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The decomposition of the hypermetric cone into L-domains

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ABSTRACT

The hypermetric cone HYP_{n+1} is the parameter space of basic Delaunay polytopes of n-dimensional lattice. If one fixes one Delaunay polytope of the lattice then there are only a finite number of possibilities for the full Delaunay tessellations. So, the cone HYP_{n+1} is the union of a finite set of L-domains, i.e. of parameter space of full Delaunay tessellations.

In this paper, we study this partition of the hypermetric cone into L-domains. In particular, we prove that the cone HYP_{n+1} of hypermetrics on $n+1$ points contains exactly $\frac{1}{2}n!$ principal L-domains. We give a detailed description of the decomposition of HYP_{n+1} for $n = 2, 3, 4$ and a computer result for $n = 5$. Remarkable properties of the root system D_4 are key for the decomposition of HYP_5.

1. Introduction

An *n-dimensional lattice* L is a subgroup of $\mathbb{R}^n$ of the form $L = v_1\mathbb{Z} + \cdots + v_n\mathbb{Z}$ with $(v_1, \ldots, v_n)$ a basis of $\mathbb{R}^n$. Let $S(c, r)$ be a sphere in $\mathbb{R}^n$ with center c and radius r. Then, $S(c, r)$ is said to be a *Delaunay sphere* in the lattice L if the following two conditions hold:

(i) $\|v - c\| \geq r$ for all $v \in L$,
(ii) the set $S(c, r) \bigcap L$ has affine rank $n + 1$.

The n-dimensional polytope P, which is defined as the convex hull of the set $S(c, r) \bigcap L$, is called a *Delaunay polytope* of rank n. The Delaunay polytopes of rank n form a face-to-face tiling of $\mathbb{R}^n$. The *Voronoi polytope* $P_V(L)$ of a lattice L is the set of points, whose closest element in L is 0. Its vertices are centers of Delaunay polytopes of L. The polytope $P_V(L)$ forms a tiling of $\mathbb{R}^n$ under translation by L, i.e. it is a *parallelohedron* (see Fig. 1).

The cones $\mathcal{S}^n_{>0}$, $\mathcal{S}^n_{\geq 0}$ are respectively the cone of positive definite, positive semidefinite $n \times n$ matrices. The *rational* closure $\mathcal{S}^n_{rat\geq 0}$ of $\mathcal{S}^n_{>0}$ is defined as the positive semidefinite matrices, whose

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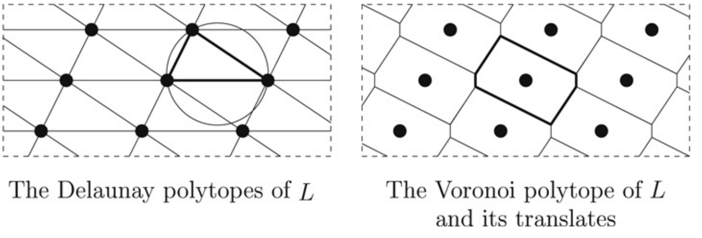

Fig. 1. A lattice $L \subset \mathbb{R}^2$ and the induced partitions.

kernel is defined by rational equalities (see [1]). Given a basis $B = (v_i)_{1\le i\le n}$ of a lattice L, we associate the Gram matrix $a = B^{\mathrm{T}}B \in \mathcal{S}^n_{>0}$. On the other hand if $a \in \mathcal{S}^n_{>0}$, then we can find an invertible real matrix B such that $a = B^{\mathrm{T}}B$, i.e. B is the basis of a lattice $L = B\mathbb{Z}^n$ with Gram matrix a. So, we can replace the study of Delaunay polytopes of L for the standard scalar product by the study of Delaunay polytopes of $\mathbb{Z}^n$ for the scalar product $x^{\mathrm{T}}ax$. If one takes another basis B' of L, then $B' = BP$ for some $P \in \mathrm{GL}_n(\mathbb{Z})$ and one has $a' = P^T aP$, i.e. a and a' are *arithmetically equivalent*. In other words, the study of n-dimensional lattices up to isometric equivalence is the same as the study of positive definite $n \times n$ symmetric matrices, up to arithmetic equivalence. In [1] it is proved that if $a \in \mathcal{S}^n_{\ge 0}$, then one can define, possibly infinite, Delaunay polytopes of $\mathbb{Z}^n$ for $x^{\mathrm{T}}ax$ if and only if $a \in \mathcal{S}^n_{rat\ge 0}$.

