A and B are biorthogonal to $\widetilde{A}$ and $\widetilde{B}$, respectively.

Conversely, let A and B be biorthogonal to $\widetilde{A}$ and $\widetilde{B}$, respectively. By Lemma 3.6, $U \cap \widetilde{U}^{\perp} = \{0\}$. And by Lemma 3.3,

$$\begin{aligned} \dim U &= \dim A + \dim B \\ &= \dim \widetilde{A} + \dim \widetilde{B} \\ &= \dim \widetilde{U}. \end{aligned}$$

Hence U is biorthogonal to $\widetilde{U}$. □

Now we are ready to consider the problem of constructing biorthogonal multiwavelets corresponding to a given pair of biorthogonal scaling vectors.

Let $(V_p)_{p \in Z}$ and $(\widetilde{V}_p)_{p \in Z}$ be biorthogonal multiresolution analyses of multiplicity n, and let $\Phi = (\phi^1, \ldots, \phi^n)^T$ and $\widetilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^n)$ be the corresponding multiscaling vectors, respectively. Assume that Φ and $\widetilde{\Phi}$ are *minimally supported* in $[-1, 1]$, that is, the nonzero restrictions to the unit interval $[0, 1]$ of the integer shifts of the scaling functions are linearly independent, none of the scaling functions are supported in $[-1, 0]$, and all of the scaling functions are supported in $[-1, 1]$. And let $k = k(V) = k(\widetilde{V})$. To simplify the notation, we let

$$\phi^i_{m,j}(x) := 2^{\frac{m}{2}} \phi^i(2^m x - j).$$

Since the scaling functions ϕ^i are supported in $[-1, 1]$, each of these can be represented as linear combination of the following functions:

$\phi^j_{1,-2},\ \phi^i_{1,-1},\ \phi^l_{1,0},\ \phi^j_{1,0},\ \phi^i_{1,1},$

where $j = k+1, \ldots, n$, $i = 1, \ldots, n$, $l = 1, \ldots, k$.

For scaling vector $\Phi = (\phi^1, \ldots, \phi^n)^T$ minimally supported in $[-1, 1]$, let

$$\begin{aligned}
\Phi_0 &:= \left(\phi^1, \ldots, \phi^k\right)^T, \\
\Phi_r &:= \left(\phi^{k+1}, \ldots, \phi^n\right)^T, \\
\Phi_{1l} &:= \left(\phi^{k+1}_{1,-2}, \ldots, \phi^n_{1,-2}, \phi^1_{1,-1}, \ldots, \phi^n_{1,-1}\right)^T, \\
\Phi_{10} &:= \left(\phi^1_{1,0}, \ldots, \phi^k_{1,0}\right)^T, \\
\Phi_{1r} &:= \left(\phi^{k+1}_{1,0}, \ldots, \phi^n_{1,0}, \phi^1_{1,1}, \ldots, \phi^n_{1,1}\right)^T.
\end{aligned}$$

Similarly we denote by $\widetilde{\Phi}_0, \widetilde{\Phi}_r, \widetilde{\Phi}_{1l}, \widetilde{\Phi}_{10}, \widetilde{\Phi}_{1r}$ for the dual scaling vector $\widetilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^n)^T$. And also for biorthogonal wavelet vectors $\Psi, \widetilde{\Psi}$ we follow the same notations as in $\Phi, \tilde{\Phi}$. Denoting the span of the components of Φ by span Φ, we let

$$\begin{aligned}
Q_0 &:= \operatorname{span} \Phi_0, \\
Q_r &:= \operatorname{span} \Phi_r, \\
Q_{1l} &:= \operatorname{span} \Phi_{1l}, \\
Q_{10} &:= \operatorname{span} \Phi_{10}, \\
Q_{1r} &:= \operatorname{span} \Phi_{1r},
\end{aligned}$$

and also $\widetilde{Q}_0, \widetilde{Q}_r, \widetilde{Q}_{1l}, \widetilde{Q}_{10}, \widetilde{Q}_{1r}$ for $\tilde{\Phi}$. Since the scaling functions $\Phi, \tilde{\Phi}$ are minimally supported in $[-1, 1]$, we have

$$\begin{aligned}
V_{1[-1,1]} &:= \{f \in V_1 \mid \operatorname{supp} f \subset [-1, 1]\} \\
&= Q_{1l} \oplus Q_{10} \oplus Q_{1r}, \\
\widetilde{V}_{1[-1,1]} &:= \{f \in \widetilde{V}_1 \mid \operatorname{supp} f \subset [-1, 1]\} \\
&= \widetilde{Q}_{1l} \oplus \widetilde{Q}_{10} \oplus \widetilde{Q}_{1r}.
\end{aligned}$$

Define three canonical projections restricted to Q_0 as

$$\begin{aligned}
\pi_l &: Q_0(\cdot - 1) \longrightarrow Q_{1l}(\cdot - 1), \\
\pi_0 &: Q_0 \longrightarrow Q_{10}, \\
\pi_r &: Q_0 \longrightarrow Q_{1r},
\end{aligned}$$

and also the duals $\tilde{\pi}_l, \tilde{\pi}_0, \tilde{\pi}_r$ of $\pi_l, \pi_0, \tilde{\pi}_r$ to $\widetilde{Q}_{1l}(\cdot - 1), \widetilde{Q}_{10}, \widetilde{Q}_{1r}$, respectively. And let

$$\begin{aligned}
m_l &:= \dim \operatorname{Ker}(\pi_l), \quad \tilde{m}_l := \dim \operatorname{Ker}(\tilde{\pi}_l), \\
m &:= \dim \left(\operatorname{Ker}(\pi_l) \cap \operatorname{Ker}(\pi_r)\right), \\
\tilde{m} &:= \dim \left(\operatorname{Ker}(\tilde{\pi}_l) \cap \operatorname{Ker}(\tilde{\pi}_r)\right) \\
m_r &:= \dim \operatorname{Ker}(\pi_r), \quad \tilde{m}_r := \dim \operatorname{Ker}(\tilde{\pi}_r).
\end{aligned}$$

These canonical projections π_l, π_0, π_r and the duals have the following properties.

Lemma 3.8. *Let $(V_p)_{p\in Z}$ and $(\widetilde{V}_p)_{p\in Z}$ be biorthogonal multiresolution analyses generated by scaling vectors $\Phi = (\phi^1, \ldots, \phi^n)^T$ and $\widetilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^n)^T$. Then*

(1) $\mathrm{Im}(\pi_l) \perp \mathrm{Im}(\tilde{\pi}_r)$ and $\mathrm{Im}(\pi_r) \perp \mathrm{Im}(\tilde{\pi}_l)$,
(2) $Q_r \perp (\mathrm{Im}(\tilde{\pi}_l) + \mathrm{Im}(\tilde{\pi}_r))$ and $\widetilde{Q}_r \perp (\mathrm{Im}(\pi_l) + \mathrm{Im}(\pi_r))$,
(3) $(\mathrm{Im}(\pi_l) + \mathrm{Im}(\pi_r)) \cap Q_r = \{0\}$ and $(\mathrm{Im}(\tilde{\pi}_l) + \mathrm{Im}(\tilde{\pi}_r)) \cap \widetilde{Q}_r = \{0\}$,
(4) $(\mathrm{Im}(\pi_l) \oplus \mathrm{Im}(\pi_r)(\cdot + 1)) \cap Q_0 = \{0\}$ and $(\mathrm{Im}(\tilde{\pi}_l) \oplus \mathrm{Im}(\tilde{\pi}_r)(\cdot + 1)) \cap \widetilde{Q}_0 = \{0\}$,
(5) $Q_0 \cap Q_r = \{0\}$ and $\widetilde{Q}_0 \cap \widetilde{Q}_r = \{0\}$.

Proof. Let $\Phi = (\phi^1, \ldots, \phi^n)^T$ and $\widetilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^n)^T$ be scaling vectors minimally supported in $[-1, 1]$. And let $\Phi_r = (\phi^{k+1}, \ldots, \phi^n)^T$ and $\widetilde{\Phi}_r = (\tilde{\phi}^{k+1}, \ldots, \tilde{\phi}^n)^T$ be biorthogonal scaling vectors supported in $[0, 1]$. Then $Q_r = \mathrm{span}\,\Phi_r$ and $\widetilde{Q}_r = \mathrm{span}\,\widetilde{\Phi}_r$ are subspaces of Q_{1r} and $\widetilde{Q}_{1r}$, respectively.

