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Existence of homoclinic orbits for 2*n*th-order nonlinear difference equations containing both many advances and retardations ☆

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ABSTRACT

By establishing a proper variational framework and using the critical point theory, we establish some new existence criteria to guarantee the 2*n*th-order nonlinear difference equation containing both many advances and retardations

$$\Delta^n\big(r(t-n)\Delta^n u(t-n)\big)+q(t)u(t)=f\big(t,u(t+n),\dots,u(t),\dots,u(t-n)\big),$$
$$n\in\mathbb{Z}(3),\ t\in\mathbb{Z},$$

has at least one or infinitely many homoclinic orbits, where r, q, f are nonperiodic in t. Our conditions on the potential are rather relaxed and some existing results in the literature are improved.

1. Introduction

The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. Since the last decade, there has been much literature on qualitative properties of difference equations, those studies over many of the branches of difference equations, such as [1,3] and references therein. In the theory of differential equations, a trajectory which is asymptotic to a constant state as $|s|\to\infty$ (s denotes the time variable) is called a homoclinic orbit. It is well known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems (see, for example, [14] and references contained therein). If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation, its perturbed system probably produce chaotic phenomenon.

For $a,b\in\mathbb{Z}$, define $\mathbb{Z}(a)=\{a,a+1,\dots\}$, $\mathbb{Z}(a,b)=\{a,a+1,\dots,b\}$ when $a\leqslant b$.

Consider the 2*n*th-order nonlinear difference equation

$$\Delta^n\big(r(t-n)\Delta^n u(t-n)\big)+q(t)u(t)=f\big(t,u(t+n),\dots,u(t),\dots,u(t-n)\big),\quad n\in\mathbb{Z}(3),\ t\in\mathbb{Z},\tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta u(t)=u(t+1)-u(t)$, $\Delta^2u(t)=\Delta(\Delta u(t))$. As usual, we say that a solution $u(t)$ of (1.1) is homoclinic (to 0) if $u(t)\to 0$ as $t\to\pm\infty$. In addition, if $u(t)\not\equiv 0$ then $u(t)$ is called a nontrivial homoclinic solution.

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We may think of (1.1) as being a discrete analogue of the 2nth-order differential equation

$$\left[r(t)x^{(n)}\right]^{(n)} + q(t)u(t) - f\left(t, x(t+n), \dots, x(t), \dots, x(t-n)\right) = 0, \quad t \in \mathbb{R}. \tag{1.2}$$

In some recent papers [10–12,17,18,28,29], the authors studied the existence of periodic and homoclinic solutions of second-order nonlinear difference equation by using the critical point theory. These papers show that the critical point method is an effective approach to the study of solutions of second-order difference equations. Compared to one-order or second-order difference equations [2,13], the study of higher-order equations has received considerably less attention (see, for example, [4,15,35] and references contained therein). But to the best knowledge of the authors, results on existence of homoclinic solutions of (1.1) have not been found in the literature. Recently, Cai and Yu [7] considered the existence of periodic solutions of special cases of (1.1):

$$\Delta^n\left(r(t-n)\Delta^n u(t-n)\right) = f\left(t, u(t)\right), \quad n \in \mathbb{Z}(3),\ t \in \mathbb{Z}. \tag{1.3}$$

In fact, there are some papers which discussed the equations containing both advance and retardation. Guo and Xu in [10] have given some criteria for the existence of periodic solutions to a class of second-order neutral differential difference equations as the following type

$$u''(s-\tau) - u(s-\tau) + f\left(s, u(s), u(s-\tau), u(s-2\tau)\right) = 0, \quad s \in \mathbb{R}.$$

Smets and Willem [27] had proved the existence of solitary waves with prescribed speed on infinite lattices of particles with nearest neighbor interaction for the following forward and backward differential difference equation

$$c^2 u''(s) = V'\left(u(s+1) - u(s)\right) - V'\left(u(s) - u(s-1)\right), \quad s \in \mathbb{R}.$$

In some recent papers [16–28], the authors studied the existence of periodic solutions and subharmonic solutions of some special forms of (1.1) by using the critical point theory. These papers show that the critical point method is an effective approach to the study of periodic solutions for difference equations. Along this direction, Ma and Guo [17] (with periodicity assumption) and [18] (without periodicity assumption) applied the critical point theory to prove the existence of homoclinic solutions of the following equation

$$\Delta\left[p(t)\Delta u(t-1)\right] - q(t)u(t) + f\left(t, u(t)\right) = 0, \tag{1.4}$$

where $t \in \mathbb{Z}$, $u \in \mathbb{R}$, $p, q : \mathbb{Z} \to \mathbb{R}$ and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$.

However, to our best knowledge, no similar results are obtained in the literature for (1.1). Since f in (1.1) depends on $u(t+n), \dots, u(t-n)$, the traditional ways of establishing the functional in [17,18] is inapplicable to our case, there is few paper discussing this point, see [33].

The main purpose of this paper is to give some sufficient conditions for the existence of homoclinic orbits of (1.1) using the critical point theory by establishing the corresponding variational structure which is different from [10–12,16–18,28–35], which seems not to have been considered in the literature.

In the present paper, motivated by the above paper [5,6,8,9,17,18,28,33,35], we will consider the homoclinic orbits of (1.1) under two assumptions on the nonlinearity f: superlinear and sublinear conditions. In fact, we will establish some new existence criteria to guarantee that Eq. (1.1) has one homoclinic solution or infinitely many homoclinic solutions under more relaxed assumptions on F. We generalize some existing results in the literature. However, our method used in this paper is quite different from [33]. Furthermore, it is worth pointing out that the Euler equation corresponding to the variational functional in [7] is only applicable to the case when n is even. When n is odd, the Euler equation corresponding to the variational functional defined in (2.3) is the following equation:

$$-\Delta^n\left(r(t-n)\Delta^n u(t-n)\right) + q(t)u(t) = f\left(t, u(t+n), \dots, u(t), \dots, u(t-n)\right), \quad n \in \mathbb{Z}(3),\ t \in \mathbb{Z}. \tag{1.5}$$

For the sake of convenience, throughout this paper, we always assume that n is even, of course, we can obtain the similar results of (1.5), we omit this course.

Our main results are the following theorems.

Theorem 1.1. *Assume that q and F satisfy the following assumptions*:

(r) *For every $t \in \mathbb{Z}$, $r(t) > 0$.*
(q) *For every $t \in \mathbb{Z}$, $q(t) > 0$, and $\lim_{|t|\to+\infty} q(t) = +\infty$.*
(F1) *There exists a function $F(t, x_n, \dots, x_0)$ which is continuously differentiable in the variable from x_n to x_0 for every $t \in \mathbb{Z}$ and satisfy*

$$\sum_{i=-n}^{0} F'_{2+n+i}(t+i, x_{n+i}, \dots, x_i) = f(t, x_n, x_{n-1}, \dots, x_0, x_{-1}, \dots, x_{-n}) \tag{1.6}$$

and

$$\left|f(t,x_n,x_{n-1},\dots,x_0,x_{-1},\dots,x_{-n})\right| = o\left(\left(\sum_{i=-n}^{n} x_i^2\right)^{1/2}\right), \quad as \left(\sum_{i=-n}^{n} x_i^2\right)^{1/2} \to 0,$$

$$\left|F(t,x_n,\dots,x_0)\right| = o\left(\sum_{i=0}^{n} x_i^2\right), \quad as \sum_{i=0}^{n} x_i^2 \to 0$$

uniformly in $t \in \mathbb{Z} \setminus J$.

(F2) *$F(t,x_n,\dots,x_0) = W(t,x_0) - H(t,x_n,\dots,x_0)$, for every $t \in \mathbb{Z}$, W, H are continuously differentiable in x_0 and $x_n,\dots,x_0$, respectively. Moreover, there is a bounded set $J \subset \mathbb{Z}$ such that*

$$H(t,x_n,\dots,x_0) \geqslant 0.$$

(F3) *There is a constant $\mu > 2$ such that*

$$0 < \mu W(t,x_0) \leqslant W_2'(t,x_0)x_0, \quad \forall (t,x_0) \in \mathbb{Z} \times \big(\mathbb{R} \setminus \{0\}\big).$$

(F4) *$H(t,0,\dots,0) \equiv 0$ and there is a constant $\varrho \in (2,\mu)$ such that*

$$\sum_{i=-n}^{0} H_{2+n+i}'(t,x_n,\dots,x_0)x_{-i} \leqslant \varrho H(t,x_n,\dots,x_0).$$

(F5) *There exists a constant b such that*

$$H(t,x_n,\dots,x_0) \leqslant b\gamma^{\varrho}, \quad for\ t \in \mathbb{Z},\ \gamma > 1,$$

where $\gamma = (\sum_{i=0}^{n} x_i^2)^{1/2}$.

Then Eq. (1.1) *possesses at least one nontrivial homoclinic solution.*

Theorem 1.2. *Assume that Eq.* (1.1) *satisfies* (r), (q), (F1)–(F5) *and the following condition*:

(F6) $$F(t,-x_n,\dots,-x_0) = F(t,x_n,\dots,x_0), \quad \forall (t,x_n,\dots,x_0) \in \mathbb{Z} \times \mathbb{R}^{n+1}.$$

Then Eq. (1.1) *possesses an unbounded sequence of homoclinic solutions.*

Theorem 1.3. *Assume that r, q and F satisfy* (r), (q), (F1), (F3)–(F5) *and the following assumption*:

(F2′) *$F(t,x_n,\dots,x_0) = W(t,x_0) - H(t,x_n,\dots,x_0)$, for every $t \in \mathbb{Z}$, W, H are continuously differentiable in x_0 and $x_n,\dots,x_0$, respectively. And*

$$\left|F(t,x_n,\dots,x_0)\right| = o\big(\gamma^2\big) \quad as\ \gamma \to 0,$$

where $\gamma = (\sum_{i=0}^{n} x_i^2)^{1/2}$ uniformly in $t \in \mathbb{Z}$.

Then Eq. (1.1) *possesses at least one nontrivial homoclinic solution.*

Theorem 1.4. *Assume that r, q and F satisfy* (r), (q), (F1), (F2′), (F3)–(F6), *then Eq.* (1.1) *possesses an unbounded sequence of homoclinic solutions.*

Theorem 1.5. *Assume that r, q and F satisfy* (r), (q), (F1), (F2′) *and satisfy the following assumptions*:

(F7) *For any $t \in \mathbb{Z}$,*

$$F(t,x_n,\dots,x_0) \geqslant F(t,x_0) \geqslant 0.$$

(F8) *For any $r > 0$, there exist $a = a(r)$, $b = b(r) > 0$ and $\nu < 2$ such that*

$$\left(2 + \frac{1}{a + b(\sum_{i=0}^{n} x_i^2)^{\nu/2}}\right) F(t,x_n,\dots,x_0) \leqslant \sum_{i=-n}^{0} F_{2+n+i}'(t,x_n,\dots,x_0)x_{-i}, \quad \forall t \in \mathbb{Z},\ \left(\sum_{i=0}^{n} x_i^2\right)^{1/2} \geqslant r.$$

(F9) *For any* $t \in \mathbb{Z}$,

$$\lim_{s\to+\infty}\Big[s^{-2}\min_{|x|=1}F(t,sx)\Big]=+\infty.$$

Then there exists an unbounded sequence of homoclinic solutions for Eq. (1.1).

Theorem 1.6. *Assume that* r, q *and* F *satisfy* (r), (q), (F1) *and the following assumption*:

(F10) $F(t,x_n,\dots,x_0)\geqslant 0$ *and there exists a constant* $\beta>2$ *such that*

$$0<\beta F(t,x_n,\dots,x_0)\leqslant\sum_{i=0}^{n}F'_{i+2}(t,x_n,\dots,x_0)x_i,$$

for all $(t,x_n,\dots,x_0)\in\mathbb{Z}\times\mathbb{R}^{n+1}\setminus\{(0,0,\dots,0)\}$.