Given a polytope P of $\mathbb{Z}^n$ the condition that it is a Delaunay polytope for the norm $x^{\mathrm{T}}ax$ translates to linear equalities and strict inequalities on the coefficients of a. An L-domain is the convex cone of all matrices $a \in \mathcal{S}^n_{>0}$ such that $\mathbb{Z}^n$ has the same Delaunay tessellation for $x^{\mathrm{T}}ax$ (Details see, for example, in [2,3,1]). Voronoi proved that the cone $\mathcal{S}^n_{>0}$ is partitioned into polyhedral L-domains. An L-domain of maximal dimension $\frac{1}{2}n(n+1)$ is called *primitive*. An L-domain is primitive if and only if the Delaunay tiling related to it consists only of simplices. Each non-primitive L-domain is an open face of the closure of a primitive one. In particular, an extreme ray of the closure of an L-domain is a non-primitive one-dimensional L-domain. The group $\mathrm{GL}_n(\mathbb{Z})$ acts on the L-domains of $\mathcal{S}^n_{>0}$ by $\mathcal{D} \mapsto P^{\mathrm{T}}\mathcal{D}P$, and there is a finite number of orbits of L-domains, called *L-types*. The geometric viewpoint is most useful for thinking, and drawings about lattice and the Gram matrix viewpoint is the most suitable to machine computations.

A *metric* on the set $\{0, 1, \dots, n\}$ is a function d such that $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, y) \le d(x, z) + d(z, y)$. A metric d is a *hypermetric* if it satisfies the inequalities

$$H_z(d) = \sum_{0\le i<j\le n} z_i z_j d(i, j) \le 0 \tag{1}$$

for all integral vectors $z \in \mathbb{Z}^{n+1}$ such that $\sum_{i=0}^n z_i = 1$. The set of all hypermetrics on n points $\{0, \dots, n-1\}$ is denoted by HYP_n.

The group $\mathrm{Sym}(n)$ acts on HYP_n; it is proved in [4] that there is no other symmetries if $n \ne 4$. It is proved in [3] that HYP_{n+1} is polyhedral, i.e. among the infinite set of inequalities of the form (1), a finite number suffices to get all facets. This result can be proved in many different ways, see [3, Theorem 14.2.1]; the second proof uses that the image $\xi(\mathrm{HYP}_{n+1})$ is the union of a finite number of L-domains. The purpose of this article is to investigate such decompositions of HYP_{n+1}.

The set of orbits of facets of HYP_n for $n \le 6$ is given in Table 1, HYP_7 has 14 orbits of facets (see [3, 5,6]) and the list is not known for $n \ge 8$. An inequality of (1) is called *k-gonal* if $\sum_{i=0}^n |z_i| = k$. 3-gonal and 5-gonal inequalities are also called *triangle* and *pentagonal* inequalities, respectively.

A Delaunay polytope P of a lattice L is called *generating* if the smallest, for the inclusion relation, lattice containing $V(P)$ is L. Moreover, if there exist a family $(v_0, \dots, v_n)$ of vertices of P such that for any $v \in L$ there exist $\alpha_i \in \mathbb{Z}$ with

$$1 = \sum_{i=0}^n \alpha_i, \qquad v = \sum_{i=0}^n \alpha_i v_i$$

Table 1
The facets of HYP_n for $n \le 6$

n	Representative of orbits of facets of HYP_n
3	$(1, 1, -1)$ (triangle inequality)
4	$(1, 1, -1, 0)$
5	$(1, 1, -1, 0, 0)$ and $(1, 1, 1, -1, -1)$ (pentagonal inequality)
6	$(1, 1, -1, 0, 0, 0)$, $(1, 1, 1, -1, -1, 0)$, $(1, 1, 1, 1, -1, -2)$ and $(2, 1, 1, -1, -1, -1)$

then P is called *basic* and $(v_0, \dots, v_n)$ is an *affine basis*. Given such an affine basis, we define the distance $d(i, j) = \|v_i - v_j\|^2$ and we have

$$H_b(d) = \sum_{0 \le i < j \le n} b_i b_j d(i, j) = \left(r^2 - \left\| \sum_{i=0}^{n} b_i v_i - c \right\|^2 \right) \le 0,$$

