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Strongly Nonlinear Quasivariational Inequalities

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In this paper, we develop the algorithms for finding the approximate solution of a strongly nonlinear quasivariational inequality and a strongly nonlinear quasicomplementarity problem and we include as special cases known results in this area. We also observe that these two problems are equivalent if the convex set involved in the formulation of these problem is a convex cone. © 1990 Academic Press, Inc

1. INTRODUCTION

In 1966, Hartman and Stampacchia [6] proved a theorem for the existence of solutions of variational inequalities. Quasivariational inequality is a generalization of variational inequality in which convex sets involved in the formulation of the variational inequality depend upon the solution of the problem. Simultaneously Cottle [3] considered complementarity problems and proved the existence of their solutions. The complementarity problems play a very important role in general equilibrium theory, economics, management sciences, and operations research. It has been shown by Karamardian [8], that if the convex set involved in a variational inequality problem and complementarity problem is a convex cone, then these problems are equivalent. In fact, variational inequality problems are more general than the complementarity problems and we include these as special cases. An important and useful generalization of the variational inequality problem is the mildly nonlinear variational inequality problem introduced and studied by Noor [10]. On the other hand, a new concept known as the quasi (implicit) complementarity problem, which includes as a special case the complementarity problem, has been studied by Dolcetta [5], Isac [7], and Noor [13]. For related work and applications, see Ahn [1], Crank [4], and Noor [14]. Noor [14] has also introduced and studied a new class of the complementarity problem which is a mildly nonlinear complementarity problem.

In this paper, we develop an algorithm for more general classes of mildly

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nonlinear variational inequalities and mildly nonlinear complementarity problems which include results of Noor [10, 14] as special cases. We also observe that these two problems are equivalent if the convex set involved in variational inequality formulation is a convex cone.

2. Preliminaries and Formulation

Let H be a Hilbert space with its dual H^*, whose norm and inner product are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. The pairing between H^* and H is denoted by $\langle\cdot,\cdot\rangle$. Let K be a nonempty closed convex subset of H, and T and A be nonlinear operators from H into H^*. Then the problem of finding $u \in K$ such that

$$\langle T(u), v-u\rangle \geqslant \langle A(u), v-u\rangle, \qquad \text{for all} \quad v \in K \tag{2.1}$$

is called the *strongly nonlinear variational inequality problem* (see Noor [10]).

If $A(u)$ is independent of u, that is, $A(u)=f$ (say), then (2.1) takes the form

$$\langle T(u), v-u\rangle \geqslant \langle f, v-u\rangle \qquad \text{for all} \quad v \in K. \tag{2.2}$$

This class of variational inequalities was considered and studied by Browder [2].

If the convex set K depends upon the solution u, then the inequality (2.1) is called the *strongly nonlinear quasivariational inequality*. More precisely, given a point-to-set mapping K from H into itself, the *strongly nonlinear quasivariational inequality problem* is to find $u \in K(u)$ such that

$$\langle T(u), v-u\rangle \geqslant \langle A(u), v-u\rangle, \qquad \text{for all} \quad v \in K(u). \tag{2.3}$$

Related to the strongly nonlinear quasivariational inequality problem, we consider and study a new class of complementarity problems. For given nonlinear operators T and A from H into H^*, we consider the problem of finding $u \in K(u)$ such that

$$T(u)+A(u) \in K^*(u) \qquad \text{and} \qquad \langle T(u)+A(u), u-m(u)\rangle = 0, \tag{2.4}$$

where m is a point-to-point mapping from H into itself and $K^*(u)$ is a polar cone of the convex cone $K(u)$, i.e.,

$$K^*(u) = \{w \in H^*/\langle w, z\rangle \geqslant 0 \text{ for all } z \in K(u)\}.$$

This type of problem is called a *strongly nonlinear quasicomplementarity problem.*

In many important applications $K(u)$ has the form

$$K(u) = m(u) + K. \tag{2.5}$$

In this case $K^*(u) = (m(u) + K)^* = m(u) \cap K^*$.

Remark 2.1. If $K(u)$ is of type (2.5), then for any u, $v \in K$,

$$P_{K(u)} v = m(u) + P_K(v - m(u)).$$

We note that if the point-to-point mapping m is zero, then (2.3) and (2.1) are equivalent and (2.4) is equivalent to finding $u \in K$ such that

$$T(u) + A(u) \in K^*(u) \qquad \text{and} \qquad \langle T(u) + A(u), u \rangle = 0. \tag{2.6}$$

The problem of type (2.6) is considered and studied by Noor [14]. For the mathematical and physical formulations of such problems, see Noor [15].