Then Eq. (1.1) *possesses at least one nontrivial homoclinic solution.*

Theorem 1.7. *Assume that* r, q *and* F *satisfy* (r), (q), (F1), (F6) *and* (F10), *then Eq.* (1.1) *possesses an unbounded sequence of homoclinic solutions.*

When F is subquadratic at infinity, as far as the authors are aware, there is no research about the existence of homoclinic solutions of (1.1). Motivated by the paper [34], the intention of this paper is that, under the assumption that F is indefinite sign and subquadratic as $|t|\to+\infty$, we will establish some existence criteria to guarantee that Eq. (1.1) has at least one homoclinic solution by using minimization theorem in critical point theory.

Theorem 1.8. *Assume that* r, q *and* F *satisfy* (r), (q) *and the following conditions*:

(F11) *There exists a functional* $F(t,x_n,\dots,x_0)$ *which satisfies* (1.6) *and there exist two constants* $1<\gamma_1<\gamma_2<2$ *and two functions* $a_1,a_2\in l^{2/(2-\gamma_1)}(\mathbb{Z},[0,+\infty))$ *such that*

$$\big|F(t,x_n,\dots,x_0)\big|\leqslant a_1(t)\left(\sum_{i=0}^{n}x_i^2\right)^{\gamma_1/2},\quad\forall(t,x_n,\dots,x_0)\in\mathbb{Z}\times\mathbb{R}^{n+1},\ \left(\sum_{i=0}^{n}x_i^2\right)^{1/2}\leqslant 1$$

and

$$\big|F(t,x_n,\dots,x_0)\big|\leqslant a_2(t)\left(\sum_{i=0}^{n}x_i^2\right)^{\gamma_2/2},\quad\forall(t,x_n,\dots,x_0)\in\mathbb{Z}\times\mathbb{R}^{n+1},\ \left(\sum_{i=0}^{n}x_i^2\right)^{1/2}\geqslant 1.$$

(F12) *There exist two functions* $b\in l^{2/(2-\gamma_1)}(\mathbb{Z},[0,+\infty))$ *and* $\varphi\in C([0,+\infty),[0,+\infty))$ *such that for every* $(t,x_n,x_{n-1},\dots,x_0,x_{-1},\dots,x_{-n})\in\mathbb{Z}\times\mathbb{R}^{2n+1}$,

$$\big|f(t,x_n,x_{n-1},\dots,x_0,x_{-1},\dots,x_{-n})\big|\leqslant b(t)\varphi\left(\left(\sum_{i=-n}^{n}x_i^2\right)^{1/2}\right),$$

where $\varphi(s)=O(s^{\gamma_1-1})$ *as* $|s|\leqslant c$, c *is a positive constant.*

(F13) *There exist* $t_0\in\mathbb{Z}$ *and two constants* $\gamma_3\in(1,2)$ *and* $\eta>0$ *such that*

$$F(t_0,x_n,\dots,x_0)\geqslant\eta\left(\sum_{i=-n}^{n}x_i^2\right)^{\gamma_3/2},\quad\forall(t,x_n,\dots,x_0)\in\mathbb{Z}\times\mathbb{R}^{n+1},\ \left(\sum_{i=-n}^{0}x_i^2\right)^{1/2}\leqslant 1.$$

Then Eq. (1.1) *possesses at least one nontrivial homoclinic solution.*

2. Preliminaries

To apply critical point theory to study the existence of homoclinic solutions of (1.1), we shall state some basic notations and lemmas, which will be used in the proofs of our main results.

Let

$$S = \{\{u(t)\}_{t\in\mathbb{Z}}\colon u(t)\in\mathbb{R},\ t\in\mathbb{Z}\},$$
$$E = \left\{u\in S\colon \sum_{t\in\mathbb{Z}}\left[r(t-1)\left(\Delta^n u(t-1)\right)^2 + q(t)\left(u(t)\right)^2\right] < +\infty\right\},$$

and for $u, v\in E$, let

$$\langle u, v\rangle = \sum_{t\in\mathbb{Z}}\left[r(t-1)\Delta^n u(t-1)\Delta^n v(t-1) + q(t)u(t)v(t)\right]. \tag{2.1}$$

Then E is a Hilbert space with the above inner product, and the corresponding norm is

$$\|u\| = \left\{\sum_{t\in\mathbb{Z}}\left[r(t-1)\left(\Delta^n u(t-1)\right)^2 + q(t)\left(u(t)\right)^2\right]\right\}^{1/2}, \quad u\in E. \tag{2.2}$$

In what follows, l_I^2 and l^2 denote the space of functions whose second powers are summable on the interval I and $\mathbb{Z}$ equipped with

$$\|u\|_I^2 = \sum_{t\in I}\left|u(t)\right|^2, \qquad \|u\|^2 = \sum_{t\in\mathbb{Z}}\left|u(t)\right|^2.$$

Let

$$l^\infty(\mathbb{Z},\mathbb{R}) = \left\{u\in S\colon \sup_{t\in\mathbb{Z}}\left|u(t)\right| < +\infty\right\}.$$

For any $n_1, n_2\in\mathbb{Z}$ with $n_1 < n_2$, we let $\mathbb{Z}(n_1, n_2) = [n_1, n_2]\cap\mathbb{Z}$; and for function $f:\mathbb{Z}\to\mathbb{R}$ and $a\in\mathbb{R}$, we set

$$\mathbb{Z}\left(f(t)\geqslant a\right) = \left\{t\in\mathbb{Z}\colon\ f(t)\geqslant a\right\}, \qquad \mathbb{Z}\left(f(t)\leqslant a\right) = \left\{t\in\mathbb{Z}\colon\ f(t)\leqslant a\right\}.$$

Let $I: E\to\mathbb{R}$ be defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \sum_{t\in\mathbb{Z}} F\left(t, u(t+n), \dots, u(t)\right). \tag{2.3}$$

If (q) and (F1) hold, then $I\in C^1(E,\mathbb{R})$ and one can easily check that

$$\left\langle I'(u), v\right\rangle = \sum_{t\in\mathbb{Z}}\left[r(t-1)\Delta^n u(n-1)\Delta^n v(n-1) + q(t)u(t)v(t) - f\left(t, u(t+n), \dots u(t), \dots u(t-n)v(t)\right)\right],$$
$$\forall u, v\in E. \tag{2.4}$$

By using

$$\Delta^n u(t-1) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}u(t+n-k-1),$$

we can compute the partial derivative as

$$\frac{\partial I(u)}{\partial u(t)} = \Delta^n\left(r(t-n)\Delta^n u(t-n)\right) + q(t)u(t) - f\left(t, u(t+n), \dots, u(t), \dots, u(t-n)\right). \tag{2.5}$$

So, the critical points of I in E are the solutions of Eq. (1.1) with $u(\pm\infty) = 0$.

We will obtain the critical points of I by the Mountain Pass Theorem and the Symmetric Mountain Pass Theorem. Since the minimax characterization it provides the critical value is important for what follows. Therefore, we state the theorem precisely.

Lemma 2.1. *(See [25].) Let E be a real Banach space and $I\in C^1(E,\mathbb{R})$ satisfy* (PS)*-condition. Suppose that I satisfies the following conditions*:

(i) $I(0) = 0$.
(ii) *There exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)}\geqslant\alpha$.*
(iii) *There exists $e\in E\setminus\bar{B}_\rho(0)$ such that $I(e)\leqslant 0$.*

Then I possesses a critical value $c \geqslant \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I\big(g(s)\big),$$

where $B_\rho(0)$ is an open ball in E of radius ρ centered at 0, and

$$\Gamma = \big\{g \in C\big([0,1], E\big):\ g(0) = 0, g(1) = e\big\}.$$

Lemma 2.2. *(See [25].) Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ with I even. Suppose that I satisfies (PS)-condition, (i), (ii) of Lemma 2.1 and the following condition:*

(iii′) *For each finite dimensional subspace $E' \subset E$, there is $r = r(E') > 0$ such that $I(u) \leqslant 0$ for $u \in E' \setminus B_r(0)$, where $B_r(0)$ is an open ball in E of radius r centered at 0.*

Then I possesses an unbounded sequence of critical values.

Lemma 2.3. *(See [19].) Let E be a real Banach space and $I \in C^1(E,\mathbb{R})$ satisfy the (PS)-condition. If I is bounded from below, then $c = \inf_E I$ is a critical value of I.*

Lemma 2.4. *For $u \in E$,*

$$\beta\|u\|_\infty^2 \leqslant \beta\|u\|_{l^2}^2 \leqslant \|u\|^2, \tag{2.6}$$

where $\beta = \inf_{t\in\mathbb{Z}} q(t)$.

Proof. Since $u \in E$, it follows that $\lim_{|t|\to\infty}|u(t)| = 0$. Hence, there exists $t^* \in \mathbb{Z}$ such that

$$\|u\|_\infty = \big|u\big(t^*\big)\big| = \max_{t\in\mathbb{Z}}\big|u(t)\big|.$$

By (q) and (2.2), we have

$$\|u\|^2 \geqslant \sum_{t\in\mathbb{Z}} q(t)\big(u(t)\big)^2 \geqslant \beta\sum_{t\in\mathbb{Z}}\big|u(t)\big|^2 \geqslant \beta\|u\|_\infty^2.$$

The proof is complete. □

Lemma 2.5. *Assume that (F3) hold. Then for every $(t,x) \in \mathbb{Z}\times\mathbb{R}$, $s^{-\mu}W(t,sx)$ is nondecreasing on $(0,+\infty)$.*

The proof of Lemma 2.5 is routine and so we omit it.

3. Proofs of theorems

Proof of Theorem 1.1. It is clear that $I(0) = 0$. We first show that I satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $c > 0$ such that

$$\big|I(u_k)\big| \leqslant c, \qquad \big\|I'(u_k)\big\|_{E^*} \leqslant \varrho c \quad \text{for } k \in \mathbb{N}. \tag{3.1}$$

From (2.2), (2.3), (2.4), (3.1), (F3) and (F4), we obtain

$$\begin{aligned}
2c + 2c\|u_k\| &\geqslant 2I(u_k) - \frac{2}{\varrho}\big\langle I'(u_k), u_k\big\rangle \\
&= \frac{\varrho-2}{\varrho}\|u_k\|^2 - 2\sum_{t\in\mathbb{Z}}\Big[W\big(t,u_k(t)\big) - \frac{1}{\varrho}W_2'\big(t,u_k(t)\big)u_k(t)\Big] + 2\sum_{t\in\mathbb{Z}} H\big(t,u_k(t+n),\ldots,u_k(t)\big) \\
&\quad - \frac{2}{\varrho}\sum_{t\in\mathbb{Z}}\sum_{i=-n}^{0} H'_{2+n+i}\big(t+i,u_k(t+n+i),\ldots,u_k(t+i)\big)u_k(t) \\
&= \frac{\varrho-2}{\varrho}\|u_k\|^2 - 2\sum_{n\in\mathbb{Z}}\Big[W\big(t,u_k(t)\big) - \frac{1}{\varrho}W_2'\big(t,u_k(t)\big)u_k(t)\Big] + 2\sum_{t\in\mathbb{Z}} H\big(t,u_k(t+n),\ldots,u_k(t)\big)
\end{aligned}$$

$$- \frac{2}{\varrho} \sum_{t\in\mathbb{Z}} \sum_{i=-n}^{0} H'_{2+n+i}\big(t, u_k(t+n), \ldots, u_k(t)\big) u_k(t-i)$$
$$\geqslant \frac{\varrho - 2}{\varrho} \|u_k\|^2, \quad k \in \mathbb{N}.$$

It follows that there exists a constant $A > 0$ such that

$$\|u_k\| \leqslant A \quad \text{for } k \in \mathbb{N}. \tag{3.2}$$

So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in E. For any given number $\varepsilon > 0$, by (F1), we can choose $\zeta > 0$ such that

$$\big| f\big(t, u(t+n), \ldots, u(t), \ldots, u(t-n)\big)\big| \leqslant \varepsilon\xi \quad \text{for } t \in \mathbb{Z} \setminus J, \qquad \big(u(t+n), \ldots, u(t), \ldots, u(t-n)\big) \in \mathbb{R}^{2n+1}, \tag{3.3}$$

where $\xi = (\sum_{i=-n}^{n}(u(t+i))^2)^{1/2} \leqslant \zeta$.