where r and c relate to the circumscribing sphere $S(c, r)$ of P. So the hypermetric inequalities correspond to the inequalities determining a family $(v_0, \dots, v_n)$ to be an affine basis of a Delaunay polytope. Moreover we have $H_b(d) = 0$ if and only if $\sum_{i=0}^{n} b_i v_i$ is a vertex of P. In other words the hypermetric cone HYP_{n+1} is the parameter space of a basic simplex in $\mathbb{Z}^n$. We refer for proofs to [7,3]. In practice, if $(v_0, \dots, v_n)$ is an affine basis, we can replace it by $v_0 = 0$, $v_i = e_i$ and call the corresponding simplex *main*. At this point we should note that the hypermetric cone is just one possibility for a parameter space of Delaunay polytopes. Following [8,9], define $M_{2,n}$ to be the space of real polynomials of n variables with degree at most 2. We then have

$$C_n = \{f \in M_{2,n} | f(x) \ge 0 \text{ for all } x \in \mathbb{Z}^n\}.$$

If P is a Delaunay polytope of rank $k \le n$, then we define

$$C_n(P) = \{f \in C_n | f(x) = 0 \text{ for all } x \in \operatorname{vert} P\}.$$

Those cones were used in [10] to find some so-called *perfect* Delaunay polytopes. Note that if $P = \{0, e_1, \dots, e_n\}$, then the cone $C_n(P)$ is isomorphic to the cone HYP_{n+1}.

The covariance map $\xi : d \to a$ transforms a hypermetric d on $n + 1$ points i, $0 \le i \le n$, of a set X into an $n \times n$ positive semidefinite symmetric matrix a as follows:

$$a_{ij} = \xi(d(i, j)) = \frac{1}{2}(d(0, i) + d(0, j) - d(i, j))$$

(see [3, Section 5.2]). The covariance ξ maps the hypermetric cone HYP_{n+1} into $\mathcal{S}^n_{rat \ge 0}$. Note that there are $n + 1$ distinct such maps depending on which point of $\{0, e_1, \dots, e_n\}$ is chosen as the zero point.

2. Decomposition methods

Recall that the Delaunay tiling related to a Gram matrix a from a primitive L-domain $\mathcal{D}$ consists of simplices. The set of Delaunay simplices of the tiling containing the common lattice point 0 is the star St_0. By translations, along $\mathbb{Z}^n$, the star St_0 determines fully the Delaunay tiling of $\mathbb{Z}^n$. The primitive L-domain $\mathcal{D}$ belongs to $\xi(\mathrm{HYP}_{n+1})$ if and only if its star St_0 contains a main simplex.

A *wall* W is an $\frac{n(n+1)}{2} - 1$ dimensional L-domain, which necessarily separates two primitive L-domains $\mathcal{D}$, $\mathcal{D}'$. Let one moves a point a from the primitive L-domain $\mathcal{D}$ to $\mathcal{D}'$ by passing through W. When $a \in W$, some pairs of simplices of St_0, which are mutually adjacent by a facet, glue into repartitioning polytopes. It is well known (see, for example, [1,2]) that an n-dimensional polytope with $n + 2$ vertices can be triangulated in exactly two ways. When the point a goes from W into the L-domain $\mathcal{D}'$, each repartitioning polytope repartitions into its other set of simplices.

Since each repartitioning polytope has $n + 2$ vertices, there is an affine dependence between its vertices. This affine dependence generates a linear equality between the coefficients a_{ij} of the Gram matrix $a \in \mathcal{S}^n_{>0}$. This equality is just the equation determining the hyperplane supporting the wall W. If the point a lies inside the L-domain $\mathcal{D}$, then this equality holds as an inequality.

Table 2
Decomposition of HYP_{n+1} into L-domains

n	# primitive L-types	# facets of HYP_{n+1}	# orbits of primitive L-domains in HYP_{n+1} under $\mathrm{Sym}(n+1)$	# primitive L-domains in HYP_{n+1}
2	1	3	1	1
3	1	12	1	3
4	3	40	5	172
5	222	210	8287	5 338 650

So, the convex hull of vertices of any pair of simplices of St_0 adjacent by a facet is a putative repartitioning polytope giving an inequality separating the L-domain $\mathcal{D}$ from another L-domain. All adjacent pairs of simplices of St_0 determine a system of inequalities describing the polyhedral cone of the primitive L-domain $\mathcal{D}$. Note that some of these inequalities define faces of $\mathcal{D}$ but not walls. If the adjacent pair of simplices contains the main simplex then the corresponding wall lies on a facet of the cone $\xi(\mathrm{HYP}_{n+1})$.