(1) By the biorthogonality of scaling vectors Φ and $\widetilde{\Phi}$, $Q_0 = \mathrm{span}\,\Phi_0$ is orthogonal to $\widetilde{Q}_0(\cdot - 1) = \mathrm{span}\,\widetilde{\Phi}_0(\cdot - 1)$. Since Q_0 is a subset of $Q_{1l} \oplus Q_{10} \oplus Q_{1r}$,

$$\widetilde{Q}_0(\cdot - 1) \subset \widetilde{Q}_{1l}(\cdot - 1) \oplus \widetilde{Q}_{10}(\cdot - 1) \oplus \widetilde{Q}_{1r}(\cdot - 1).$$

But since $\widetilde{Q}_{1l}(\cdot - 1) = \widetilde{Q}_{1r}$ and $\langle \phi^i_{1,k}, \tilde{\phi}^j_{1,l} \rangle = \delta_{ij}\delta_{kl}$, $\mathrm{Im}(\pi_r)$ is orthogonal to $\mathrm{Im}(\tilde{\pi}_l)$. Similarly we can prove the orthogonality of $\mathrm{Im}(\pi_l)$ and $\mathrm{Im}(\tilde{\pi}_r)$.

(2) Since Φ_0 and $\widetilde{\Phi}_0$ are orthogonal to $\widetilde{\Phi}_r$ and Φ_r, respectively, $\mathrm{Im}(\pi_r)$ and $\mathrm{Im}(\tilde{\pi}_r)$ are orthogonal to $\widetilde{Q}_r$ and Q_r, respectively. Also since $\Phi_0(\cdot - 1)$ and $\widetilde{\Phi}_0(\cdot - 1)$ are orthogonal to $\widetilde{\Phi}_r$ and Φ_r, respectively, $\mathrm{Im}(\pi_l)$ and $\mathrm{Im}(\tilde{\pi}_l)$ are orthogonal to $\widetilde{Q}_r$ and Q_r, respectively.

(3) Since $\mathrm{Im}(\pi_l)$ and $\mathrm{Im}(\pi_r)$ are orthogonal to $\widetilde{Q}_r$, by Lemma 3.3

$$\mathrm{Im}(\pi_l) \cap Q_r = \{0\}, \quad \mathrm{Im}(\pi_r) \cap Q_r = \{0\}.$$

That is, $(\mathrm{Im}(\pi_l) + \mathrm{Im}(\pi_r)) \cap Q_r = \{0\}$. Similarly, we can obtain

$$(\mathrm{Im}(\tilde{\pi}_l) + \mathrm{Im}(\tilde{\pi}_r)) \cap \widetilde{Q}_r = \{0\}.$$

(4) Q_0 and $\widetilde{Q}_0$ are biorthogonal subspaces of $V_1[-1, 1]$. Since $\widetilde{\Phi}_0$ is orthogonal to $\Phi_0(\cdot + 1)$ and $\Phi_0(\cdot - 1)$, by Lemma 3.3

$$\mathrm{Im}(\pi_l) \oplus \mathrm{Im}(\pi_r)(\cdot + 1) \cap Q_0 = \{0\}.$$

Similarly, we have $\mathrm{Im}(\tilde{\pi}_l) \oplus \mathrm{Im}(\tilde{\pi}_r)(\cdot + 1) \cap \widetilde{Q}_0 = \{0\}$.

(5) Q_0 and Q_r are biorthogonal to $\widetilde{Q}_0$ and $\widetilde{Q}_r$, respectively. Since $Q_0 \perp \widetilde{Q}_r$ and $Q_r \perp \widetilde{Q}_0$, by Lemma 3.3 $Q_0 \cap Q_r = \{0\}$ and $\widetilde{Q}_0 \cap \widetilde{Q}_r = \{0\}$. □

Let $\Phi = (\phi^1, \ldots, \phi^n)^T$ and $\widetilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^n)^T$ be biorthogonal scaling vectors minimally supported in $[-1, 1]$, and let $\Psi = (\psi^1, \ldots, \psi^n)^T$ and $\widetilde{\Psi} = (\tilde{\psi}^1, \ldots, \tilde{\psi}^n)^T$ be the corresponding biorthogonal wavelet vectors minimally supported in $[-1, 1]$. The spans of wavelet functions are denoted by

$$R_0 = \mathrm{span}\,\Psi_0,$$
$$R_r = \mathrm{span}\,\Psi_r,$$

and also $\widetilde{R_0}$, $\widetilde{R}_r$ for $\widetilde{\Psi}$. Define canonical projections restricted to R_0 as

$$\eta_l : R_0(\cdot - 1) \to Q_{1l}(\cdot - 1),$$
$$\eta_0 : R_0 \to Q_{10},$$
$$\eta_r : R_0 \to Q_{1r},$$

and also the duals $\tilde{\eta}_l, \tilde{\eta}_0, \tilde{\eta}_r$.

Theorem 2. *Let $(V_P)_{P\in Z}$ and $(\widetilde{V}_P)_{P\in Z}$ be biorthogonal multiresolution analyses generated by scaling vectors $\Phi = (\phi^1, \ldots, \phi^n)^T$ and $\widetilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^n)^T$. And let $\Psi = (\psi^1, \ldots, \psi^n)^T$ and $\widetilde{\Psi} = (\tilde{\psi}^1, \ldots, \tilde{\psi}^n)^T$ be biorthogonal wavelet vectors associated with Φ and $\widetilde{\Phi}$, respectively. If* $\mathrm{Im}(\pi_l) \cap \mathrm{Im}(\pi_r) = \{0\}$ [resp., $\mathrm{Im}(\tilde{\pi}_l) \cap \mathrm{Im}(\tilde{\pi}_r) = \{0\}$], *then the number* $\tilde{\kappa}_r$ [resp., κ_r] *of wavelet function* $\tilde{\psi}^i$ [resp., ψ^i] *supported in* $[0, 1]$ *is*

$$\tilde{\kappa}_r \leqslant n - 2k + m_l + m_r \ \left[\text{resp., } \kappa_r \leqslant n - 2k + \tilde{m}_l + \tilde{m}_r\right].$$

Proof. Let $\mathrm{Im}(\pi_l) \cap \mathrm{Im}(\pi_r) = \{0\}$ and let $\Psi = (\psi^1, \ldots, \psi^n)^T$ and $\widetilde{\Psi} = (\tilde{\psi}^1, \ldots, \tilde{\psi}^n)^T$ be biorthogonal wavelet vectors. We can assume that $\widetilde{\Psi}$ has $n - \tilde{k}$ wavelet functions $\tilde{\psi}^i$ supported in $[0, 1]$, $i = \tilde{k} + 1, \ldots, n$. Suppose that $\tilde{k} < 2k - m_l - m_r$. Then $\psi^{\tilde{k}+1}, \ldots, \psi^n$ are wavelet functions biorthogonal to $\tilde{\psi}^{\tilde{k}+1}, \ldots, \tilde{\psi}^n$, but not supported in $[0, 1]$ in general. If we note that $\tilde{\psi}^{\tilde{k}+1}, \ldots, \tilde{\psi}^n$ are supported in $[0, 1]$, it can be seen that $\eta_r(\psi^{\tilde{k}+1}), \ldots, \eta_r(\psi^n)$ are in fact biorthogonal to these functions. Let

$$W_0 = \mathrm{span}\left(\eta_r\left(\psi^{\tilde{k}+1}\right), \ldots, \eta_r(\psi^n)\right),$$
$$\widetilde{W}_0 = \mathrm{span}\left(\tilde{\psi}^{\tilde{k}+1}, \ldots, \tilde{\psi}^n\right).$$