Since $q(t) \to \infty$, we can also choose an integer $\Pi > \max\{|k|\colon k \in J\}$ such that

$$q(t) \geqslant \frac{(2n+1)A^2}{\zeta^2}, \quad |t| \geqslant \Pi. \tag{3.4}$$

By (3.2) and (3.4), we have

$$\big|u_k(t)\big|^2 = \frac{1}{q(t)} q(t)\big(u_k(t)\big)^2 \leqslant \frac{\zeta^2}{(2n+1)A^2} \|u_k\|^2 \leqslant \frac{\zeta^2}{2n+1} \quad \text{for } |t| \geqslant \Pi,\ k \in \mathbb{N}. \tag{3.5}$$

Since $u_k \rightharpoonup u_0$ in E, it is easy to verify that $u_k(t)$ converges to $u_0(t)$ pointwise for all $t \in \mathbb{Z}$, that is

$$\lim_{k\to\infty} u_k(t) = u_0(t), \quad \forall t \in \mathbb{Z}. \tag{3.6}$$

Hence, we have by (3.5) and (3.6)

$$\big|u_0(t)\big|^2 \leqslant \frac{\zeta^2}{2n+1} \quad \text{for } |t| \geqslant \Pi. \tag{3.7}$$

It follows from (3.6) and the continuity of $f(t, u(t+1), \ldots, u(t), \ldots, u(t-n))$ on $u(t+1), \ldots, u(t), \ldots, u(t-n)$ that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{t=-\Pi}^{\Pi} \big| f\big(t, u_k(t+n), \ldots, u_k(t), \ldots, u_k(t-n)\big) - f\big(t, u_0(t+n), \ldots, u_0(t), \ldots, u_0(t-n)\big)\big| < \varepsilon \quad \text{for } k \geqslant k_0. \tag{3.8}$$

On the other hand, it follows from (F1), (2.6), (3.2), (3.3), (3.5) and (3.7) that

$$\begin{aligned}
&\sum_{|t|>\Pi} \big| f\big(t, u_k(t+n), \ldots, u_k(t), \ldots, u_k(t-n)\big) - f\big(t, u_0(t+n), \ldots, u_0(t), \ldots, u_0(t-n)\big)\big| \big|u_k(t) - u_0(t)\big| \\
&\quad \leqslant \sum_{|t|>\Pi} \big(\big| f\big(t, u_k(t+1), \ldots, u_k(t), \ldots, u_k(t-n)\big)\big| + \big| f\big(t, u_0(t+1), \ldots, u_0(t), \ldots, u_0(t-n)\big)\big|\big)\big(\big|u_k(t)\big| + \big|u_0(t)\big|\big) \\
&\quad \leqslant \varepsilon \sum_{|t|>\Pi} \left[\left(\sum_{i=-n}^{n} \big(u_k(t+i)\big)^2 \right)^{1/2} + \left(\sum_{i=-n}^{n} \big(u_0(t+i)\big)^2 \right)^{1/2} \right] \big(\big|u_k(t)\big| + \big|u_0(t)\big|\big) \\
&\quad \leqslant (2n+1)\varepsilon \sum_{t\in\mathbb{Z}} \big(\big|u_k(t)\big| + \big|u_0(t)\big|\big)\big(\big|u_k(t)\big| + \big|u_0(t)\big|\big) \\
&\quad \leqslant (4n+2)\varepsilon \sum_{t\in\mathbb{Z}} \big(\big|u_k(t)\big|^2 + \big|u_0(t)\big|^2\big) \\
&\quad \leqslant \frac{(4n+2)\varepsilon}{\beta} \big(A^2 + \|u_0\|^2\big).
\end{aligned} \tag{3.9}$$

Since ε is arbitrary, combining (3.8) with (3.9) we get

$$\begin{aligned}
&\sum_{t\in\mathbb{Z}} \big| f\big(t, u_k(t+n), \ldots, u_k(t), \ldots, u_k(t-n)\big) - f\big(t, u_0(t+n), \ldots, u_0(t), \ldots, u_0(t-n)\big)\big|, \\
&\big|u_k(t) - u_0(t)\big| \to 0 \quad \text{as } k \to \infty.
\end{aligned} \tag{3.10}$$

It follows from (2.2) and (2.4) that

$$\big\langle I'(u_k)-I'(u_0),u_k-u_0\big\rangle=\|u_k-u_0\|^2-\sum_{t\in\mathbb{Z}}\big(f\big(t,u_k(t+1),\dots,u_k(t),\dots,u_k(t-n)\big)\\ -f\big(t,u_0(t+1),\dots,u_0(t),\dots,u_0(t-n)\big),u_k(t)-u_0(t)\big). \tag{3.11}$$

Since $\langle I'(u_k)-I'(u_0),u_k-u_0\rangle\to 0$, it follows from (3.10) and (3.11) that $u_k\to u_0$ in E. Hence, I satisfies the (PS)-condition.

We now show that there exist constants $\rho,\alpha>0$ such that I satisfies the assumption (ii) of Lemma 2.1 with these constants. By (F1), there exists $\eta\in(0,1)$ such that

$$\big|F\big(t,u(t+n),\dots,u(t)\big)\big|\leqslant\frac{\beta}{8(n+1)}\sum_{i=0}^{n}\big(u(t+i)\big)^2\quad\text{for } t\in\mathbb{Z}\setminus J,\ \left(\sum_{i=0}^{n}\big(u(t+i)\big)^2\right)^{1/2}\leqslant\eta. \tag{3.12}$$

Set

$$M=\sup\big\{W(t,u)\ \big|\ t\in J,\ u\in\mathbb{R},\ |u|=1\big\}, \tag{3.13}$$

and

$$\delta=\min\big\{\big(\beta/(8M+1)\big)^{(\mu-2)},\eta\big\}.$$

If $\|u\|=\sqrt{\beta}\delta:=\rho$, then by Lemma 2.4, $|u(t)|\leqslant\delta\leqslant\eta<1$ for $t\in\mathbb{Z}$. By (q), (3.13) and Lemma 2.4, we have

$$\begin{aligned}\sum_{t\in J}W\big(t,u(t)\big)&\leqslant\sum_{t\in J,u(t)\neq 0}W\left(t,\frac{u(t)}{|u(t)|}\right)\big|u(t)\big|^{\mu}\\ &\leqslant M\sum_{t\in J}\big|u(t)\big|^{\mu}\\ &\leqslant M\delta^{\mu-2}\sum_{t\in J}\big(u(t)\big)^2\\ &\leqslant\frac{M\delta^{\mu-2}}{\beta}\sum_{t\in J}q(t)\big(u(t)\big)^2\\ &\leqslant\frac{1}{8}\sum_{t\in J}q(t)\big(u(t)\big)^2.\end{aligned} \tag{3.14}$$

Set $\alpha=\beta\delta^2/4$. Hence, from (2.1), (3.12), (3.14), (q), (F1) and (F2), we have

$$\begin{aligned}I(u)&=\frac{1}{2}\|u\|^2-\sum_{t\in\mathbb{Z}}F\big(t,u(t+n),\dots,u(t)\big)\\ &=\frac{1}{2}\|u\|^2-\sum_{t\in\mathbb{Z}\setminus J}F\big(t,u(t+n),\dots,u(t)\big)-\sum_{t\in J}F\big(t,u(t+n),\dots,u(t)\big)\\ &\geqslant\frac{1}{2}\|u\|^2-\frac{\beta}{8(n+1)}\sum_{t\in\mathbb{Z}\setminus J}\sum_{i=0}^{n}\big(u(t+i)\big)^2-\sum_{t\in J}W\big(t,u(t)\big)+\sum_{t\in J}H\big(t,u(t+n),\dots,u(t)\big)\\ &\geqslant\frac{1}{2}\|u\|^2-\frac{\beta}{8}\sum_{t\in\mathbb{Z}}\big(u(t)\big)^2-\frac{1}{8}\sum_{t\in J}q(t)\big(u(t)\big)^2\\ &\geqslant\frac{1}{2}\|u\|^2-\frac{1}{8}\sum_{t\in\mathbb{Z}}q(t)\big(u(t)\big)^2-\frac{1}{8}\sum_{t\in J}q(t)\big(u(t)\big)^2\\ &\geqslant\frac{1}{2}\|u\|^2-\frac{1}{8}\|u\|^2-\frac{1}{8}\|u\|^2\\ &=\frac{1}{4}\|u\|^2\\ &=\alpha.\end{aligned} \tag{3.15}$$

(3.15) shows that $\|u\|=\rho$ implies that $I(u)\geqslant\alpha$, i.e., I satisfies assumption (ii) of Lemma 2.1.

Finally, it remains to show that I satisfies assumption (iii) of Lemma 2.1. Take $\omega \in E$ such that

$$\big|\omega(t)\big| = \begin{cases} 1, & \text{for } |t| \leqslant 1, \\ 0, & \text{for } |t| \geqslant 2, \end{cases} \tag{3.16}$$

and $|\omega(t)| \leqslant 1$ for $|t| \in (1,2)$.

For any $u \in E$, it follows from (2.4) and (F5) that

$$\begin{aligned} \sum_{t=-2}^{2} H\big(t, u(t+n), \ldots, u(t)\big) &= \sum_{\{t\in\mathbb{Z}(-2,2),\ (\sum_{i=0}^{n}(u(t+i))^2)^{1/2}>1\}} H\big(t, u(t+n), \ldots, u(t)\big) \\ &\quad + \sum_{\{t\in\mathbb{Z}(-2,2)\ (\sum_{i=0}^{n}(u(t+i))^2)^{1/2}\leqslant 1\}} H\big(t, u(t+n), \ldots, u(t)\big) \\ &\leqslant b \sum_{\{t\in\mathbb{Z}(-2,2),\ (\sum_{i=0}^{n}(u(t+i))^2)^{1/2}>1\}} \left(\sum_{i=0}^{n}\big(u(t+i)\big)^2\right)^{\frac{\varrho}{2}} \\ &\quad + \sum_{\{t\in\mathbb{Z}(-2,2),\ (\sum_{i=0}^{n}(u(t+i))^2)^{1/2}\leqslant 1\}} \big|H\big(t, u(t+n), \ldots, u(t)\big)\big| \\ &\leqslant (n+1)^{\frac{\varrho}{2}+1} b \sum_{t\in\mathbb{Z}} \big|u(t)\big|^{\varrho} + \sum_{\{t\in\mathbb{Z}(-2,2),\ (\sum_{i=0}^{n}(u(t+i))^2)^{1/2}\leqslant 1\}} \big|H\big(t, u(t+n), \ldots, u(t)\big)\big| \\ &\leqslant (n+1)^{\frac{\varrho}{2}+1} \beta^{\frac{\varrho}{2}} b \|u\|^{\varrho} + \sum_{\{t\in\mathbb{Z}(-2,2),\ (\sum_{i=0}^{n}(u(t+i))^2)^{1/2}\leqslant 1\}} \big|H\big(t, u(t+n), \ldots, u(t)\big)\big| \\ &= M_0 \|u\|^{\varrho} + M_1, \end{aligned} \tag{3.17}$$

where

$$M_0 = (n+1)^{\frac{\varrho}{2}+1} \beta^{\frac{\varrho}{2}} b, \qquad M_1 = \sum_{\{t\in\mathbb{Z}(-2,2),\ (\sum_{i=0}^{n}(u(t+i))^2)^{1/2}\leqslant 1\}} \big|H\big(t, u(t+n), \ldots, u(t)\big)\big|.$$