Using the above system of inequalities, one can define all extreme rays of $\mathcal{D}$, and then all facets of $\mathcal{D}$. The sum of Gram matrices lying on extreme rays of $\mathcal{D}$ is an interior point $a(\mathcal{D})$ of $\mathcal{D}$ uniquely related to this L-domain (see [1] for more details). Hence, L-domains $\mathcal{D}$ and $\mathcal{D}'$ belong to the same L-type if and only if $a(\mathcal{D}') = P^{\mathrm{T}}a(\mathcal{D})P$ for some $P \in \mathrm{GL}_n(\mathbb{Z})$, i.e. $a(\mathcal{D}')$ and $a(\mathcal{D})$ are arithmetically equivalent.

The algorithm for enumerating primitive L-domains in $\xi(\mathrm{HYP}_{n+1})$ works as follows. One takes a primitive Gram matrix $a \in \xi(\mathrm{HYP}_{n+1})$. There is a standard algorithm which, for a given $a \in \mathcal{S}^n_{>0}$, constructs its simplicial Delaunay tiling. (For example, one can take a from a principal L-domain, described in following sections).

Using the star St_0 of the Delaunay tiling, the algorithm, for each pair of adjacent simplices, determines the corresponding inequality. By the system of obtained inequalities, the algorithm finds all extreme rays of the domain $\mathcal{D}$ of a, computes the interior central ray $a(\mathcal{D})$ and finds all facets of $\mathcal{D}$. The L-domain $\mathcal{D}$ is put in the list $\mathcal{L}$ of primitive L-domains in $\xi(\mathrm{HYP}_{n+1})$.

Let F be a facet of $\mathcal{D}$ which does not lie on a facet of $\xi(\mathrm{HYP}_{n+1})$. For each repartitioning polytope related to the facet F of $\mathcal{D}$, the algorithm finds another partition into simplices. This gives the Delaunay tiling of the primitive L-domain $\mathcal{D}'$, which is neighboring to $\mathcal{D}$ by the facet F. The algorithm finds all extreme rays of $\mathcal{D}'$, the ray $a(\mathcal{D}')$ and tests if it is arithmetically equivalent to $a(\mathcal{D})$ for some $\mathcal{D}$ from $\mathcal{L}$. If not, one puts $\mathcal{D}'$ in $\mathcal{L}$. The algorithm stops when all neighboring L-domains are equivalent to ones in $\mathcal{L}$. This algorithm is very similar to the one in [11] for the decomposition of the metric cone into T-domains and belongs to the class of *graph traversal algorithms*.

We give some details of the partition of HYP_{n+1} into primitive L-domains in Table 2.

We now expose another enumeration method of the orbits of primitive L-domains in HYP_{n+1}. Consider a primitive L-domain $\mathcal{D}$. The group $\mathrm{Stab}(\mathcal{D}) = \{P \in \mathrm{GL}_n(\mathbb{Z}) : P^{\mathrm{T}}a(\mathcal{D})P = a(\mathcal{D})\}$ is a finite group, which permutes the translation classes of simplices of the Delaunay decomposition of $\mathcal{D}$. It splits the translation classes of simplices into different orbits. Let S be a basic simplex in $\mathcal{D}$; if one chooses the coordinates such that $\mathrm{vert}\, S = \{0, e_1, \dots, e_n\}$ then one obtains an L-domain $\mathcal{D}_S$, whose image by ξ^{-1} is included in HYP_{n+1}. A permutation of the vertex set of S induces a permutation in HYP_{n+1} as well. So, two cones $\xi^{-1}(\mathcal{D}_S)$ and $\xi^{-1}(\mathcal{D}_{S'})$ are equivalent under $\mathrm{Sym}(n+1)$ if and only if S and S' belong to the same orbit of translation classes of simplices under $\mathrm{Stab}(\mathcal{D})$. Therefore from the list of L-types in dimension n, one obtains the orbits of L-domains in HYP_{n+1}.