Then $\dim W_0 = \dim \widetilde{W}_0 = n - \tilde{k}$. By the orthogonality of Φ and $\widetilde{\Psi}$, $\mathrm{Im}(\pi_r)$ and Q_r are orthogonal to $\widetilde{W}_0$. By the orthogonality of $\Phi(\cdot - 1)$ and $\widetilde{\Psi}$, $\mathrm{Im}(\pi_l)$ is also orthogonal to $\widetilde{W}_0$. It implies that by Lemma 3.3 and 3.8,

$$(\mathrm{Im}(\pi_l) \oplus \mathrm{Im}(\pi_r) \oplus Q_r) \cap W_0 = \{0\}.$$

But $\mathrm{Im}(\pi_l), \mathrm{Im}(\pi_r), Q_r$ and W_0 are subspaces of Q_{1r}. Since $\mathrm{Ker}(\pi_l) = m_l$ and $\mathrm{Ker}(\pi_r) = m_r$, $\dim(\mathrm{Im}(\pi_l)) = k - m_l$ and $\dim(\mathrm{Im}(\pi_r)) = k - m_r$ and moreover $\dim Q_r = n - k$ and $\dim W_0 = n - \tilde{k}$. Hence $\mathrm{Im}(\pi_l) \oplus \mathrm{Im}(\pi_r) \oplus Q_r \oplus W_0$ is a subspace of Q_{1r}, and

$$\begin{aligned}\dim(\mathrm{Im}(\pi_l) \oplus \mathrm{Im}(\pi_r) \oplus Q_r \oplus W_0) &= k - m_l + k - m_r + n - k + n - \tilde{k}\\ &= 2n + k - m_l - m_r - \tilde{k}\\ &> 2n - k,\end{aligned}$$

which contradicts to $\dim Q_{1r} = 2n - k$. Therefore $\widetilde{\Psi}$ has at most $n - 2k + m_l + m_r$ wavelet functions supported in $[0, 1]$. Similarly we can show that Ψ has at most $n - 2k + \tilde{m}_l + \tilde{m}_r$ wavelet functions supported in $[0, 1]$. □

If $(V_p)_{p\in Z}$ is an orthogonal multiresolution analysis, then $\mathrm{Im}(\pi_l)$ and $\mathrm{Im}(\pi_r)$ are orthogonal. Thus we have the following corollary from the theorem.

Corollary 3.1 [9]. *Let $(V_n)_{n\in Z}$ be an orthogonal multiresolution analysis generated by scaling vector $\Phi = (\phi^1, \ldots, \phi^n)^T$ minimally supported in* $[-1, 1]$. *And let $\Psi = (\psi^1, \ldots, \psi^n)^T$ be a wavelet vector associated with the scaling vector Φ. Then the number of wavelet functions ψ^i supported in* $[0, 1]$ *is at most $n - 2k + m_l + m_r$.*

To prove our main theorem, we need the following lemma.

Lemma 3.9. *Let U and $\widetilde{U}$ be biorthogonal subspaces of finite dimension, and let $\widetilde{A}$ be a subspace of $\widetilde{U}$. Then there exists a subspace $B \subset U$ of dimension* $\dim \widetilde{U} - \dim \widetilde{A}$ *which is orthogonal to $\widetilde{A}$.*

Proof. Let $u = (u_1, \ldots, u_n)^T$ and $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_n)^T$ be biorthogonal bases for U and $\widetilde{U}$, respectively. And let $c : U \to R^n, \tilde{c} : \widetilde{U} \to R^n$ be coordinate functions, defined by $c(X^T u) = X, \tilde{c}(\widetilde{X}^T \tilde{u}) = \widetilde{X}$

for column vectors $X, \widetilde{X} \in R^n$. Let B be a subspace of U, defined by $B = c^{-1}(\tilde{c}(\widetilde{A})^{\perp})$. To show B is orthogonal to $\widetilde{A}$, let $v = \alpha^T u \in B, \alpha \in R^n$. Then $c(v) = \alpha \in \tilde{c}(\widetilde{A})^{\perp}$. If $\tilde{v} = \tilde{\alpha}^T \tilde{u} \in \widetilde{A}$ is any element, $\tilde{\alpha} \in R^n$, then $\tilde{c}(\tilde{v}) = \tilde{\alpha} \in \tilde{c}(\widetilde{A})$ is orthogonal to $c(v) = \alpha$. By the biorthogonality of u and $\tilde{u}$,

$$\langle v, \tilde{v} \rangle = \left\langle \alpha^T u, \tilde{\alpha}^T \tilde{u} \right\rangle = \alpha^T \tilde{\alpha} = 0.$$

Hence $B \perp \widetilde{A}$. □

Theorem 3. *Let $(V_p)_{p\in Z}$ and $(\widetilde{V}_p)_{p\in Z}$ be biorthogonal multiresolution analysis generated by scaling vectors $\Phi = (\phi^1, \ldots, \phi^n)^T$ and $\widetilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^n)^T$. Then $\mathrm{Im}(\pi_l) + \mathrm{Im}(\pi_r)$ is biorthogonal to $\mathrm{Im}(\tilde{\pi}_l) + \mathrm{Im}(\tilde{\pi}_r)$ if and only if $\gamma = \dim(\mathrm{Im}(\pi_l) + \mathrm{Im}(\pi_r)) = \dim(\mathrm{Im}(\tilde{\pi}_l) + \mathrm{Im}(\tilde{\pi}_r))$ and there exist biorthogonal wavelet vectors Ψ, $\widetilde{\Psi}$ each of which has exactly $n - \gamma$ wavelet functions supported in $[0, 1]$.*

Proof. Suppose $\mathrm{Im}(\pi_l) + \mathrm{Im}(\pi_r)$ is biorthogonal to $\mathrm{Im}(\tilde{\pi}_l) + \mathrm{Im}(\tilde{\pi}_r)$. Then $\dim(\mathrm{Im}(\pi_l) + \mathrm{Im}(\pi_r)) = \dim(\mathrm{Im}(\tilde{\pi}_l) + \mathrm{Im}(\tilde{\pi}_r))$ and

$$(\mathrm{Im}(\pi_l) + \mathrm{Im}(\pi_r)) \cap (\mathrm{Im}(\tilde{\pi}_l) + \mathrm{Im}(\tilde{\pi}_r))^{\perp} = \{0\},$$

which implies that by Lemma 3.1 and 3.5,

$$\mathrm{Im}(\pi_l) \cap \mathrm{Im}(\pi_r) = \{0\},$$
$$\mathrm{Im}(\tilde{\pi}_l) \cap \mathrm{Im}(\tilde{\pi}_r) = \{0\}.$$

Let $c : Q_{1r} \to R^{2n-k}$ be a coordinate system defined by $c(X^T \Phi_{1r}) = X$ for column vectors $X \in R^{2n-k}$. Similarly, we define a coordinate system $\tilde{c} : \tilde{Q}_{1r} \to R^{2n-k}$. Let

$$A = \mathrm{Im}(\pi_l) \oplus \mathrm{Im}(\pi_r) \oplus Q_r,$$
$$\tilde{A} = \mathrm{Im}(\tilde{\pi}_l) \oplus \mathrm{Im}(\tilde{\pi}_r) \oplus \widetilde{Q}_r$$

be subspaces of biorthogonal spaces Q_{1r} and $\widetilde{Q}_{1r}$, respectively. Then $\dim A = \dim \tilde{A} = n - k + \gamma$. By Lemma 3.9, there exist subspaces B and $\widetilde{B}$ of Q_{1r} and $\widetilde{Q}_{1r}$, respectively, such that $A \perp \widetilde{B}$, $B \perp \widetilde{A}$, and $\dim B = \dim \widetilde{B} = n - \gamma$. Now we note that $\mathrm{Im}(\pi_l) \oplus \mathrm{Im}(\pi_r)$ and Q_r are biorthogonal to $\mathrm{Im}(\tilde{\pi}_l) \oplus \mathrm{Im}(\tilde{\pi}_r)$ and $\tilde{Q}_r$, respectively. Moreover, $\mathrm{Im}(\pi_l) \oplus \mathrm{Im}(\pi_r)$ and Q_r are orthogonal to $\tilde{Q}_r$ and $\mathrm{Im}(\tilde{\pi}_l) \oplus \mathrm{Im}(\tilde{\pi}_r)$, respectively. Thus by Lemma 3.7, A is biorthogonal to $\tilde{A}$. Since $B \perp \widetilde{A}$ and $\widetilde{B} \perp A$, $A \cap B = \{0\}$ and $\widetilde{A} \cap \widetilde{B} = \{0\}$. It follows that

$$Q_{1r} = \mathrm{Im}(\pi_l) \oplus \mathrm{Im}(\pi_r) \oplus Q_r \oplus B,$$
$$\tilde{Q}_{1r} = \mathrm{Im}(\tilde{\pi}_l) \oplus \mathrm{Im}(\tilde{\pi}_r) \oplus \tilde{Q}_r \oplus \tilde{B}.$$