For $\sigma > 1$, by Lemma 2.3(i) and (3.16), we have

$$\sum_{t=-1}^{1} W\big(t, \sigma\omega(t)\big) \geqslant \sigma^{\mu} \sum_{t=-1}^{1} W\big(t, \omega(t)\big) = m\sigma^{\mu}, \tag{3.18}$$

where $m = \sum_{t=-1}^{1} W(t, \omega(t)) > 0$. By (2.1), (3.16), (3.17) and (3.18), we have for $\sigma > 1$,

$$\begin{aligned} I(\sigma\omega) &= \frac{1}{2}\|\sigma\omega\|^2 + \sum_{t\in\mathbb{Z}} \big[H\big(t, \sigma\omega(t+n), \ldots, \sigma\omega(t)\big) - W\big(t, \sigma\omega(t)\big)\big] \\ &\leqslant \frac{\sigma^2}{2}\|\omega\|^2 + \sum_{t=-2}^{2} H\big(t, \sigma\omega(t+n), \ldots, \sigma\omega(t)\big) - \sum_{t=-1}^{1} W\big(t, \sigma\omega(t)\big) \\ &\leqslant \frac{\sigma^2}{2}\|\omega\|^2 + M_0\sigma^{\varrho}\|\omega\|^{\varrho} + M_1 - m\sigma^{\mu}. \end{aligned} \tag{3.19}$$

Since $\mu > \varrho \geqslant 2$ and $m > 0$, (3.19) implies that there exists $\sigma_0 > 1$ such that $\|\sigma_0\omega\| > \rho$ and $I(\sigma_0\omega) < 0$. Set $e = \sigma_0\omega(t)$. Then $e \in E$, $\|e\| = \|\sigma_0\omega\| > \rho$ and $I(e) = I(\sigma_0\omega) < 0$. By Lemma 2.1, I possesses a critical value $d \geqslant \alpha$ given by

$$d = \inf_{g\in\Gamma} \max_{s\in[0,1]} I\big(g(s)\big), \tag{3.20}$$

where

$$\Gamma = \big\{g \in C\big([0,1], E\big):\ g(0) = 0,\ g(1) = e\big\}.$$

Hence, there exists $u^* \in E$ such that

$$I\big(u^*\big) = d, \quad \text{and} \quad I'\big(u^*\big) = 0. \tag{3.21}$$

Then function u^* is a desired classical solution of Eq. (1.1). Since $d > 0$, u^* is a nontrivial homoclinic solution. The proof is complete. □

Proof of Theorem 1.2. The main idea of the proof is the same as in [28]. (F6) implies that I is even. In view of the proof of Theorem 1.1, we see that $I \in C^1(X, \mathbb{R})$, and I satisfies (PS)-condition and assumptions (i) and (ii) of Lemma 2.2. To apply Lemma 2.2, it suffices to prove that I satisfies (iii′) of Lemma 2.2.

Now, we prove (iii). Let E' be a finite dimensional subspace of E. Since all norms of a finite dimensional normed space are equivalent, so there is a constant $c > 0$ such that

$$\|u\| \leqslant c\|u\|_\infty \quad \text{for } u \in E'. \tag{3.22}$$

Assume that $\dim E' = m$ and $u_1, u_2, \ldots, u_m$ are the basis of E' such that

$$\langle u_i, u_j\rangle = \begin{cases} c^2, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j = 1, 2, \ldots, m. \tag{3.23}$$

Since $u_i \in E$, we can choose an integer $\Pi_1 > \max\{|k|\colon\ k \in J\}$ such that

$$\left|u_i(t)\right| < \frac{\eta}{m}, \quad |t| > \Pi_1,\ i = 1, 2, \ldots, m. \tag{3.24}$$

Set $\Theta = \{u \in E'\colon\ \|u\| = c\}$. Then for $u \in \Theta$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$ such that

$$u(t) = \sum_{i=1}^{m} \lambda_i u_i(t) \quad \text{for } t \in \mathbb{Z}, \tag{3.25}$$

it follows that

$$c^2 = \|u\|^2 = \langle u, u\rangle = \sum_{i=1}^{m} \lambda_i^2 \langle u_i, u_i\rangle = c^2 \sum_{i=1}^{m} \lambda_i^2,$$

which implies that $|\lambda_i| \leqslant 1$ for $i = 1, 2, \ldots, m$. Hence, for $u \in \Theta$, let $|u(t_0)| = \|u\|_\infty$, then by (3.22) and (3.25) we have

$$1 \leqslant \|u\|_\infty = \left|u(t_0)\right| \leqslant \sum_{i=1}^{m} |\lambda_i|\left|u_i(t_0)\right| \leqslant \sum_{i=1}^{m}\left|u_i(t_0)\right|, \quad u \in \Theta. \tag{3.26}$$

This shows that there exists $i_0 \in \{1, 2, \ldots, m\}$ such that $|u_{i_0}(t_0)| \geqslant 1/m$, which, together with (3.24), implies that $|n_0| \leqslant \Pi_1$. Set

$$\tau = \min\left\{W(t, v)\colon\ |t| \leqslant \Pi_1,\ |v| = 1\right\}. \tag{3.27}$$

Since $W(t, v) > 0$ for all $t \in \mathbb{Z}$ and $v \in \mathbb{R} \setminus \{0\}$, and $W(t, v)$ is continuous in v, so $\tau > 0$. It follows from (3.26), (3.27) and Lemma 2.5 that

$$\sum_{t=-\Pi_1}^{\Pi_1} W\big(t, u(t)\big) \geqslant W\big(t_0, u(t_0)\big) \geqslant W\left(t_0, \frac{u(t_0)}{|u(t_0)|}\right)\left|u(t_0)\right|^\mu \geqslant \Big[\min_{|x|=1} W(t_0, x)\Big]\left|u(t_0)\right|^\mu \geqslant \tau \quad \text{for } u \in \Theta. \tag{3.28}$$

For any $u \in E$, it follows from (2.2) and (F5) that

$$\begin{aligned}
\sum_{t=-\Pi_1}^{\Pi_1} H\big(t, u(t+n), \ldots, u(t)\big) = {} & \sum_{\{t \in \mathbb{Z}(-\Pi_1, \Pi_1),\ (\sum_{i=0}^{n}(u(t+i))^2)^{1/2} > 1\}} H\big(t, u(t+n), \ldots, u(t)\big) \\
& + \sum_{\{t \in \mathbb{Z}(-\Pi_1, \Pi_1),\ (\sum_{i=0}^{n}(u(t+i))^2)^{1/2} \leqslant 1\}} H\big(t, u(t+n), \ldots, u(t)\big) \\
\leqslant {} & b \sum_{\{t \in \mathbb{Z}(-\Pi_1, \Pi_1),\ (\sum_{i=0}^{n}(u(t+i))^2)^{1/2} > 1\}} \left(\sum_{i=0}^{n} u(t+i)^2\right)^{\frac{\varrho}{2}} \\
& + \sum_{t=-\Pi_1}^{\Pi_1} \max_{(\sum_{i=0}^{n}(u(t+i))^2)^{1/2} \leqslant 1} \left|H\big(t, u(t+n), \ldots, u(t)\big)\right|
\end{aligned}$$

$$\leqslant (n+1)^{\frac{\varrho}{2}+1} b \sum_{t\in\mathbb{Z}} \left|u(t)\right|^{\varrho} + \sum_{t=-\Pi_1}^{\Pi_1} \max_{(\sum_{i=0}^{n}(u(t+i))^2)^{1/2}\leqslant 1} \left|H\big(t, u(t+n), \ldots, u(t)\big)\right|$$

$$\leqslant (n+1)^{\frac{\varrho}{2}+1} \beta^{-\frac{\varrho}{2}} b \|u\|^{\varrho} + \sum_{t=-\Pi_1}^{\Pi_1} \max_{(\sum_{i=0}^{n}(u(t+i))^2)^{1/2}\leqslant 1} \left|H\big(t, u(t+n), \ldots, u(t)\big)\right|$$

$$= M_0\|u\|^{\varrho} + M_1, \tag{3.29}$$

where

$$M_0 = (n+1)^{\frac{\varrho}{2}+1}\beta^{-\frac{\varrho}{2}} b, \qquad M_1 = \sum_{t=-\Pi_1}^{\Pi_1} \max_{(\sum_{i=0}^{n}(u(t+i))^2)^{1/2}\leqslant 1} \left|H\big(t, u(t+n), \ldots, u(t)\big)\right|.$$

From (3.14), (3.24), (3.25), (3.28), (3.29) and Lemma 2.5, we have for $u \in \Theta$ and $\sigma > 1$,

$$\begin{aligned}
I(\sigma u) &= \frac{\sigma^2}{2}\|u\|^2 - \sum_{t\in\mathbb{Z}} F\big(t, \sigma u(t+n), \ldots, \sigma u(t)\big) \\
&= \frac{\sigma^2}{2}\|u\|^2 - \sum_{|t|>\Pi_1} F\big(t, \sigma u(t+n), \ldots, \sigma u(t)\big) - \sum_{|t|\leqslant\Pi_1} F\big(t, \sigma u(t+n), \ldots, \sigma u(t)\big) \\
&\leqslant \frac{\sigma^2}{2}\|u\|^2 + \sum_{|t|>\Pi_1}\sum_{i=0}^{n} \frac{\beta\sigma^2}{8(n+1)}\big(u(t+i)\big)^2 - \sum_{|t|\leqslant\Pi_1} F\big(t, \sigma u(t+n), \ldots, \sigma u(t)\big) \\
&\leqslant \frac{\sigma^2}{2}\|u\|^2 + \frac{\sigma^2}{8}\|u\|^2 - \sum_{|t|\leqslant\Pi_1} F\big(t, \sigma u(t+n), \ldots, \sigma u(t)\big) \\
&= \frac{\sigma^2}{2}\|u\|^2 + \frac{\sigma^2}{8}\|u\|^2 - \sum_{|t|\leqslant\Pi_1} W\big(n, \sigma u(n)\big) + \sum_{|t|\leqslant\Pi_1} H\big(t, \sigma u(t+n), \ldots, \sigma u(t)\big) \\
&\leqslant \frac{\sigma^2}{2}\|u\|^2 + \frac{\sigma^2}{8}\|u\|^2 + \sigma^{\varrho}\big(M_0\|u\|^{\varrho} + M_1\big) - \tau\sigma^{\mu} \\
&= \frac{(c\sigma)^2}{2} + \frac{c^2\sigma^2}{8} + M_0(c\sigma)^{\varrho} + M_1\sigma^{\varrho} - \tau\sigma^{\mu}.
\end{aligned} \tag{3.30}$$

Since $\mu > \varrho > 2$, we deduce that there is $\sigma_0 = \sigma_0(c, M_1, M_2, \tau) = \sigma_0(E') > 1$ such that

$$I(\sigma u) < 0 \quad \text{for } u \in \Theta \text{ and } \sigma \geqslant \sigma_0.$$