3. Dicings, rank 1 extreme rays of an *L*-domain and of *HYP*$_{n+1}$

We denote by $b^{\mathrm{T}}c$ the scalar product of column vectors b and c. A vector $v \in \mathbb{Z}^n$ is called *primitive* if the greatest common divisor of its coefficients is 1; such a vector defines a family of parallel hyperplanes $v^{\mathrm{T}}x = \alpha$ for $\alpha \in \mathbb{Z}$. In the same way, a *vector family* $\mathcal{V} = (v_i)_{1\le i\le M}$ of primitive vectors defines M families of parallel hyperplanes. A vector family $\mathcal{V}$ is called a *lattice dicing* if for any n independent vectors $v_{i_1}, \dots, v_{i_n} \in \mathcal{V}$ the vertices of the hyperplane arrangement $v_{i_j}^{\mathrm{T}}x = \alpha_i$ form the lattice $\mathbb{Z}^n$ (see an example on Fig. 1). This is equivalent to say that any n independent vectors v_i have determinant ± 1, i.e. the vector family is *unimodular*. Given a dicing, the connected components of the complement of $\mathbb{R}^n$ by the hyperplane arrangement form a partition of $\mathbb{R}^n$ by polytopes. It is proved in [12] that the polytopes of a lattice dicing defined by $\mathcal{V} = (v_i)_{1\le i\le M}$ are Delaunay polytopes for the matrices belonging to the L-domain generated by the rank 1-forms $(v_i v_i^{\mathrm{T}})_{1\le i\le M}$, whose corresponding quadratic form is $f_{v_i}(x) = (v_i^{\mathrm{T}} x)^2$. The reverse is also proved there, i.e. any L-domain, whose extreme rays have rank 1 has its Delaunay tessellation being a lattice dicing. Such L-domains are called *dicing domains*; they are simplicial, i.e. their dimension is equal to their number of extreme rays.

Our exposition of matroid theory is limited here to what is useful for the comprehension of the paper and we refer to [13–15] for more details. Given a graph G of vertex-set $\{1, \dots, n\}$ we associate to every edge $e = (i, j)$ a vector v_e, which is equal to 1 in position i, -1 in position j and 0 otherwise. The vector family $\mathcal{V}(G) = (v_e)_{e\in E(G)}$ is unimodular and is called the *graphic unimodular system of the graph G*. Given a unimodular n-dimensional system U of m vectors, for any basis $B \subseteq U$, we can write $U = B(I_n, A)$, where A is a totally unimodular matrix and (I_n, A) is the concatenation of I_n and A. The matrix $(-A^{\mathrm{T}}, I_{m-n})$ defines a unimodular system, which is called the *dual of U* and denoted $\mathrm{Dual}(U)$. Given a graph G of vertex set $V(G)$ and edge set $E(G)$, we choose an orientation on every edge e and associate to it a coordinate x_e. We define a vector space V to be the set of vector $v \in \mathbb{R}^{E(G)}$ satisfying for all $x \in V(G)$ to the *vertex cut equation*

$$0 = \sum_{y\in N(x)} v_{(x,y)} \epsilon_{(x,y)}$$

with $N(x)$ the neighbors of x and $\epsilon_{(x,y)} = 1$ if the orientation of the edge (x, y) goes from x to y and -1 otherwise. Take $v_1, \dots, v_N$ a basis of the space and denote by $CoGr(G)$ the *cographic unimodular system of the graph G* defined to be the vector system obtained by taking the transpose of the matrix $(v_1, \dots, v_N)$. The unimodular systems $CoGr(G)$ and $\mathrm{Dual}(Gr(G))$ are isomorphic. In [13] a general method for describing unimodular vector families is given using graphic, cographic unimodular systems and a special unimodular system named E_5 (or R_{10} as in [16]).

Given a finite set E, a *matroid* $M = M(E)$ is a family $\mathcal{C}(M)$ of subset of X called *circuits* such that:

- for $C_1, C_2 \in \mathcal{C}(M)$, it holds $C_1 \not\subseteq C_2$, $C_2 \not\subseteq C_1$ if $C_1 \neq C_2$;
- if $e \in C_1 \bigcap C_2$, then there is $C_3 \in \mathcal{C}(M)$ such that $C_3 \subseteq C_1 \bigcup C_2 - \{e\}$.

A set of vectors q_e, $e \in E$, represents a matroid $M(E)$ if, for any circuit $C \in \mathcal{C}(M)$, the equality $\sum_{e\in C} q_e = 0$ holds. A matroid, is called *regular* if it admits a representation as a unimodular system of vectors. If $M(E)$ is a *graphic* or *cographic* matroid of a graph G with a set E of edges, then circuits of M are *cycles* or *cuts* of G, respectively.