Since Q_{1r} is biorthogonal to $\tilde{Q}_{1r}$, by Lemma 3.7 B is biorthogonal to $\tilde{B}$. Thus we can take biorthogonal bases $\Psi_r = (\psi^{\gamma+1}, \ldots, \psi^n)^T$ and $\widetilde{\Psi}_r = (\tilde{\psi}^{\gamma+1}, \ldots, \tilde{\psi}^n)^T$ of B and $\widetilde{B}$, respectively. To complete the proof, it remains only on the construction of wavelet vectors Ψ_0 and $\widetilde{\Psi}_0$. By Lemma 3.8, $V_1[-1, 1]$ and $\widetilde{V}_1[-1, 1]$ have the following subspaces

$$X = Q_0 \oplus \mathrm{Im}(\pi_l) \oplus \mathrm{Im}(\pi_r)(\cdot + 1) \oplus Q_r \oplus Q_r(\cdot + 1) \oplus R_r \oplus R_r(\cdot + 1)$$

and

$$\widetilde{X} = \widetilde{Q}_0 \oplus \mathrm{Im}(\tilde{\pi}_l) \oplus \mathrm{Im}(\tilde{\pi}_r)(\cdot + 1) \oplus \widetilde{Q}_r \oplus \widetilde{Q}_r(\cdot + 1) \oplus \widetilde{R}_r \oplus \widetilde{R}_r(\cdot + 1),$$

respectively. And the dimensions of X is

$$\dim X = k + (k - m_l) + (k - m_r) + 2(n - k) + 2(n - \gamma)$$
$$= 4n - k - \gamma.$$

Similarly, $\dim \widetilde{X} = 4n - k - \gamma$. By Lemma 3.9, there exists a subspace $\bar{R_0}$ of $V_1[-1, 1]$ such that $\bar{R_0} \perp \widetilde{X}$, and

$$\dim \bar{R}_0 = \dim \widetilde{V}_1[-1, 1] - \dim \widetilde{X}$$
$$= \gamma.$$

By Lemma 3.8, $\mathrm{Im}(\pi_l)$ and $\mathrm{Im}(\pi_r)$ are orthogonal to $\mathrm{Im}(\tilde{\pi}_r)$ and $\mathrm{Im}(\tilde{\pi}_l)$, respectively. Since $\mathrm{Im}(\pi_l) + \mathrm{Im}(\pi_r)$ is biorthogonal to $\mathrm{Im}(\tilde{\pi}_l) + \mathrm{Im}(\tilde{\pi}_r)$, by Lemma 3.7 $\mathrm{Im}(\pi_l)$ and $\mathrm{Im}(\pi_r)$ are biorthogonal to $\mathrm{Im}(\tilde{\pi}_l)$ and $\mathrm{Im}(\tilde{\pi}_r)$, respectively. Thus $\mathrm{Im}(\pi_r)(\cdot + 1)$ is biorthogonal to $\mathrm{Im}(\tilde{\pi}_r)(\cdot + 1)$. If we note that $\mathrm{Im}(\pi_l)$ and $\mathrm{Im}(\pi_r)(\cdot + 1)$ are orthogonal to $\mathrm{Im}(\tilde{\pi}_r)(\cdot + 1)$ and $\mathrm{Im}(\tilde{\pi}_l)$, then by Lemma 3.7 $\mathrm{Im}(\pi_l) \oplus \mathrm{Im}(\pi_r)(\cdot + 1)$ is biorthogonal to $\mathrm{Im}(\tilde{\pi}_l) \oplus \mathrm{Im}(\tilde{\pi}_r)(\cdot + 1)$. Since $Q_0 \oplus Q_r \oplus Q_r(\cdot + 1) \oplus R_r \oplus R_r(\cdot + 1)$ is also biorthogonal to $\tilde{Q}_0 \oplus \tilde{Q}_r \oplus Q_r(\cdot + 1) \oplus R_r \oplus R_r(\cdot + 1)$, by Lemma 3.7 X is biorthogonal to $\tilde{X}$. By the orthogonality of $\bar{R_0}$ and $\tilde{\bar{R_0}}$ to $\tilde{X}$ and X resp.,

$$\bar{R_0} \cap X = \tilde{\bar{R_0}} \cap \tilde{X} = \phi.$$

Since $\dim V_1[-1, 1] = \dim \widetilde{V}_1[-1, 1] = 4n - k$,

$$V_1[-1, 1] = X \oplus \bar{R_0},$$
$$\widetilde{V}_1[-1, 1] = \widetilde{X} \oplus \tilde{\bar{R_0}}.$$

The biorthogonality of $V_1[-1, 1]$ and $\widetilde{V}_1[-1, 1]$ implies by Lemma 3.7 that $\bar{R_0}$ is biorthogonal to $\tilde{\bar{R_0}}$. Hence we can take biorthogonal bases $\Psi_0 = \{\psi^1, \ldots, \psi^\gamma\}^T$ and $\widetilde{\Psi}_0 = \{\widetilde{\Psi}^1, \ldots, \widetilde{\Psi}^\gamma\}^T$ of $\bar{R_0}$ and $\tilde{\bar{R_0}}$, respectively. Now the remaining condition for Ψ and $\tilde{\Psi}$ to be wavelet vectors is

$$\left\langle \Psi_0, \widetilde{\Psi}_0^T(\cdot \pm 1)\right\rangle = 0,$$

which is equivalent to the following conditions:

$$\mathrm{Im}(\eta_l) \perp \mathrm{Im}(\tilde{\eta}_r),$$
$$\mathrm{Im}(\tilde{\eta}_l) \perp \mathrm{Im}(\eta_r).$$

To show that $\mathrm{Im}(\eta_l)$ and $\mathrm{Im}(\eta_r)$ are orthogonal to $\mathrm{Im}(\tilde{\eta}_r)$ and $\mathrm{Im}(\tilde{\eta}_l)$, respectively, we note that

$$Q_{1r} = \mathrm{Im}(\pi_r) \oplus \mathrm{Im}(\pi_l) \oplus Q_r \oplus R_r,$$
$$\widetilde{Q}_{1r} = \mathrm{Im}(\tilde{\pi}_r) \oplus \mathrm{Im}(\tilde{\pi}_l) \oplus \widetilde{Q}_r \oplus \widetilde{R}_r,$$

are biorthogonal subspaces. And by the constructions of Ψ_0 and $\widetilde{\Psi}_0$, $\mathrm{Im}(\eta_r)$ is orthogonal to $\mathrm{Im}(\tilde{\pi}_l) \oplus \widetilde{Q}_r \oplus \widetilde{R}_r$, which is orthogonal to $\mathrm{Im}(\pi_r)$. Also $\mathrm{Im}(\tilde{\eta}_r)$ is orthogonal to $\mathrm{Im}(\pi_l) \oplus Q_r \oplus R_r$, which is orthogonal to $\mathrm{Im}(\tilde{\pi}_r)$. By Lemma 3.7, $\mathrm{Im}(\pi_r)$ and $\mathrm{Im}(\pi_l) \oplus Q_r \oplus R_r$ are biorthogonal to $\mathrm{Im}(\tilde{\pi}_r)$ and $\mathrm{Im}(\tilde{\pi}_l) \oplus \widetilde{Q}_r \oplus \widetilde{R}_r$, respectively. Hence

$$\mathrm{Im}(\eta_r) \subset \mathrm{Im}(\pi_r),$$
$$\mathrm{Im}(\tilde{\eta}_r) \subset \mathrm{Im}(\tilde{\pi}_r).$$

Similarly,

$$\mathrm{Im}(\eta_l) \subset \mathrm{Im}(\pi_l),$$
$$\mathrm{Im}(\tilde{\eta}_l) \subset \mathrm{Im}(\tilde{\pi}_l).$$

Since $\mathrm{Im}(\pi_r) \perp \mathrm{Im}(\tilde{\pi}_l)$ and $\mathrm{Im}(\pi_l) \perp \mathrm{Im}(\tilde{\pi}_r)$, $\mathrm{Im}(\eta_r) \perp \mathrm{Im}(\tilde{\eta}_l)$ and $\mathrm{Im}(\eta_l) \perp \mathrm{Im}(\tilde{\eta}_r)$. That is, $\langle \Psi_0, \widetilde{\Psi}_0^T(\cdot \pm 1)\rangle = 0$. Therefore

$$\Psi = \Psi_0 \cup \Psi_r = \left(\psi^1, \ldots, \psi^\gamma, \psi^{\gamma+1}, \ldots, \psi^n\right)^T$$

and

$$\widetilde{\Psi} = \widetilde{\Psi}_0 \cup \widetilde{\Psi}_r = \left(\tilde{\psi}^1, \ldots, \tilde{\psi}^\gamma, \tilde{\psi}^{\gamma+1}, \ldots, \tilde{\psi}^n\right)^T$$

are biorthogonal wavelet vectors.