That is

$$I(u) < 0 \quad \text{for } u \in E' \text{ and } \|u\| \geqslant c\sigma_0.$$

This shows that (iii) of Lemma 2.2 holds. By Lemma 2.2, I possesses an unbounded sequence $\{d_k\}_{k\in\mathbb{N}}$ of critical values with $d_k = I(u_k)$, where u_k is such that $I'(u_k) = 0$ for $k = 1, 2, \ldots$. If $\{\|u_k\|\}_{k\in\mathbb{N}}$ is bounded, then there exists $B > 0$ such that

$$\|u_k\| \leqslant B \quad \text{for } k \in \mathbb{N}. \tag{3.31}$$

By a similar fashion for the proof of (3.4) and (3.7), for the given η in (3.12), there exists $\Pi_2 > \max\{|k|\colon k \in J\}$ such that

$$\left|u_k(t)\right| \leqslant \eta \quad \text{for } |t| \geqslant \Pi_2,\ k \in \mathbb{N}. \tag{3.32}$$

Thus, from (2.1), (2.3), (3.12), (3.31) and (3.32), we have

$$\begin{aligned}
\frac{1}{2}\|u_k\|^2 &= d_k + \sum_{t\in\mathbb{Z}} F\big(t, u_k(t+n), \ldots, u_k(t)\big) \\
&= d_k + \sum_{|t|>\Pi_2} F\big(t, u_k(t+n), \ldots, u_k(t)\big) + \sum_{t=-\Pi_2}^{\Pi_2} F\big(t, u_k(t+n), \ldots, u_k(t)\big)
\end{aligned}$$

$$\geqslant d_k - \frac{\beta}{8(n+1)} \sum_{|t|>\Pi_2} \sum_{i=0}^{n} \big(u_k(t+i)\big)^2 - \sum_{n=-\Pi_2}^{\Pi_2} H\big(t, u_k(t+n), \ldots, u_k(t)\big)$$

$$\geqslant d_k - \frac{1}{8}\|u_k\|^2 - \sum_{n=-\Pi_2}^{\Pi_2} \max_{|u_k|\leqslant B/\sqrt{\beta}} \big|H\big(t, u_k(t+n), \ldots, u_k(t)\big)\big|. \tag{3.33}$$

It follows that

$$d_k \leqslant \frac{5}{8}\|u_k\|^2 + \sum_{t=-\Pi_2}^{\Pi_2} \max_{|u_k|\leqslant B/\sqrt{\beta}} \big|H\big(t, u_k(t+n), \ldots, u_k(t)\big)\big| < +\infty.$$

This contradicts to the fact that $\{d_k\}_{k\in\mathbb{N}}$ is unbounded, and so $\{\|u_k\|\}_{k\in\mathbb{N}}$ is unbounded. The proof is complete. □

Proof of Theorems 1.3 and 1.4. In the proof of Theorem 1.1, the condition that $H(t, u(t+n), \ldots, u(t)) \geqslant 0$ for $(t, u(t+n), \ldots, u(t)) \in J \times \mathbb{R}^{n+1}$, $\gamma = (\sum_{i=0}^{n} u^2(t+i))^{1/2} \leqslant 1$ in (F1) is only used in the proofs of assumption (ii) of Lemma 2.1. Therefore, we only prove assumption (ii) of Lemma 2.1 still hold using (F2′) instead of (F2). By (F2′), it follows that

$$\big|F\big(t, u(t+n), \ldots, u(t)\big)\big| \leqslant \frac{\beta}{4(n+1)} \sum_{i=0}^{n} u^2(t+i) \quad \text{for } t \in \mathbb{Z},\ \left(\sum_{i=0}^{n} u^2(t+i)\right)^{1/2} \leqslant \eta. \tag{3.34}$$

If $\|u\| = \sqrt{\beta}\eta := \rho$, then by Lemma 2.4, $|u(t)| \leqslant \eta$ for $t \in \mathbb{Z}$. Set $\alpha = \beta\eta^2/4$. Hence, from (2.2), (2.3), (3.34) and Lemma 2.4, we have

$$I(u) = \frac{1}{2}\|u\|^2 - \sum_{t\in\mathbb{Z}} F\big(t, u(t+n), \ldots, u(t)\big) \geqslant \frac{1}{2}\|u\|^2 - \frac{\beta}{4}\sum_{t\in\mathbb{Z}} u^2(t) \geqslant \frac{1}{2}\|u\|^2 - \frac{1}{4}\|u\|^2 = \frac{1}{4}\|u\|^2 = \alpha. \tag{3.35}$$

(3.35) shows that $\|u\| = \rho$ implies that $I(u) \geqslant \alpha$, i.e., assumption (ii) of Lemma 2.2 holds. The proof of Theorems 1.3 and 1.4 is completed. □

Proof of Theorem 1.5. We first show that I satisfies condition (C). Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a (C) sequence of I, that is, $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $(1 + \|u_k\|)\|I'(u_k)\| \to 0$ as $k \to +\infty$. Then it follows from (2.1) and (2.2) that

$$C_1 \geqslant 2I(u_k) - \big\langle I'(u_k), u_k\big\rangle$$

$$= \sum_{t\in\mathbb{Z}} \left[\sum_{i=-n}^{0} F'_{2+n+i}\big(t, u_k(t+n), \ldots, u_k(t)\big)u_k(t-i) - 2F\big(t, u_k(t+n), \ldots, u_k(t)\big)\right]. \tag{3.36}$$

It follows from (F8) that there exists $\eta \in (0, 1)$ such that (3.34) holds. By (F8), we have

$$\sum_{i=-n}^{0} F'_{2+n+i}\big(t, u(t+n), \ldots, u(t)\big)u(t-i) > 2F\big(t, u_k(t+n), \ldots, u(t)\big) \geqslant 0$$

$$\text{for } \big(t, u(t+n), \ldots, u(t)\big) \in \mathbb{Z} \times \mathbb{R}^{n+1},\ k \in \mathbb{N}, \tag{3.37}$$

and for $t \in \mathbb{Z}$, $\sum_{i=0}^{n} u^2(t+i) \geqslant \eta^2$, we have

$$F\big(t, u_k(t+n), \ldots, u_k(t)\big) \leqslant \left[a + b\left(\sum_{i=0}^{n} u^2(t+i)\right)^{\nu/2}\right]$$

$$\times \left[\sum_{i=-n}^{0} F'_{2+n+i}\big(t, u_k(t+n), \ldots, u_k(t)\big)u_k(t-i) - 2F\big(t, u_k(t+n), \ldots, u_k(t)\big)\right]. \tag{3.38}$$

It follows from Lemma 2.4, (2.1), (2.3), (3.34), (3.36), (3.37) and (3.38) that

$$\frac{1}{2}\|u_k\|^2 = I(u_k) + \sum_{t\in\mathbb{Z}} F\big(t, u_k(t+n), \ldots, u(t)\big)$$

$$= I(u_k) + \sum_{t\in\mathbb{Z}((\sum_{i=0}^{n} u^2(t+i))^{1/2}\leqslant\eta)} F\big(t, u_k(t+n), \ldots, u(t)\big)$$

$$
\begin{aligned}
&+ \sum_{t\in\mathbb{Z}((\sum_{i=0}^{n}u^2(t+i))^{1/2}>\eta)} F\big(t,u_k(t+n),\dots,u(t)\big)\\
&\leqslant I(u_k)+\frac{\beta}{4(n+1)}\sum_{t\in\mathbb{Z}((\sum_{i=0}^{n}u^2(t+i))^{1/2}\leqslant\eta)}\sum_{i=0}^{n}u^2(t+i)\\
&\quad+\sum_{t\in\mathbb{Z}((\sum_{i=0}^{n}u^2(t+i))^{1/2}>\eta)}\left[a+b\left(\sum_{i=0}^{n}u^2(t+i)\right)^{\nu/2}\right]\\
&\quad\times\left[\sum_{i=-n}^{0}F'_{2+n+i}\big(t,u_k(t+n),\dots,u_k(t)\big)u_k(t-i)-2F\big(t,u_k(t+n),\dots,u_k(t)\big)\right]\\
&\leqslant C_2+\frac{1}{4}\|u_k\|^2+\sum_{t\in\mathbb{Z}}\left[a+b\left(\sum_{i=0}^{n}u^2(t+i)\right)^{\nu/2}\right]\\
&\quad\times\left[\sum_{i=-n}^{0}F'_{2+n+i}\big(t,u_k(t+n),\dots,u_k(t)\big)u_k(t-i)-2F\big(t,u_k(t+n),\dots,u_k(t)\big)\right]\\
&\leqslant C_2+\frac{1}{4}\|u_k\|^2+\big(a+b(n+1)\|u_k\|_\infty^{\nu}\big)\\
&\quad\times\left[\sum_{i=-n}^{0}F'_{2+n+i}\big(t,u_k(t+n),\dots,u_k(t)\big)u_k(t-i)-2F\big(t,u_k(t+n),\dots,u_k(t)\big)\right]\\
&\leqslant C_2+\frac{1}{4}\|u_k\|^2+C_1\big(a+b(n+1)\|u_k\|_\infty^{\nu}\big)\\
&\leqslant C_2+\frac{1}{4}\|u_k\|^2+C_1\big\{a+\beta^{-\nu/2}b(n+1)\|u_k\|^{\nu}\big\},\quad k\in\mathbb{N}.
\end{aligned}
\tag{3.39}
$$

Since $\nu<2$, it follows from (3.39) that $\{\|u_k\|\}_{k\in\mathbb{N}}$ is bounded. Similar to the proof of Theorem 1.1, we can prove that $\{u_k\}$ has a convergent subsequence in E. Hence, I satisfies condition (C).

It is obvious that I is even and $I(0)=0$ and so assumption (i) of Lemma 2.1 holds. The proof of assumption (ii) of Lemma 2.1 is the same as in the proof of Theorem 1.2.

Now, we prove assumption (iii) of Lemma 2.2. Let E' be a finite dimensional subspace of E. Since all norms of a finite dimensional normed space are equivalent, so there is a constant $c>0$ such that (3.22) holds. Assume that $\dim E'=m$ and $u_1,u_2,\dots,u_m$ is the basis of E' such that (3.23) holds. Let η,Π_1 and Θ be the same as in the proof of Theorem 1.2. Then (3.24), (3.25) and (3.26) hold. For the Π_1 given in the proof of Theorem 1.2, by (F7), there exists $\sigma_0=\sigma_0(c,\Pi_1)>1$ such that

$$
s^{-2}\min_{|u|=1}F\big(t,su(t)\big)\geqslant c^2\quad\text{for } s\geqslant\sigma_0,\ t\in\mathbb{Z}(-\Pi_1,\Pi_1). \tag{3.40}
$$

For $u\in\Theta$, it follows from (3.24) and (3.26) that there exists $t_0=t_0(u)\in\mathbb{Z}(-\Pi_1,\Pi_1)$ such that

$$
1\leqslant\big|u(t_0)\big|=\|u\|_\infty. \tag{3.41}
$$

It follows from (2.3), (3.37), (3.40) and (3.41) that

$$
\begin{aligned}
I(\sigma u)&=\frac{\sigma^2}{2}\|u\|^2-\sum_{t\in\mathbb{Z}}F\big(t,\sigma u(t+n),\dots,\sigma u(t)\big)\\
&\leqslant\frac{\sigma^2}{2}\|u\|^2-F\big(t_0,\sigma u(t_0+n),\dots,\sigma u(t_0)\big)\\
&\leqslant\frac{\sigma^2}{2}\|u\|^2-\min_{|x|=1}F\big(t_0,\sigma\big|u(t_0)\big|x\big)\\
&\leqslant\frac{(c\sigma)^2}{2}-(c\sigma)^2\big|u(t_0)\big|^2\\
&\leqslant\frac{(c\sigma)^2}{2}-(c\sigma)^2\\
&=-\frac{(c\sigma)^2}{2},\quad u\in\Theta,\ \sigma\geqslant\sigma_0.
\end{aligned}
\tag{3.42}
$$

We deduce that there is $\sigma_0 = \sigma_0(c, \Pi_1) = \sigma_0(E') > 1$ such that

$$I(\sigma u) < 0 \quad \text{for } u \in \Theta \text{ and } \sigma \geqslant \sigma_0.$$

That is

$$I(u) < 0 \quad \text{for } u \in E' \text{ and } \|u\| \geqslant c\sigma_0.$$

This shows that condition (iii) of Lemma 2.2 holds. By Lemma 2.2, I possesses an unbounded sequence $\{d_k\}_{k\in\mathbb{N}}$ of critical values with $d_k = I(u_k)$, where u_k is such that $I'(u_k) = 0$ for $k = 1, 2, \ldots$. From (2.3) and (3.37), we have

$$\frac{1}{2}\|u_k\|^2 = d_k + \sum_{t\in\mathbb{Z}} F\big(t, u_k(t+n), \ldots, u(t)\big) \geqslant d_k, \quad k \in \mathbb{N}.$$

Since $\{d_k\}_{k\in\mathbb{N}}$ is unbounded, it follows that $\{\|u_k\|\}_{k\in\mathbb{N}}$ is unbounded. The proof is complete. □

Proof of Theorem 1.6 and 1.7. By a fashion similar to the proofs of Theorem 1.1, Theorem 1.2 and the process in [33], we can prove Theorem 1.6 and Theorem 1.7, respectively. The detailed proofs are omitted. □

Proof of Theorem 1.8. Our proof will be divided into five steps.