Now, we define Λ, a canonical isomorphism for H^* onto H by

$$\langle f, v \rangle = (\Lambda f, v) \qquad \text{for all} \quad v \in H, \quad f \in H^*. \tag{2.7}$$

Then $\|\Lambda\|_{H^*} = 1 = \|\Lambda^{-1}\|_H$.

DEFINITION 2.1. An operator $T: K \to H^*$ is called

(i) *Lipschitz continuous*, if there exists a constant $\alpha > 0$ such that

$$\|T(u) - T(v)\| \leqslant \alpha \|u - v\|, \qquad \text{for all} \quad u, v \in K;$$

(ii) *strongly monotone*, if there exists a constant $\beta > 0$ such that

$$\langle T(u) - T(v), v - u \rangle \geqslant \beta \, \|u - v\|^2, \qquad \text{for all} \quad u, v \in K.$$

Now we state two lemmas which will be used in the proof of our main results.

LEMMA 2.1. [9]. *Let K be a convex subset of H. Then, given $z \in H$, we have*

$$x = P_K z$$

if and only if

$$x \in K\colon (x - z, y - x) \geqslant 0 \qquad \textit{for all} \quad y \in K,$$

where P_K is a projection of H into K.

LEMMA 2.2 [9]. *P_K is nonexpansive, i.e.,*

$$\|P_K u - P_K v\| \leqslant \|u - v\| \qquad \text{for all} \quad u, v \in H.$$

We make the following hypothesis.

CONDITION N. *We assume that $\alpha > \lambda$ and $\beta > \alpha$, where α and β are a Lipschitz constant and a strongly monotone constant, respectively, of T, and λ is a Lipschitz constant of A.*

3. MAIN RESULTS

Algorithm 3.1. For any given $u \in K(u)$, compute

$$u_{n+1} = m(u_n) + P_K(u_n - \xi A(T(u_n) - A(u_n)) - m(u_n)), \tag{3.1}$$

for some positive constant ξ.

Now we will prove that the approximate solution obtained from the iterative scheme (3.1) converges strongly to the exact solution of (2.3).

THEOREM 3.1. *Let T be strongly monotone and Lipschitz continuous and A be Lipschitz continuous. If the mapping m is Lipschitz continuous, Condition N holds, and u_{n+1} and u are solutions of (3.1) and (2.3), respectively, then*

$$u_{n+1} \text{ strongly converges to } u \text{ in } H,$$

for $0 < \xi < 2\mu/(\alpha - \lambda)$, where α, λ, and μ are Lipschitz constants of T, A, and m, respectively, and β is a monotonicity constant of T.

For the proof of this theorem we need the following lemma.

LEMMA 3.1. *$u \in K(u)$ is a solution of (2.3) if and only if $u \in K(u)$ satisfies the relation*

$$u = m(u) + P_K(u - \xi A(T(u) - A(u)) - m(u)), \tag{3.2}$$

for some positive constant ξ.

Proof. Suppose that $u \in K(u)$ satisfies (2.3) then it is equivalent to finding $u \in K(u)$ such that

$$(A(T(u) - A(u)), v - u) \geqslant 0, \qquad \text{for all} \quad v \in K(u)$$

or

$$(u - (u - \xi A(T(u) - A(u)), v - u)) \geqslant 0, \qquad \text{for all} \quad v \in K(u).$$

By Lemma 2.1, $u \in K(u)$ satisfies (2.3) and is equivalent to finding $u \in K(u)$ such that

$$\begin{aligned} u &= P_{K(u)}(u - \xi A(T(u) - A(u))) \\ &= m(u) + P_K(u - \xi A(T(u) - A(u)) - m(u)). \end{aligned}$$