Conversely, let $\Psi = (\psi^1, \ldots, \psi^n)$ and $\widetilde{\Psi} = (\tilde{\psi}^1, \ldots, \tilde{\psi}^n)$ be biorthogonal wavelet vectors such that $\psi^{\gamma+1}, \ldots, \psi^n$ and $\tilde{\psi}^{\gamma+1}, \ldots, \tilde{\psi}^n$ are only supported in $[0, 1]$. Let $R_r = \text{span}(\psi^{\gamma+1}, \ldots, \psi^n)$ and $\widetilde{R}_r = \text{span}(\tilde{\psi}^{\gamma+1}, \ldots, \tilde{\psi}^n)$. Then R_r and Q_r are biorthogonal to $\widetilde{R}_r$ and $\widetilde{Q}_r$, respectively. Since $R_r \perp \widetilde{Q}_r$ and $Q_r \perp \widetilde{R}_r$, by Lemma 3.7 $R_r \oplus Q_r$ is biorthogonal to $\widetilde{R}_r \oplus \widetilde{Q}_r$. And also since $R_r \oplus Q_r \perp (\text{Im}(\tilde{\pi}_l) + \text{Im}(\tilde{\pi}_r))$ and $\widetilde{R}_r \oplus \widetilde{Q}_r \perp (\text{Im}(\pi_l) + \text{Im}(\pi_r))$, by Lemma 3.3 $R_r \oplus Q_r \cap (\text{Im}(\pi_l) + \text{Im}(\pi_r)) = \{0\}$ and $\widetilde{R}_r \oplus \widetilde{Q}_r \cap (\text{Im}(\tilde{\pi}_l) + \text{Im}(\tilde{\pi}_r)) = \{0\}$. Hence

$$\begin{aligned}\dim R_r \oplus Q_r \oplus (\text{Im}(\pi_l) + \text{Im}(\pi_r)) &= \dim \widetilde{R}_r \oplus \widetilde{Q}_r \oplus (\text{Im}(\tilde{\pi}_l) + \text{Im}(\tilde{\pi}_r)) \\ &= (n - \gamma) + (n - k) + \gamma = 2n - k.\end{aligned}$$

If we note that $R_r \oplus Q_r \oplus (\text{Im}(\pi_l) + \text{Im}(\pi_r))$ and $\widetilde{R}_r \oplus \widetilde{Q}_r \oplus (\text{Im}(\tilde{\pi}_l) + \text{Im}(\tilde{\pi}_r))$ are subspaces of Q_{1r} and $\widetilde{Q}_{1r}$, respectively, and $\dim Q_{1r} = \dim \widetilde{Q}_{1r} = 2n - k$, then

$$Q_{1r} = R_r \oplus Q_r \oplus (\text{Im}(\pi_l) + \text{Im}(\pi_r)), \quad \widetilde{Q}_{1r} = \widetilde{R}_r \oplus \widetilde{Q}_r \oplus (\text{Im}(\tilde{\pi}_l) + \text{Im}(\tilde{\pi}_r)).$$

Since Q_{1r} is biorthogonal to $\widetilde{Q}_{1r}$, by Lemma 3.7 $\text{Im}(\pi_l) + \text{Im}(\pi_r)$ is biorthogonal to $\text{Im}(\tilde{\pi}_l) + \text{Im}(\tilde{\pi}_r)$. □

If Φ is an orthogonal scaling vector, then $\text{Im}(\pi_l) + \text{Im}(\pi_r)$ is biorthogonal to itself, and $\text{Im}(\pi_l) \cap \text{Im}(\pi_r)$. Thus we have the following corollary.

Corollary 3.2 [9]. *Let $(V_p)_{p \in Z}$ be orthogonal multiresolution analysis generated by scaling vector $\Phi = (\phi^1, \ldots, \phi^n)^T$, minimally supported in $[-1, 1]$. Then there exists a wavelet vector $\Psi = (\psi^1, \ldots, \psi^n)^T$ which has exactly $n - \dim \text{Im}(\pi_l) + \dim \text{Im}(\pi_r)$ wavelet functions ψ^i supported in $[0, 1]$.*

4. Applications

In this section, we consider the computation algorithm to find multiwavelet coefficients corresponding to a given multiscaling coefficients by using the developed theorem of multiwavelet construction in previous chapter. The dilation Eqs. (1) and (2) of biorthogonal scaling and wavelet vectors $\Phi, \widetilde{\Phi}, \Psi, \widetilde{\Psi}$ can be modified as

$$\begin{aligned}\Phi_0 &= C_{0l}\Phi_{1l}(\cdot + 1) + C_{00}\Phi_{10} + C_{0r}\Phi_{1r}, \\ \Phi_r &= C_{rr}\Phi_{1r}, \\ \Psi_0 &= D_{0l}\Phi_{1l}(\cdot + 1) + D_{00}\Phi_{10} + D_{0r}\Phi_{1r}, \\ \Psi_r &= D_{rr}\Phi_{1r},\end{aligned}$$

and the duals

$$\begin{aligned}\widetilde{\Phi}_0 &= \widetilde{C}_{0l}\widetilde{\Phi}_{1l}(\cdot + 1) + \widetilde{C}_{00}\widetilde{\Phi}_{10} + \widetilde{C}_{0r}\widetilde{\Phi}_{1r}, \\ \widetilde{\Phi}_r &= \widetilde{C}_{rr}\widetilde{\Phi}_{1r}, \\ \widetilde{\Psi}_0 &= \widetilde{D}_{0l}\widetilde{\Phi}_{1l}(\cdot + 1) + \widetilde{D}_{00}\widetilde{\Phi}_{10} + \widetilde{D}_{0r}\widetilde{\Psi}_{1r}, \\ \widetilde{\Phi}_r &= \widetilde{D}_{rr}\widetilde{\Phi}_{1r}.\end{aligned}$$

These matrix coefficients of dilation equation of biorthogonal scaling vectors can be composed as matrices L and $\widetilde{L}$ called biorthogonal low pass filters, as follows:

$$L := \begin{bmatrix} C_{0l} & C_{00} & C_{0r} \\ \mathbf{0} & \mathbf{0} & C_{rr} \end{bmatrix}$$

and

$$\widetilde{L} := \begin{bmatrix} \widetilde{C}_{0l} & \widetilde{C}_{00} & \widetilde{C}_{0r} \\ \mathbf{0} & \mathbf{0} & \widetilde{C}_{rr} \end{bmatrix}.$$

Also matrix coefficients of dilation equation of biorthogonal wavelet vectors can be composed as matrices H and $\widetilde{H}$ called biorthogonal high pass filters, as follows:

$$H := \begin{bmatrix} D_{0l} & D_{00} & D_{0r} \\ \mathbf{0} & \mathbf{0} & D_{rr} \end{bmatrix}$$

and

$$\widetilde{H} := \begin{bmatrix} \widetilde{D}_{0l} & \widetilde{D}_{00} & \widetilde{D}_{0r} \\ \mathbf{0} & \mathbf{0} & \widetilde{D}_{rr}. \end{bmatrix}.$$

By these matrix coefficients, Theorem 3 can be rewritten as follows.

Theorem 4. *Let $(V_p)_{p\in Z}$ and $(\widetilde{V}_p)_{p\in Z}$ be biorthogonal multiresolution analyses generated by scaling vectors $\Phi = (\phi^1, \ldots, \phi^n)^T$ and $\widetilde{\Phi} = (\tilde{\phi}^1, \ldots, \tilde{\phi}^n)^T$. Then* range $C_{0l}^T +$ range C_{0r}^T *is biorthogonal to* range $\widetilde{C}_{0l}^T +$ range $\widetilde{C}_{0r}^T$ *if and only if* $\gamma = \dim(\text{range } C_{0l}^T + \text{range } C_{0r}^T) = \dim(\text{range } \widetilde{C}_{0l}^T + \text{range } \widetilde{C}_{0r}^T)$ *and there exist biorthogonal wavelet vectors Ψ, $\widetilde{\Psi}$ each of which has exactly $n - \gamma$ wavelet functions supported in* $[0, 1]$.