Step 1: We first verify that the functional $I : E \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \sum_{t\in\mathbb{Z}} F\big(t, u(t+n), \ldots, u(t)\big), \quad \forall u \in E \tag{3.43}$$

is well defined and of class $C^1(E, \mathbb{R})$ and (2.4) holds. Furthermore, the critical points of I in E are solutions of (1.1) with $u(\pm\infty) = 0$.

For any $u \in E$, there exists an integer $\Pi > 0$ such that $|u(t)| < 1$ for $|t| > \Pi$. It follows from (F11) and Hölder inequality that

$$\begin{aligned}
\sum_{|t|>\Pi} \big|F\big(t, u(t+n), \ldots, u(t)\big)\big| &\leqslant \sum_{|t|>\Pi} a_1(t) \left(\sum_{|t|>\Pi} \sum_{i=0}^{n} \big(u(t+i)\big)^2 \right)^{\gamma_1/2} \\
&\leqslant (n+1)^{\gamma_1/2} \left(\sum_{|t|>\Pi} \big|a_1(t)\big|^{2/(2-\gamma_1)} \right)^{(2-\gamma_1)/2} \left(\sum_{t\in\mathbb{Z}} \big(u(t)\big)^2 \right)^{\gamma_1/2} \\
&\leqslant \beta^{-\gamma_1} (n+1)^{\gamma_1/2} \|a_1\|_{(2-\gamma_1)/2} \|u\|^{\gamma_1},
\end{aligned} \tag{3.44}$$

and so I defined by (2.3) is well defined on E. By (F12), there exists $M_1 > 0$ such that

$$\varphi\big(|x|\big) \leqslant M_1 |x|^{\gamma_1 - 1}, \quad \forall x \in \mathbb{R},\ |x| \leqslant 1. \tag{3.45}$$

For any $u, v \in E$, there exists an integer $\Pi_1 > 0$ such that $|u(t+i)| + |v(t+i)| < 1$ for $|t| > \Pi_1$, $i = 0, \ldots n$. Then for any sequence $\{\theta_t\}_{t\in\mathbb{Z}} \subset \mathbb{R}$ with $|\theta_t| < 1$ for $t \in \mathbb{Z}$ and any number $h \in (0, 1)$, by (F12), (3.45) and Lemma 2.4, we have

$$\begin{aligned}
&\sum_{t\in\mathbb{Z}} \left| \sum_{i=0}^{n} F'_{2+n-i}\big(t, u(t+n) + \theta_t h v(t+n), \ldots, u(t+i) + \theta_t v(t+i), \ldots, u(t) + \theta_t h v(t)\big) v(t+i) \right| \\
&\quad = \sum_{|t|\leqslant\Pi_1} \left| \sum_{i=0}^{n} F'_{2+n-i}\big(t, u(t+n) + \theta_t h v(t+n), \ldots, u(t+i) + \theta_t v(t+i), \ldots, u(t) + \theta_t h v(t)\big) v(t+i) \right| \\
&\qquad + \sum_{|t|>\Pi_1} \left| \sum_{i=0}^{n} F'_{2+n-i}\big(t, u(t+n) + \theta_t h v(t+n), \ldots, u(t+i) + \theta_t v(t+i), \ldots, u(t) + \theta_t h v(t)\big) v(t+i) \right| \\
&\quad \leqslant \sum_{|t|\leqslant\Pi_1} \left| \sum_{i=0}^{n} F'_{2+n-i}\big(t, u(t+n) + \theta_t h v(t+n), \ldots, u(t+i) + \theta_t v(t+i), \ldots, u(t) + \theta_t h v(t)\big) \right| \big|v(t+i)\big| \\
&\qquad + \sum_{|t|>\Pi_1} \left| \sum_{i=0}^{n} F'_{2+n-i}\big(t, u(t+n) + \theta_t h v(t+n), \ldots, u(t+i) + \theta_t v(t+i), \ldots, u(t) + \theta_t h v(t)\big) \right| \big|v(t+i)\big|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{|t|\leqslant \Pi_1} \max_{|x(t+i)|\leqslant \|u(t+i)\|_\infty+\|v(t+i)\|_\infty} \left|\sum_{i=0}^{n} F'_{2+n-i}\big(t,x(t+n),\ldots,x(t+i),\ldots,x(t)\big)\right| \big|v(t+i)\big| \\
&\quad + M_1 \sum_{|t|>\Pi_1} b(t)\left(\sum_{i=-n}^{n}\big(u^2(t+i)+v^2(t+i)\big)\right)^{(\gamma_1-1)/2} \big|v(t+i)\big| \\
&\leqslant \sum_{|t|\leqslant \Pi_1} \max_{|x|\leqslant \|u\|_\infty+\|v\|_\infty} \big|f\big(t,x(t+n),\ldots,x(t),\ldots,x(t-n)\big)\big|\big|v(t+i)\big| \\
&\quad + M_1 \sum_{|t|>\Pi_1} b(t)\left(\sum_{i=-n}^{n}\big(u(t+i)\big)^{\gamma_1-1}+\big(v(t+i)\big)^{\gamma_1-1}\right)\big|v(t+i)\big| \\
&\leqslant \sum_{|t|\leqslant \Pi_1} \max_{|x|\leqslant \|u\|_\infty+\|v\|_\infty} \big|f\big(t,x(t+n),\ldots,x(t),\ldots,x(t-n)\big)\big|\big|v(t+i)\big| \\
&\quad + M_1\left(\sum_{|t|>\Pi_1}\big|b(t)\big|^2\sum_{i=-n}^{n}\big(u(t+i)\big)^{2(\gamma_1-1)}\right)^{1/2}\left(\sum_{|t|>\Pi_1}\big(v(t+i)\big)^2\right)^{1/2} \\
&\quad + M_1\left(\sum_{|t|>\Pi_1}\big|b(t)\big|^2\big(v(t+i)\big)^{2(\gamma_1-1)}\right)^{1/2}\left(\sum_{|t|>\Pi_1}\sum_{i=-n}^{n} v\big((t+i)\big)^2\right)^{1/2} \\
&\leqslant \sum_{|t|\leqslant \Pi_1} \max_{|x|\leqslant \|u\|_\infty+\|v\|_\infty} \big|f\big(t,u(t+n),\ldots,u(t),\ldots,u(t-n)\big)\big|\big|v(t+i)\big| \\
&\quad + M_1\beta^{-1/2}\left(\sum_{|n|>\Pi_1}\big|b(t)\big|^{2/(2-\gamma_1)}\right)^{(2-\gamma_1)/2}\left(\sum_{|t|>\Pi_1}\sum_{i=-n}^{n}\big(u(t+i)\big)^2\right)^{(\gamma_1-1)/2}\|v\| \\
&\quad + M_1\beta^{-1/2}\left(\sum_{|n|>\Pi_1}\big|b(t)\big|^{2/(2-\gamma_1)}\right)^{(2-\gamma_1)/2}\left(\sum_{|t|>\Pi_1}\sum_{i=-n}^{n}\big(v(t+i)\big)^2\right)^{(\gamma_1-1)/2}\|v\| \\
&\leqslant \sum_{|t|\leqslant \Pi_1} \max_{|x|\leqslant \|u\|_\infty+\|v\|_\infty} \big|f\big(t,u(t+n),\ldots,u(t),\ldots,u(t-n)\big)\big|\big|v(t+i)\big| \\
&\quad + M_1(2n+1)^{(\gamma_1-1)/2}\beta^{-\gamma_1/2}\|b\|_{2/(2-\gamma_1)}\big(\|u\|^{\gamma_1-1}+\|v\|^{\gamma_1-1}\big)\|v\| \\
&< +\infty. \qquad (3.46)
\end{aligned}
$$

Then by (2.3) and (3.46), we have

$$
\begin{aligned}
\big\langle I'(u),v\big\rangle &= \lim_{h\to 0^+}\frac{I(u+hv)-I(u)}{h} \\
&= \lim_{h\to 0^+}\frac{1}{h}\left\{\frac{\|u+hv\|^2-\|u\|^2}{2}\right. \\
&\quad \left. -\sum_{t\in\mathbb{Z}}\big[F\big(t,u(t+n)+hv(t+n),\ldots,u(t)+hv(t)\big)-F\big(t,u(t+n),\ldots,u(t)\big)\big]\right\} \\
&= \lim_{h\to 0^+}\left[\langle u,v\rangle+\frac{h\|v\|^2}{2}\right. \\
&\quad \left. -\sum_{t\in\mathbb{Z}}\sum_{i=0}^{n}F'_{2+n-i}\big(t,u(t+n)+\theta_t hv(t+n),\ldots,u(t+i)+\theta_t v(t+i),\ldots,u(t)+\theta_t hv(t)\big)v(t+i)\right] \\
&= \langle u,v\rangle-\sum_{t\in\mathbb{Z}} f\big(t,u(t+n),\ldots,u(t),\ldots,u(t-n)\big)v(t) \\
&= \sum_{t\in\mathbb{Z}}\big[r(t-1)\Delta^n u(t-1)\Delta^n v(t-1)+q(t)u(t)v(t)-f\big(t,u(t+n),\ldots,u(t),\ldots,u(t-n)\big)v(t)\big].
\end{aligned}
$$

This shows that (2.4) holds. Then $\langle I'(u), v\rangle = 0$ for all $v \in E$ if and only if

$$\begin{aligned}&\Delta^n\big(r(t-n)\Delta^n u(t-n)\big) + q(t)u(t)\\&\quad = f\big(t, u(t+n), u(t+n-1), \dots, u(t), u(t-1), \dots, u(t-n)\big), \quad t \in \mathbb{Z}.\end{aligned}$$

So, the critical points of I in E are the solutions of Eq. (1.1) with $u(\pm\infty) = 0$.