Proof of Theorem 3.1. By Lemma 3.1, we know that $u \in K(u)$ satisfying (2.3) is also a solution of (3.2) and conversely. Thus from (3.1) and (3.2), we obtain

$$\begin{aligned} \|u_{n+1} - u\| &= \|m(u_n) + P_K(u_n - \xi A(T(u_n) - A(u_n)) - m(u_n)) - m(u) \\ &\quad - P_K(u - \xi A(T(u) - A(u)) - m(u))\| \\ &\leqslant \|m(u_n) - m(u)\| + \|u_n - \xi A(T(u_n) - A(u_n)) - m(u_n) \\ &\quad - u + \xi A(T(u) - A(u)) + m(u)\| \\ &\leqslant 2\,\|m(u_n) - m(u)\| + \|u_n - u - \xi A(T(u_n) - T(u)) \\ &\quad + \xi A(A(u_n) - A(u))\| \\ &\leqslant 2\,\|m(u_n) - m(u)\| + \|u_n - u - \xi A(T(u_n) - T(u))\| \\ &\quad + \xi\,\|A(A(u_n) - A(u))\| \\ &\leqslant 2\mu\,\|u_n - u\| + \|u_n - u - \xi A(T(u_n) - T(u))\| + \xi\lambda\,\|u_n - u\|. \end{aligned}$$

By using strong monotonicity and Lipschitz continuity of T, we have

$$\begin{aligned} \|u_n - u + \xi A(T(u_n) - T(u))\|^2 &= \|u_n - u\|^2 - 2\xi(u_n - u, A(T(u_n) - T(u))) \\ &\quad + \xi^2\,\|A(T(u_n) - T(u))\|^2 \\ &\leqslant \|u_n - u\|^2 - 2\xi\langle u_n - u, T(u_n) - T(u)\rangle + \xi^2\,\|T(u_n) - T(u)\|^2 \\ &\leqslant \|u_n - u\|^2 - 2\xi\beta\,\|u_n - u\|^2 + \xi^2\alpha^2\,\|u_n - u\|^2 \\ &= (1 - 2\xi\beta + \xi^2\alpha^2)\,\|u_n - u\|^2. \end{aligned}$$

Thus,

$$\begin{aligned} \|u_{n+1} - u\| &\leqslant 2\mu\,\|u_n - u\| + \sqrt{(1 + 2\xi\beta + \xi^2\alpha^2)}\,\|u_n - u\| + \xi\lambda\,\|u_n - u\| \\ &\quad (2\mu + \xi\lambda + \sqrt{(1 - 2\xi\beta + \xi^2\alpha^2)})\,\|u_n - u\| \end{aligned}$$

or

$$\|u_{n+1} - u\| \leqslant \theta\,\|u_n - u\|,$$

where $\theta = 2\mu + \xi\lambda + \sqrt{(1 - 2\xi\beta + \xi^2\alpha^2)} < 1$ for $0 < \xi < 2\mu/(\alpha - \lambda)$, $\alpha > \lambda$ and $\beta > \alpha$.

By iteration, we get

$$\|u_{n+1}-u\| \leqslant \theta^n \|u_1 - u\|.$$

Since $\theta < 1$, we find that u_{n+1} strongly converges to u in H.

Remark 3.1. If a point-to-point mapping m is zero, then Algorithm 3.1 is the same as Algorithm 2.1 [14].

THEOREM 3.2. *If K is a convex cone in H, then u is a solution of the relation*

$$\langle T(u)+A(u), v-u\rangle \geqslant 0, \qquad \forall v \in K(u) \tag{3.3}$$

if and only if u satisfies (2.3), *where $K(u)$ is defined by* (2.5).

Proof. It is similar to the proof of Theorem 3.1 in [13].

Since the solutions of (3.3) and (2.3) are the same if $K(u)$ is defined by (2.5) and K is a convex cone, the algorithm for finding the approximate solution of a strongly nonlinear quasicomplementarity problem is as follows:

Algorithm 3.2. For any given $u \in K(u)$, compute

$$u_{n+1} = m(u_n) + P_K(u_n - vA(T(u_n) + A(u_n)) - m(u_n)),$$

for some positive constant v.

If $K(u)$ is independent of the solution, that is, $K(u) = K$, then Algorithm 3.2 reduces to Algorithm 3.1 [14].

Note added in proof. The authors would like to mention that a paper by M. A. Noor, "Generalized Quasi Mildly Non-linear Variational Inequalities: Variational Methods in Engineering" (edited by C. A. Brebbia, Springer-Verlag, Berlin/Heidelberg/New York, 1985, 3-3–3-11) on a related theme has just recently come to their notice.

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