Given above matrix coefficients C of dilation equations of biorthogonal scaling vectors, by the theorem we can find matrix coefficients D of dilation equations to construct corresponding wavelet vectors. The algorithm to compute wavelet matrix coefficients of dilation equations can be summerized by the proof of Theorem 3 as follows.

(Algorithm to construct biorthogonal multiwavelets.)

1. Choose any scaling vector so that range C_{0l}^T + range C_{0r}^T is biorthogonal to range $\widetilde{C}_{0l}^T$ + range $\widetilde{C}_{0r}^T$.
2. Find any matrix $\tilde{D}_{rr}$ such that range$[C_{0l}^T, C_{0r}^T, C_{rr}^T]$ is orthogonal complement of range$[\tilde{D}_{rr}^T]$.
3. Find any matrix D_{rr} such that range$[\tilde{C}_{0l}^T, \tilde{C}_{0r}^T, \tilde{C}_{rr}^T]$ is orthogonal complement of range$[D_{rr}^T]$.
4. $D_{rr} := (D_{rr}\tilde{D}_{rr}^T)^{-1}D_{rr}$.
5. Find any matrix $[\tilde{D}_{0l}, \tilde{D}_{00}, \tilde{D}_{0r}]$ such that range $[(C_{0l}, C_{00}, C_{0r})^T,\ (\mathbf{0}, \mathbf{0}, C_{0l})^T,\ (C_{0r}, \mathbf{0}, \mathbf{0})^T,\ (\mathbf{0}, \mathbf{0}, C_{rr})^T,\ (C_{rr}, \mathbf{0}, \mathbf{0})^T,\ (\mathbf{0}, \mathbf{0}, D_{rr})^T,\ (D_{rr}, \mathbf{0}, \mathbf{0})^T]$ is orthogonal complement of range $[\tilde{D}_{0l}, \tilde{D}_{00}, \tilde{D}_{0r}]^T$.
6. Find any matrix $[D_{0l}, D_{00}, D_{0r}]$ such that range $[(\tilde{C}_{0l}, \tilde{C}_{00}, \tilde{C}_{0r})^T,\ (\mathbf{0}, \mathbf{0}, \tilde{C}_{0l})^T,\ (\tilde{C}_{0r}, \mathbf{0}, \mathbf{0})^T,\ (\mathbf{0}, \mathbf{0}, \tilde{C}_{rr})^T,\ (\tilde{C}_{rr}^T, \mathbf{0}, \mathbf{0})^T,\ (\mathbf{0}, \mathbf{0}, \tilde{D}_{rr})^T,\ (\tilde{D}_{rr}, \mathbf{0}, \mathbf{0})^T]$ is orthogonal complement of range $[D_{0l}, D_{00}, D_{0r}]^T$.
7. $[D_{0l}, D_{00}, D_{0r}] := ([D_{0l}, D_{00}, D_{0r}][\tilde{D}_{0l}, \tilde{D}_{00}, \tilde{D}_{0r}]^T)^{-1}[D_{0l}, D_{00}, D_{0r}]$.
8. The corresponding high pass filters H and $\widetilde{H}$:

$$H = \begin{bmatrix} D_{0l} & D_{00} & D_{0r} \\ \mathbf{0} & \mathbf{0} & D_{rr} \end{bmatrix},$$

$$\widetilde{H} = \begin{bmatrix} \widetilde{D}_{0l} & \widetilde{D}_{00} & \widetilde{D}_{0r} \\ \mathbf{0} & \mathbf{0} & \widetilde{D}_{rr} \end{bmatrix}.$$

By using this algorithm, let us compute the biorthogonal wavelet coefficients corresponding to the biorthogonal scaling vectors in the following examples.

Example 4.1. From the symmetric biorthogonal scaling vectors constructed by Hardin and Marasovich [7], we have the following matrix coefficients:

$$
\begin{aligned}
C_{0l} &= \left[\frac{-\sqrt{2-5u}(1+2u)}{12\sqrt{3}},\ \frac{-1+4u}{6\sqrt{2}},\ \frac{\sqrt{2-5u}(5-2u)}{12\sqrt{3}}\right],\\
C_{00} &= \left[\frac{1}{\sqrt{2}}\right],\\
C_{0r} &= \left[\frac{\sqrt{2-5u}(5-2u)}{12\sqrt{3}},\ \frac{-1+4u}{6\sqrt{2}},\ \frac{\sqrt{2-5u}(1+2u)}{12\sqrt{3}}\right],\\
C_{rr} &= \left[\frac{2+u}{3\sqrt{2}},\ \frac{-2(-1+u)}{\sqrt{6-15u}},\ \frac{2+u}{3\sqrt{2}}\right],
\end{aligned}
$$

and the dual matrix coefficients

$$
\begin{aligned}
\widetilde{C}_{0l} &= \left[\frac{3\sqrt{6-15u}u}{4(2-5u)^2},\ \frac{2+u}{\sqrt{2}(-4+10u)},\ \frac{\sqrt{6-15u}(4-7u)}{4(2-5u)^2}\right],\\
\widetilde{C}_{00} &= \left[\frac{1}{\sqrt{2}}\right],\\
\widetilde{C}_{0r} &= \left[\frac{\sqrt{6-15u}(4-7u)}{4(2-5u)^2},\ \frac{2+u}{\sqrt{2}(-4+10u)},\ \frac{3\sqrt{6-15u}u}{4(2-5u)^2}\right],\\
\widetilde{C}_{rr} &= \left[\frac{-1+4u}{\sqrt{2}(-2+5u)},\ \frac{-2(-1+u)}{\sqrt{6-15u}},\ \frac{-1+4u}{\sqrt{2}(-2+5u)}\right].
\end{aligned}
$$

If we combine above matrix coefficients of biorthogonal scaling vectors, then we obtain the following biorthogonal low pass filter L

$$
\begin{bmatrix}
\frac{-\sqrt{2-5u}(1+2u)}{12\sqrt{3}} & \frac{-1+4u}{6\sqrt{2}} & \frac{\sqrt{2-5u}(5-2u)}{12\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2-5u}(5-2u)}{12\sqrt{3}} & \frac{-1+4u}{6\sqrt{2}} & \frac{\sqrt{2-5u}(1+2u)}{12\sqrt{3}}\\
0 & 0 & 0 & 0 & \frac{2+u}{3\sqrt{2}} & \frac{-2(-1+u)}{\sqrt{6-15u}} & \frac{2+u}{3\sqrt{2}}
\end{bmatrix}
\tag{3}
$$

and the dual low pass filter $\widetilde{L}$

$$
\begin{bmatrix}
\frac{3\sqrt{6-15u}u}{4(2-5u)^2} & \frac{2+u}{\sqrt{2}(-4+10u)} & \frac{\sqrt{6-15u}(4-7u)}{4(2-5u)^2} & \frac{1}{\sqrt{2}} & \frac{\sqrt{6-15u}(4-7u)}{4(2-5u)^2} & \frac{2+u}{\sqrt{2}(-4+10u)} & \frac{3\sqrt{6-15u}u}{4(2-5u)^2}\\
0 & 0 & 0 & 0 & \frac{-1+4u}{\sqrt{2}(-2+5u)} & \frac{-2(-1+u)}{\sqrt{6-15u}} & \frac{-1+4u}{\sqrt{2}(-2+5u)}
\end{bmatrix}.
\tag{4}
$$

Now we will find the matrix coefficients of biorthogonal wavelet vectors by using the proposed computational algorithm.

(1) Since the determinant of matrix $[C_{0l}^T, C_{0r}^T]^T[\widetilde{C}_{0l}^T, \widetilde{C}_{0r}^T]$ is $\frac{1}{16}$, it is nonsingular for any u.

(2) We can find that C_{0l}, C_{0r} and $\widetilde{C}_{0l}$, $\widetilde{C}_{0r}$ are all nonzero for every $u \in (-1, 1/7)$. Hence m_l, m_r, m, and $\tilde{m}_l$, $\tilde{m}_r$, $\tilde{m}$ are all zero for $u \in (-1, 1/7)$. And since $\widetilde{C}_{1l}$, $\widetilde{C}_{10}$ are zero, $k = 1$. It means that our biorthogonal wavelet functions does not have supports in [0, 1].