Step 2: Let's prove now that I' is continuous. Let $u_k \to u$ in E. For any $\varepsilon \in (0, \sqrt{\beta})$, we can choose an integer $\Pi_\varepsilon > 0$ such that

$$\left\{\sum_{|t|>\Pi_\varepsilon}\left[r(t-1)\big(\Delta^n u_k(t)\big)^2 + q(t)\big(u_k(t)\big)^2\right]\right\}^{1/2} < \varepsilon, \quad k \in \mathbb{N}, \tag{3.47}$$

and

$$\left\{\sum_{|t|>\Pi_\varepsilon}\left[r(t-1)\big(\Delta^n u(t)\big)^2 + q(t)\big(u(t)\big)^2\right]\right\}^{1/2} < \varepsilon. \tag{3.48}$$

For any $v \in E$, from Lemma 2.4, (2.4), (3.45), (3.47), (3.48), (F12) and Hölder inequality, we have

$$\begin{aligned}
&\big|\langle I'(u_k) - I'(u), v\rangle\big|\\
&\leqslant \left|\sum_{t\in\mathbb{Z}}\left[r(t-1)\big(\Delta^n u_k(t-n) - \Delta^n u(t-n)\big)\Delta^n v(t-n) + q(t)\big(u_k(t) - u(t)\big)v(t)\right]\right|\\
&\quad + \sum_{t\in\mathbb{Z}}\Big|\Big(f\big(t, u_k(t+n), \dots, u_k(t), \dots, u_k(t-n)\big) - f\big(t, u(t+n), \dots, u(t), \dots, u(t-n)\big)\Big)v(t)\Big|\\
&= \big|\langle u_k - u, v\rangle\big| + \sum_{t\in\mathbb{Z}}\Big|f\big(t, u_k(t+n), \dots, u_k(t), \dots, u_k(t-n)\big) - f\big(t, u(t+n), \dots, u(t), \dots, u(t-n)\big)\Big|\big|v(t)\big|\\
&\leqslant \|u_k - u\|\|v\| + \sum_{|t|\leqslant\Pi_\varepsilon}\Big|f\big(t, u_k(t+n), \dots, u_k(t), \dots, u_k(t-n)\big) - f\big(t, u(t+n), \dots, u(t), \dots, u(t-n)\big)\Big|\big|v(t)\big|\\
&\quad + \sum_{|t|>\Pi_\varepsilon}\Big(\big|f\big(t, u_k(t+n), \dots, u_k(t), \dots, u_k(t-n)\big)\big| + \big|f\big(t, u(t+n), \dots, u(t), \dots, u(t-n)\big)\big|\Big)\big|v(t)\big|\\
&\leqslant o(1) + M_1\sum_{|t|>\Pi_\varepsilon} b(t)\left(\left(\sum_{i=-n}^{n}\big(u_k(t+i)\big)^2\right)^{(\gamma_1-1)/2} + \left(\sum_{i=-n}^{n}\big(u(t+i)\big)^2\right)^{(\gamma_1-1)/2}\right)\big|v(t)\big|\\
&\leqslant o(1) + M_1\beta^{-1/2}\left(\sum_{|t|>\Pi_1}\big|b(t)\big|^{2/(2-\gamma_1)}\right)^{(2-\gamma_1)/2}\left(\sum_{|t|>\Pi_1}\sum_{i=-n}^{n}\big(u_k(t+i)\big)^2\right)^{(\gamma_1-1)/2}\|v\|\\
&\quad + M_1\beta^{-1/2}\left(\sum_{|n|>\Pi_1}\big|b(t)\big|^{2/(2-\gamma_1)}\right)^{(2-\gamma_1)/2}\left(\sum_{|t|>\Pi_1}\sum_{i=-n}^{n}\big(u(t+i)\big)^2\right)^{(\gamma_1-1)/2}\|v\|\\
&\leqslant o(1) + 2(2n+1)^{(\gamma_1-1)/2}M_1\beta^{-\gamma_1/2}\|b\|_{2/(2-\gamma_1)}\|v\|\varepsilon^{\gamma_1-1}, \quad \text{for sufficiently large } k,
\end{aligned}$$

which, since ε is arbitrary, implies the continuity of I'. The proof is complete.

Step 3: In view of Lemma 2.3, $I \in C^1(E, \mathbb{R})$. In what follows, we first show that I is bounded from below. By Lemma 2.4, (F11), (2.3) and Hölder inequality, we have

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} F\big(t, u(t+n), \dots, u(t)\big)\\
&= \frac{1}{2}\|u\|^2 - \sum_{\mathbb{Z}((\sum_{i=0}^{n}(u(t+i))^2)^{1/2}\leqslant 1)} F\big(t, u(t+n), \dots, u(t)\big) - \sum_{\mathbb{Z}((\sum_{i=0}^{n}(u(t+i))^2)^{1/2}>1)} F\big(t, u(t+n), \dots, u(t)\big)\\
&\geqslant \frac{1}{2}\|u\|^2 - \sum_{\mathbb{Z}((\sum_{i=0}^{n}(u(t+i))^2)^{1/2}\leqslant 1)} a_1(t)\left(\sum_{i=0}^{n}\big(u(t+i)\big)^2\right)^{\gamma_1/2} - \sum_{\mathbb{Z}(\sum_{i=0}^{n}(u(t+i))^2>1)} a_2(t)\left(\sum_{i=0}^{n}\big(u(t+i)\big)^2\right)^{\gamma_2/2}
\end{aligned}$$

$$\geqslant \frac{1}{2}\|u\|^2-\Bigg(\sum_{\mathbb{Z}((\sum_{i=0}^n(u(t+i))^2)^{1/2}\leqslant 1)}\big|a_1(t)\big|^{2/(2-\gamma_1)}\Bigg)^{(2-\gamma_1)/2}\Bigg(\sum_{\mathbb{Z}(\sum_{i=0}^n(u(t+i))^2\leqslant 1)}\sum_{i=0}^n\big(u(t+i)\big)^2\Bigg)^{\gamma_1/2}$$

$$-\Bigg(\sum_{\mathbb{Z}((\sum_{i=0}^n(u(t+i))^2)^{1/2}>1)}\big|a_2(t)\big|^{2/(2-\gamma_1)}\Bigg)^{(2-\gamma_1)/2}\Bigg(\sum_{\mathbb{Z}((\sum_{i=0}^n(u(t+i))^2)^{1/2}>1)}\Bigg(\sum_{i=0}^n\big(u(t+i)\big)^2\Bigg)^{\gamma_2/\gamma_1}\Bigg)^{\gamma_1/2}$$

$$\geqslant \frac{1}{2}\|u\|^2-(n+1)^{\gamma_1/2}\beta^{-\gamma_1/2}\Bigg(\sum_{\mathbb{Z}((\sum_{i=0}^n(u(t+i))^2)^{1/2}\leqslant 1)}\big|a_1(t)\big|^{2/(2-\gamma_1)}\Bigg)^{(2-\gamma_1)/2}\|u\|^{\gamma_1}$$

$$-\big[2(n+1)\big]^{\gamma_2/2}\beta^{-\gamma_1/2}\|u\|_\infty^{\gamma_2-\gamma_1}\Bigg(\sum_{\mathbb{Z}((\sum_{i=0}^n(u(t+i))^2)^{1/2}>1)}\big|a_2(t)\big|^{2/(2-\gamma_1)}\Bigg)^{(2-\gamma_1)/2}\|u\|^{\gamma_1}$$

$$\geqslant \frac{1}{2}\|u\|^2-(n+1)^{\gamma_1/2}\beta^{-\gamma_1/2}\Bigg(\sum_{\mathbb{Z}((\sum_{i=0}^n(u(t+i))^2)^{1/2}\leqslant 1)}\big|a_1(t)\big|^{2/(2-\gamma_1)}\Bigg)^{(2-\gamma_1)/2}\|u\|^{\gamma_1}$$

$$-\big[2(n+1)\big]^{\gamma_2/2}\beta^{-\gamma_2/2}\Bigg(\sum_{\mathbb{Z}((\sum_{i=0}^n(u(t+i))^2)^{1/2}>1)}\big|a_2(t)\big|^{2/(2-\gamma_1)}\Bigg)^{(2-\gamma_1)/2}\|u\|^{\gamma_2}$$

$$\geqslant \frac{1}{2}\|u\|^2-(n+1)^{\gamma_1/2}\beta^{-\gamma_1/2}\|a_1\|_{2/(2-\gamma_1)}\|u\|^{\gamma_1}-\big[2(n+1)\big]^{\gamma_2/2}\beta^{-\gamma_2/2}\|a_2\|_{2/(2-\gamma_1)}\|u\|^{\gamma_2}. \tag{3.49}$$

Since $1<\gamma_1<\gamma_2<2$, (3.49) implies that $I(u)\to+\infty$ as $\|u\|\to+\infty$. Consequently, I is bounded from below.

Step 4: We prove that I satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}}\subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k)\to 0$ as $k\to+\infty$. Then by Lemma 2.4 and (3.49), there exists a constant $A>0$ such that

$$\|u_k\|_\infty\leqslant\beta^{-\frac{1}{2}}\|u_k\|\leqslant A,\quad k\in\mathbb{N}. \tag{3.50}$$

So passing to a subsequence if necessary, it can be assumed that $u_k\rightharpoonup u_0$ in E. It is easy to verify that $u_k(t)$ converses to $u_0(t)$ pointwise for all $n\in\mathbb{Z}$, that is

$$\lim_{k\to\infty}u_k(t)=u_0(t),\quad\forall k\in\mathbb{Z}. \tag{3.51}$$

Hence, we have by (3.50) and (3.51)

$$\|u_0\|_\infty\leqslant A. \tag{3.52}$$

By (F12), there exists $M_2>0$ such that

$$\varphi\big(|x|\big)\leqslant M_2|x|^{\gamma_1-1},\quad\forall x\in\mathbb{R},\ |x|\leqslant A. \tag{3.53}$$

For any given number $\varepsilon>0$, by (F12), we can choose an integer $\Pi>0$ such that

$$\Bigg(\sum_{|t|>\Pi}\big(b(t)\big)^{2/(2-\gamma_1)}\Bigg)^{(2-\gamma_1)/2}<\varepsilon. \tag{3.54}$$

It follows from (3.52) and the continuity of $f(t,u(t+n),\dots,u(t),\dots,u(t-n))$ on u that there exists $k_0\in\mathbb{N}$ such that

$$\sum_{t=-\Pi}^{\Pi}\big|f\big(t,u_k(t+n),\dots,u_k(t),\dots,u_k(t-n)\big)-f\big(t,u_0(t+n),\dots,u_0(t),\dots,u_0(t-n)\big)\big|\big|u_k(t)-u_0(t)\big|<\varepsilon$$
$$\text{for } k\geqslant k_0. \tag{3.55}$$

On the other hand, it follows from (3.50), (3.52), (3.53), (3.54) and (F12) that

$$\sum_{|t|>\Pi}\big|f\big(t,u_k(t+n),\dots,u_k(t),\dots,u_k(t-n)\big)-f\big(t,u_0(t+n),\dots,u_0(t),\dots,u_0(t-n)\big)\big|\big|u_k(t)-u_0(t)\big|$$

$$\leqslant\sum_{|t|>\Pi}b(t)\Bigg[\varphi\Bigg(\sum_{i=-n}^n u_k^2(t+i)\Bigg)+\varphi\Bigg(\sum_{i=-n}^n u_0^2(t+i)\Bigg)\Bigg]\big(\big|u_k(t)\big|+\big|u_0(t)\big|\big)$$

$$\leqslant M_2 \sum_{|t|>\Pi} b(t)\left(\left(\sum_{i=-n}^{n} u_k^2(t+i)\right)^{\frac{\gamma_1-1}{2}} + \left(\sum_{i=-n}^{n} u_0^2(t+i)\right)^{\frac{\gamma_1-1}{2}}\right)\left(\left|u_k(t)\right| + \left|u_0(t)\right|\right)$$

$$\leqslant 2M_2 \sum_{|t|>\Pi} b(t)\left(\left(\sum_{i=-n}^{n} u_k^2(t+i)\right)^{\frac{\gamma_1}{2}} + \left(\sum_{i=-n}^{n} u_0^2(t+i)\right)^{\frac{\gamma_1}{2}}\right)$$

$$\leqslant 2M_2\left(\sum_{|t|>\Pi} \left(b(t)\right)^{2/(2-\gamma_1)}\right)^{(2-\gamma_1)/2}\left(\sum_{t\in\mathbb{Z}}\sum_{i=-n}^{n} u_k^2(t+i) + \sum_{t\in\mathbb{Z}}\sum_{i=-n}^{n} u_0^2(t+i)\right)^{\frac{\gamma_1}{2}}$$

$$\leqslant 2(2n+1)^{\frac{\gamma_1}{2}} M_2\beta^{-\gamma_1/2}\left(\sum_{|t|>\Pi} \left(b(t)\right)^{2/(2-\gamma_1)}\right)^{(2-\gamma_1)/2}\left[\|u_k\|^{\gamma_1} + \|u_0\|^{\gamma_1}\right]$$

$$\leqslant 2(2n+1)^{\frac{\gamma_1}{2}} M_2\beta^{-\gamma_1/2}\left[\beta^{\gamma_1/2}A^{\gamma_1} + \|u_0\|^{\gamma_1}\right]\varepsilon, \quad k\in\mathbb{N}. \tag{3.56}$$

Since ε is arbitrary, combining (3.55) with (3.56) we get

$$\sum_{t\in\mathbb{Z}}\left(f\left(t, u_k(t+n), \ldots, u_k(t), \ldots, u_k(t-n)\right) - f\left(t, u_0(t+n), \ldots, u_0(t), \ldots, u_0(t-n)\right), u_k(t) - u_0(t)\right) \to 0$$

as $k \to \infty$.

It follows from (2.4) that

$$\left\langle I'(u_k) - I'(u_0), u_k - u_0\right\rangle = \|u_k - u_0\|^2 - \sum_{t\in\mathbb{Z}}\left(f\left(t, u_k(t+n), \ldots, u_k(t), \ldots u_k(t-n)\right)\right.$$
$$\left. - f\left(t, u_0(t+n)\right), \ldots, u_0(t), \ldots, u_0(t-n)\right)\left(u_k(t) - u_0(t)\right).$$

Since $\langle I'(u_k) - I'(u_0), u_k - u_0\rangle \to 0$, it follows from (3.9) and (3.10) that $u_k \to u_0$ in E. Hence, I satisfies (PS)-condition.

By Lemma 2.3, $c = \inf_E I(u)$ is a critical value of I, that is there exists a critical point $u^* \in E$ such that $I(u^*) = c$.

Step 5: Finally, we show that $u^* \neq 0$. Let $u_0(t_0) = 1$ and $u_0(t) = 0$ for $t \neq t_0$. Then by (F11), (F13) and (2.3), we have

$$I(su_0) = \frac{s^2}{2}\|u_0\|^2 - \sum_{t\in\mathbb{Z}} F\left(t, su_0(t+n), \ldots, su_0(t)\right)$$
$$= \frac{s^2}{2}\|u_0\|^2 - F\left(t_0, su_0(t_0+n), \ldots, su_0(t_0)\right)$$
$$\leqslant \frac{s^2}{2}\|u_0\|^2 - \eta s^{\gamma_3}\left|u_0(t)\right|^{\gamma_3}, \quad 0 < s < 1. \tag{3.57}$$

Since $1 < \gamma_3 < 2$, it follows from (3.57) that $I(su_0) < 0$ for $s > 0$ small enough. Hence $I(u^*) = c < 0$, therefore u^* is nontrivial critical point of I, and so $u^* = u^*(n)$ is a nontrivial homoclinic solution of (1.1). The proof is complete. □

4. Examples

In this section, we give some examples to illustrate our results.

Example 4.1. In Eq. (1.1), let $r(t) > 0$, $q(t) \to +\infty$ as $|t| \to +\infty$ and

$$F\left(t, u(t+n), \ldots, u(t)\right) = \left[\left|u(t)\right|^{4+|t|} - \left(\sum_{i=0}^{n} u^2(t+i)\right)^{(5+3|t|)/2(4+|t|)}\right].$$

Let $\mu = 4$, $\varrho = 3$, $J = \{-3, -2, -1, 0, 1, 2, 3\}$ and

$$W\left(t, u(t)\right) = \left|u(t)\right|^{4+|t|}, \qquad H\left(t, u(t+n), \ldots, u(t)\right) = \left(\sum_{i=0}^{n} u^2(t+i)\right)^{(5+3|t|)/2(4+|t|)}.$$

Then it is easy to verify that all conditions of Theorem 1.2 are satisfied. By Theorem 1.2, Eq. (1.1) has an unbounded sequence of homoclinic solutions.

Example 4.2. In Eq. (1.1), let $r(t) > 0$, $q(t) \to +\infty$ as $|t| \to +\infty$ and

$$F\big(t, u(t+n), \dots, u(t)\big) = \left(\sum_{k=1}^{m} a_k \big|u(t)\big|^{\mu_k} - \sum_{j=1}^{p} b_j \left(\sum_{i=0}^{n} u^2(t+i)\right)^{\varrho_j/2}\right),$$

where $\mu_1 > \mu_2 > \cdots > \mu_m > \varrho_1 > \varrho_2 > \cdots > \varrho_k > 2$, $a_k, b_j > 0$, $k = 1, 2, \dots, m;\ j = 1, 2, \dots, p$. Let $\mu = \mu_m$, $\varrho = \varrho_1$, and

$$W\big(t, u(t)\big) = \sum_{k=1}^{m} a_k \big|u(t)\big|^{\mu_k}, \qquad H\big(t, u(t+n), \dots, u(t)\big) = \sum_{j=1}^{p} b_j \left(\sum_{i=0}^{n} u^2(t+i)\right)^{\varrho_j/2}.$$

Then it is easy to verify that all conditions of Theorem 1.4 are satisfied. By Theorem 1.4, system (1.1) has an unbounded sequence of homoclinic solutions.

Example 4.3. In Eq. (1.1), let

$$F\big(t, u(t+n), \dots, u(t)\big) = q(t) \sum_{i=0}^{n} u^2(t+i) \ln\left[1 + \left(\sum_{i=0}^{n} u^2(t+i)\right)^{1/2}\right],$$

where $q : \mathbb{Z} \to (0, \infty)$ such that $q(t) \to +\infty$ as $|t| \to +\infty$. Since

$$\begin{aligned}
&\sum_{i=-n}^{0} F'_{2+n+i}\big(t, u(t+n), \dots, u(t)\big) u(t-i) \\
&\quad = q(t)\left[2 \sum_{i=0}^{n} u^2(t+i) \ln\left[1 + \left(\sum_{i=0}^{n} u^2(t+i)\right)^{1/2}\right] + \frac{(\sum_{i=0}^{n} u^2(t+i))^{3/2}}{1 + (\sum_{i=0}^{n} u^2(t+i))^{1/2}}\right] \\
&\quad \geqslant \left(2 + \frac{1}{1 + (\sum_{i=0}^{n} u^2(t+i))^{1/2}}\right) F\big(t, u(t+n), \dots, u(t)\big) \geqslant 0, \quad \forall \big(t, u(t+n), \dots, u(t)\big) \in \mathbb{Z} \times \mathbb{R}^{n+1}.
\end{aligned}$$

This shows that (F8) holds with $a = b = \nu = 1$. In addition, for any $t \in \mathbb{Z}$,

$$s^{-2} \min_{|u|=1} F(t, su) = s^{-2} \min_{|u|=1} \big[q(t)|su|^2 \ln\big(1 + |su|\big)\big] = q(t) \ln(1+s) \to +\infty, \quad s \to +\infty.$$

This shows that (F9) also holds. It is easy to verify that assumptions (q), (F1) and (F7) of Theorem 1.5 are satisfied. By Theorem 1.5, Eq. (1.1) has an unbounded sequence of homoclinic solutions.

Example 4.4. In Eq. (1.1), let $r(t) > 0$, $q(t) \to +\infty$ as $|t| \to +\infty$ and

$$F\big(t, u(t+n), \dots, u(t)\big) = \big(1 + 2\sin^2 t\big)\left(\sum_{i=0}^{n} u^2(t+i)\right)^{\frac{\beta}{2}}.$$

By a fashion similar to the computation in [33], it is easy to verify that all conditions of Theorem 1.6 are satisfied. By Theorem 1.6, Eq. (1.1) has an unbounded sequence of homoclinic solutions.

Example 4.5. In Eq. (1.1), let $r(t) > 0$, $q : \mathbb{Z} \to (0, \infty)$ such that $q(t) \to +\infty$ as $|t| \to +\infty$ and

$$F\big(t, u(t+n), \dots, u(t)\big) = \frac{\cos t}{1 + \sum_{i=0}^{n} |t+i|}\left(\sum_{i=0}^{n} u^2(t+i)\right)^{2/3} + \frac{\sin t}{1 + \sum_{i=0}^{n} |t+i|}\left(\sum_{i=0}^{n} u^2(t+i)\right)^{3/4}.$$

Then

$$\begin{aligned}
&f\big(t, u(t+n), \dots, u(t), \dots, u(t-n)\big) \\
&\quad = \frac{4\sum_{i=0}^{n} \cos(t-i)}{3(1 + \sum_{i=0}^{n} |t-i|)}\left(\sum_{i=-n}^{n} u^2(t+i)\right)^{-1/3} u(t) + \frac{3\sum_{i=0}^{n} \sin(t-i)}{2(1 + \sum_{i=0}^{n} |t-i|)}\left(\sum_{i=-n}^{n} u^2(t+i)\right)^{-1/4} u(t),
\end{aligned}$$

$$\left|F\big(t,u(t+n),\ldots,u(t)\big)\right| \leqslant \frac{2(\sum_{i=0}^{n} u^2(t+i))^{2/3}}{1+|t|}, \quad \forall\big(t,u(t+n),\ldots,u(t)\big) \in \mathbb{Z}\times\mathbb{R}^{n+1},\ \sum_{i=0}^{n} u^2(t+i) \leqslant 1,$$

$$\left|F\big(t,u(t+n),\ldots,u(t)\big)\right| \leqslant \frac{2(\sum_{i=0}^{n} u^2(t+i))^{3/4}}{1+|t|}, \quad \forall\big(t,u(t+n),\ldots,u(t)\big) \in \mathbb{Z}\times\mathbb{R}^{n+1},\ \sum_{i=0}^{n} u^2(t+i) > 1$$

and for every $t \in \mathbb{Z}$, $(u(t+n),\ldots,u(t),\ldots,u(t-n)) \in \mathbb{R}^{2n+1}$, we have

$$\left|f\big(t,u(t+n),\ldots,u(t),\ldots,u(t-n)\big)\right| \leqslant \frac{8(n+1)(\sum_{i=-n}^{n} u^2(t+i))^{1/6} + 9(n+1)(\sum_{i=-n}^{n} u^2(t+i))^{1/4}}{6(1+|t|)}.$$

We can choose t_0 such that

$$\cos t_0 > 0, \qquad \sin t_0 > 0.$$

Let

$$\eta = \frac{\cos t_0}{1+\sum_{i=0}^{n}|t_0+i|} + \frac{\sin t_0}{1+\sum_{i=0}^{n}|t_0+i|}.$$

Then

$$F\big(t_0,u(t+n),\ldots,u(t)\big) \geqslant \eta\left(\sum_{i=0}^{n} u^2(t+i)\right)^{3/4}, \quad \forall\big(u(t+n),\ldots,u(t)\big) \in \mathbb{R}^{n+1},\ \left(\sum_{i=0}^{n} u^2(t+i)\right)^{1/2} \leqslant 1.$$

These show that all conditions of Theorem 1.8 are satisfied, where

$$1 < \frac{4}{3} = \gamma_1 < \gamma_2 = \gamma_3 = \frac{3}{2} < 2, \qquad a_1(t) = a_2(t) = b(t) = \frac{2}{1+|t|}, \qquad \varphi(s) = \frac{8(n+1)s^{1/3} + 9(n+1)s^{1/2}}{12}.$$

By Theorem 1.8, Eq. (1.1) has at least a nontrivial homoclinic solution.

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