(3) To find the possible matrix coefficients of biorthogonal wavelet vector through step 5 and step 6, we put

$$[X, Y] := [D_{0l}, D_{00}, D_{0r}]^T = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_3 & x_2 & x_1 \\ y_1 & y_2 & y_3 & y_4 & -y_3 & -y_2 & -y_1 \end{bmatrix}^T,$$

$$[\widetilde{X}, \widetilde{Y}] := [\widetilde{D}_{0l}, \widetilde{D}_{00}, \widetilde{D}_{0r}]^T = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_3 & \tilde{x}_2 & \tilde{x}_1 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 & \tilde{y}_4 & -\tilde{y}_3 & -\tilde{y}_2 & -\tilde{y}_1 \end{bmatrix}^T,$$

which must satisfy the following linear system of equations

$$\widetilde{A}[X\ Y] = \mathbf{0},$$
$$A[\widetilde{X}\ \widetilde{Y}] = \mathbf{0},$$
$$[X\ Y]^T [\widetilde{X}\ \widetilde{Y}] = I,$$

where

$$A = \begin{bmatrix} C_{0l} & C_{00} & C_{0r} \\ C_{rl} & C_{r0} & C_{rr} \\ \mathbf{0} & \mathbf{0} & C_{0l} \\ C_{0r} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and

$$\widetilde{A} = \begin{bmatrix} \widetilde{C}_{0l} & \widetilde{C}_{00} & \widetilde{C}_{0r} \\ \widetilde{C}_{rl} & \widetilde{C}_{r0} & \widetilde{C}_{rr} \\ \mathbf{0} & \mathbf{0} & \widetilde{C}_{0l} \\ \widetilde{C}_{0r} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

With the help of *Mathematica*, we have th following general biorthogonal high pass filters H and $\widetilde{H}$ by the computation algorithm:

$$H = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_3 & x_2 & x_1 \\ y_1 & y_2 & y_3 & y_4 & -y_3 & -y_2 & -y_1 \end{bmatrix} \tag{5}$$

and

$$\widetilde{H} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_3 & \tilde{x}_2 & \tilde{x}_1 \\ \tilde{y}_1 & \tilde{y}_2 & \tilde{y}_3 & \tilde{y}_4 & -\tilde{y}_3 & -\tilde{y}_2 & -\tilde{y}_1 \end{bmatrix}, \tag{6}$$

where

$$x_1 = \frac{4+u-14u^2}{-96\tilde{x}_3+240u\tilde{x}_3}, \qquad y_1 = \frac{4+u-14u^2}{-48\tilde{y}_3+120u\tilde{y}_3},$$

$$x_2 = -\frac{\sqrt{12-30u}(-1+4u)(-4+7u)}{48(2-5u)^2\tilde{x}_3}, \qquad y_2 = \frac{-\left(\sqrt{12-30u}(-1+4u)(-4+7u)\right)}{24(2-5u)^2\tilde{y}_3},$$

$$x_3 = \frac{20-43u+14u^2}{96\tilde{x}_3-240u\tilde{x}_3}, \qquad y_3 = \frac{20-43u+14u^2}{48\tilde{y}_3-120u\tilde{y}_3},$$

$$x_4 = \frac{\sqrt{12-30u}(-4+7u)}{8(2-5u)^2\tilde{x}_3}, \qquad y_4 = 0,$$

$$\tilde{x}_1 = \frac{3u\tilde{x}_3}{4-7u}, \qquad \tilde{y}_1 = \frac{3u\tilde{y}_3}{4-7u},$$
$$\tilde{x}_2 = \frac{\sqrt{12-30u}(2+u)\tilde{x}_3}{3(-4+7u)}, \qquad \tilde{y}_2 = \frac{\sqrt{12-30u}(2+u)\tilde{y}_3}{3(-4+7u)},$$
$$\tilde{x}_4 = \frac{2\sqrt{\frac{2}{3}}(2-5u)^{\frac{3}{2}}\tilde{x}_3}{-4+7u}, \qquad \tilde{y}_4 = 0.$$

By using this general formula of high pass filter, we can construct several biorthogonal wavelet vectors with free variables $\tilde{x}_3$ and $\tilde{y}_3$, given biorthogonal scaling vector.

For example, we give filters of biorthogonal wavelet vectors determined by particular values of u and free variables.

(1) If we take $u = -\frac{1}{5}$ and $\tilde{x}_3 = \frac{9}{20}$, $\tilde{y}_3 = \frac{9\sqrt{2}}{20}$ then we obtain the orthogonal GHM multiwavelet filters introduced by Geronimo et al. [4,10].

(2) If we take $u = 0$ and $\tilde{x}_3 = \frac{5}{6\sqrt{6}}$, $\tilde{y}_3 = \frac{\sqrt{3}}{2}$ then we obtain the biorthogonal multiwavelet filters introduced by Hardin and Marasovich [7].

Remark 4.1. By step 2, $m_l, \tilde{m}_l, m_r, \tilde{m}_r, m, \tilde{m}$ are all zero for $\forall u \in (-1, \frac{1}{7})$, and $k = 1$. Thus by Theorem 2, none of the corresponding wavelet functions are supported in $[0, 1]$. Hence for Hardin and Marasovich's symmetric biorthogonal scaling vectors with low pass filters L and $\tilde{L}$ in (3) and (4), all the corresponding symmetric or antisymmetric biorthogonal wavelet vectors are given by high pass filters H and $\tilde{H}$ in (5) and (6).

Example 4.2. Let $\Phi(x) = (\phi_1, \phi_2)^T$ and $\tilde{\Phi}(x) = (\tilde{\phi}_1, \tilde{\phi}_2)^T$ be biorthogonal scaling vectors which are supported in $[-1, 1]$, satisfying the following equations [7,8]:

$$\Phi(x) = P_{-1}\Phi(2x+1) + P_0\Phi(2x) + P_1\Phi(2x-1),$$
$$\tilde{\Phi}(x) = P_{-1}\tilde{\Phi}(2x+1) + P_0\tilde{\Phi}(2x) + P_1\tilde{\Phi}(2x-1),$$

where

$$P_{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ -1 & -\frac{2}{5} \end{bmatrix}, \quad P_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad P_1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{5} \\ 1 & -\frac{2}{5} \end{bmatrix}$$

$$\tilde{P}_{-1} = \begin{bmatrix} \frac{1}{2} & \frac{5}{4} \\ \frac{-7}{16} & -\frac{35}{32} \end{bmatrix}, \quad \tilde{P}_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad \tilde{P}_1 = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} \\ \frac{7}{16} & -\frac{35}{32} \end{bmatrix}.$$

Then

$$C_{0l} = P_{-1}, \quad C_{00} = P_0, \quad C_{0r} = P_1,$$
$$\tilde{C}_{0l} = \tilde{P}_{-1}, \quad \tilde{C}_{00} = \tilde{P}_0, \quad \tilde{C}_{0r} = \tilde{P}_1,$$
$$C_{rr} = \tilde{C}_{rr} = \mathbf{0}.$$

Hence

$$\kappa = \kappa(\Phi) = \kappa(\tilde{\Phi}) = 2,$$
$$\mathrm{Im}(\pi_l) = \mathrm{span}\{[1/2, 1/5]\}, \quad \mathrm{Im}(\pi_r) = \mathrm{span}\{[1/2, -1/5]\},$$
$$\mathrm{Im}(\tilde{\pi}_l) = \mathrm{span}\{[1/2, 5/4]\}, \quad \mathrm{Im}(\tilde{\pi}_r) = \mathrm{span}\{[1/2, -5/4]\}.$$

And $\mathrm{Im}(\pi_l) + \mathrm{Im}(\pi_r) = \mathbf{R}^2$ is biorthogonal to $\mathrm{Im}(\tilde{\pi}_l) + \mathrm{Im}(\tilde{\pi}_r) = \mathbf{R}^2$. Since $\gamma = 2$, all the corresponding biorthogonal wavelet functions are not supported in $[0, 1]$. By the computational algorithm, we can compute complete biorthogonal wavelet functions. With the help of Mathematica, we get the general form of biorthogonal wavelet vectors determined by the following coefficients:

$$D_{-1} = \begin{bmatrix} \frac{5-40\tilde{y}_2 y_3}{16\tilde{x}_2} & \frac{1-8\tilde{y}_2 y_3}{8\tilde{x}_2} \\ \frac{5y_3}{2} & y_3 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} \frac{5(-1+8\tilde{y}_2 y_3)(\tilde{y}_2-\tilde{y}_6)}{2\tilde{y}_2(3\tilde{y}_2-5\tilde{y}_6)} & \frac{35(-1+8\tilde{y}_2 y_3)(\tilde{y}_2+\tilde{y}_6)}{32\tilde{x}_2(3\tilde{y}_2-5\tilde{y}_6)} \\ \frac{1-20\tilde{y}_2 y_3+20y_3\tilde{y}_6}{3\tilde{y}_2-5\tilde{y}_6} & -\frac{7(-1+5\tilde{y}_2 y_3+5y_3\tilde{y}_6)}{4(3\tilde{y}_2-5\tilde{y}_6)} \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -\frac{5(-1+8\tilde{y}_2 y_3)(5\tilde{y}_2-3\tilde{y}_6)}{16\tilde{x}_2(3\tilde{y}_2-5\tilde{y}_6)} & \frac{(-1+8\tilde{y}_2 y_3)(5\tilde{y}_2-3\tilde{y}_6)}{8\tilde{x}_2(3\tilde{y}_2-5\tilde{y}_6)} \\ \frac{2-25\tilde{y}_2 y_3+15y_3\tilde{y}_6}{-6\tilde{y}_2+10\tilde{y}_6} & \frac{2-25\tilde{y}_2 y_3+15y_3\tilde{y}_6}{15\tilde{y}_2-25\tilde{y}_6} \end{bmatrix},$$

$$\tilde{D}_{-1} = \begin{bmatrix} \tilde{x}_2 & \frac{5\tilde{x}_2}{2} \\ \tilde{y}_2 & \frac{5\tilde{y}_2}{2} \end{bmatrix},$$

$$\tilde{D}_0 = \begin{bmatrix} -\frac{8\tilde{x}_2(-1+5\tilde{y}_2 y_3+5y_3\tilde{y}_6)}{-5+40\tilde{y}_2 y_3} & \frac{8\tilde{x}_2(-1+20\tilde{y}_2 y_3-20y_3\tilde{y}_6)}{-5+40\tilde{y}_2 y_3} \\ -\tilde{y}_2-\tilde{y}_6 & 4(\tilde{y}_2-\tilde{y}_6) \end{bmatrix},$$

$$\tilde{D}_1 = \begin{bmatrix} \frac{\tilde{x}_2(-3+40y_3\tilde{y}_6)}{-5+40\tilde{y}_2 y_3} & \frac{\tilde{x}_2(3-40y_3\tilde{y}_6)}{-2+16\tilde{y}_2 y_3} \\ \frac{\tilde{y}_6}{2} & -\frac{5\tilde{y}_6}{2} \end{bmatrix},$$

which have free variables $y_3, \tilde{x}_2, \tilde{y}_2, \tilde{y}_6$. In particular, by taking $y_3 = -2\sqrt{7}/35$, $\tilde{y}_2 = -\sqrt{7}/16$, $\tilde{y}_6 = \sqrt{7}/16$, $\tilde{x}_2 = 1/2$ we obtain a pair of biorthogonal wavelet vectors with the following coefficients:

$$D_{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} \\ -\frac{\sqrt{7}}{7} & -\frac{2\sqrt{7}}{35} \end{bmatrix},\quad D_0 = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{\sqrt{7}}{7} \end{bmatrix},\quad D_1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{5} \\ \frac{\sqrt{7}}{7} & -\frac{2\sqrt{7}}{35} \end{bmatrix},$$

$$\tilde{D}_{-1} = \begin{bmatrix} \frac{1}{2} & \frac{5}{4} \\ -\frac{\sqrt{7}}{16} & -\frac{5\sqrt{7}}{32} \end{bmatrix},\quad \tilde{D}_0 = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{\sqrt{7}}{2} \end{bmatrix},\quad \tilde{D}_1 = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} \\ \frac{\sqrt{7}}{16} & -\frac{5\sqrt{7}}{32} \end{bmatrix}.$$

Note that the corresponding wavelet vectors are the same as in Example 1 in [8].

Example 4.3. Let $\Phi(x) = (\phi_1, \phi_2)^T$ be orthogonal scaling vector which is supported in $[-1, 1]$, satisfying the following equations [12,8]:

$$\Phi(x) = P_{-1}\Phi(2x+1) + P_0\Phi(2x) + P_1\Phi(2x-1),$$

where

$$P_{-1} = \begin{bmatrix} 0 & \frac{2+\sqrt{7}}{4} \\ 0 & \frac{2-\sqrt{7}}{4} \end{bmatrix},\quad P_0 = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4\sqrt{2}} \end{bmatrix},\quad P_1 = \begin{bmatrix} \frac{2-\sqrt{7}}{4} & 0 \\ \frac{2+\sqrt{7}}{4} & 0 \end{bmatrix}.$$

Then

$$C_{0l} = P_{-1},\quad C_{00} = P_0,\quad C_{0r} = P_1,\quad C_{rr} = \mathbf{0}.$$

Since $\gamma = \mathrm{Im}(\pi_l)+\mathrm{Im}(\pi_r) = 2$, all the corresponding orthogonal wavelet functions are not supported in [0, 1]. By the computational algorithm, we can compute all wavelet functions. With the help of Mathematica, we get the general form of wavelet vector determined by the following coefficients:

$$D_{-1} = \begin{bmatrix} 0 & \frac{\left(-15\sqrt{2}+24\sqrt{2}\,y_3^2-16\,y_3\sqrt{5-8\,y_3^2}\right)\sqrt{45+56\,y_3^2-48\sqrt{2}\,y_3\sqrt{5-8\,y_3^2}}}{20\left(-9+40\,y_3^2\right)} \\ 0 & y_3 \end{bmatrix},$$

$$D_0 = \begin{bmatrix} a & b \\ \frac{-2\left(4+\sqrt{7}\right)y_3+\sqrt{2}\left(-1+\sqrt{7}\right)\sqrt{5-8\,y_3^2}}{10} & \frac{2\left(-4+\sqrt{7}\right)y_3-\sqrt{2}\left(1+\sqrt{7}\right)\sqrt{5-8\,y_3^2}}{10} \end{bmatrix},$$

$$D_1 = \begin{bmatrix} \frac{\sqrt{\frac{45}{2}+28\,y_3^2-24\sqrt{2}\,y_3\sqrt{5-8\,y_3}}}{10} & 0 \\ \frac{3\,y_3+\sqrt{2}\sqrt{5-8\,y_3^2}}{5} & 0 \end{bmatrix},$$

where

$$a = \frac{\sqrt{45+56\,y_3^2-48\sqrt{2}\,y_3\sqrt{5-8\,y_3^2}}\left(3\sqrt{2}\left(4+\sqrt{7}\right)+8\sqrt{2}\left(-4+\sqrt{7}\right)y_3^2+8\left(1+\sqrt{7}\right)y_3\sqrt{5-8\,y_3^2}\right)}{20\left(-9+40\,y_3^2\right)},$$

$$b = -\frac{\left(\sqrt{45+56\,y_3^2-48\sqrt{2}\,y_3\sqrt{5-8\,y_3^2}}\left(3\sqrt{2}\left(-4+\sqrt{7}\right)+8\sqrt{2}\left(4+\sqrt{7}\right)y_3^2+8\left(-1+\sqrt{7}\right)y_3\sqrt{5-8\,y_3^2}\right)\right)}{20\left(-9+40\,y_3^2\right)},$$

which have a free variable y_3. In particular, if we take $y_3 = \frac{1}{4}$, then we obtain a pair of biorthogonal wavelet vectors with the following coefficients:

$$D_{-1} = \begin{bmatrix} 0 & \frac{3}{4} \\ 0 & \frac{1}{4} \end{bmatrix}, \quad D_0 = \begin{bmatrix} -\frac{2+\sqrt{7}}{4} & -\frac{2-\sqrt{7}}{4} \\ -\frac{-2-\sqrt{7}}{4} & -\frac{2+\sqrt{7}}{4} \end{bmatrix}, \quad D_1 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{3}{4} & 0 \end{bmatrix}.$$

Note that the corresponding wavelet vector is the same as in Example 2 in [8].

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