A graph G is *plane* if it is embedded in the 2-plane such that any two edges are non-crossings. A plane graph defines a partition of the plane into faces delimited by edges. The *dual graph* G^* is the graph defined by faces with an edge between two faces if they share an edge. Then $(G^*)^* = G$, and there is a bijection between (intersecting) edges of G and G^* such that each cut of G corresponds to a cycle of G^*, and vice versa. In other words, the cographic matroid of G and the graphic matroid of G^* are isomorphic.

The only rank 1 extreme rays of the cone HYP_{n+1} are *cut metrics*. For $n \le 5$, the hypermetric cone HYP_{n+1} coincides with the cut cone CUT_{n+1} which is the cone hull of $2^n - 1$ cut metrics. Denote $N = \{1, \dots, n\}$; if $S \subset N$, $S \neq \emptyset$ then the cut metric δ_S on $X = \{0\} \bigcup N$ is defined as follows:

$$\delta_S(i,j) = 1 \quad \text{if } \left|\{ij\} \bigcap S\right| = 1, \quad \text{and} \quad \delta_S(i,j) = 0, \quad \text{otherwise}.$$

The covariance map ξ transforms the cut metric δ_S into the following *correlation* matrix $p(S)$ of rank 1:

$$p_{ij}(S) = 1 \quad \text{if } \{ij\} \subset S, \quad \text{and} \quad p_{ij}(S) = 0, \quad \text{otherwise, where } 1 \le i, j \le n.$$

The quadratic form corresponding to the correlation matrix $p(S)$ is

$$f_S(x) = \xi(\delta_S)(x) = \sum p_{ij}(S) x_i x_j = \left(\sum_{i\in S} x_i \right)^2 .$$

So, $f_S = f_q$ with $f_q(x) = (q^T x)^2$ and $q = \sum_{i\in S} b_i := b(S)$ the incidence vector of the set S. In summary:

Lemma 1. *A vector $q \in \mathbb{Z}^n$ determines an extreme ray f_q of $\xi(\mathrm{HYP}_{n+1})$ if and only if $q = b(S)$ for some $S \subseteq N$, $S \neq \emptyset$.*

By Lemma 1, if U determines a dicing domain in $\xi(\mathrm{HYP}_n)$, then the set of coordinates of vectors from U in the basis $\mathcal{B}$ forms a unimodular matrix with (0, 1)-coefficients. Note that the columns of any (0, 1)-matrix are incidence vectors of subsets of a set. Since any unimodular system U determines a dicing L-domain $\mathcal{D}(U)$, we have the following proposition:

Proposition 1. *Let $\mathcal{D}(U)$ be a dicing domain determined by a unimodular set U. The following assertions are equivalent:*

(i) *$\mathcal{D}(U)$ lies in $\xi(\mathrm{HYP}_n)$;*
(ii) *U is represented by a (0, 1)-matrix.*

4. The principal *L*-domain

There is a unique primitive L-type, whose L-domains are dicing domains of maximal dimension $\frac{1}{2}n(n+1)$ [17,18]. Voronoi calls this L-type *principal*. Each principal L-domain is simplicial and all its $\frac{1}{2}n(n+1)$ extreme rays have rank 1. The set Q of vectors q determining extreme rays f_q of a principal L-domain forms a maximal unimodular system. This system is the classical unimodular root system A_n representing the graphic matroid of the complete graph K_{n+1} on $n+1$ vertices.

In our case, when a principal L-domain is contained in $\xi(\mathrm{HYP}_{n+1})$, its extreme rays belong to the set $\{p(S) : S \subseteq N, S \neq \emptyset\}$ of extreme rays of the cone $\xi(\mathrm{HYP}_{n+1})$. Hence, the vectors q have the form $b(S)$ for $S \subseteq N$. We shall find all subsets of these vectors representing the graphic matroid of K_{n+1}. We orient edges of K_{n+1} into *arcs* and relate a vector $b(S)$ to each arc of the directed graph K_{n+1} such that, for any directed circuit C in K_{n+1}, the following equality holds

$$\sum_{e\in C} \varepsilon_e b(S_e) = 0. \tag{2}$$

Here $b(S_e)$ is the vector related to the arc e and $\varepsilon_e = 1$ if the directions of e and C coincide, and $\varepsilon_e = -1$, otherwise.

Given a chain of equally directed arcs labeled by one-element set, a subchain of this chain determines a (0, 1)-characteristic vector. It is known that the set of characteristic vectors of a set of connected subchains determines a graphic unimodular system. We show below that such graphical systems are contained in the set $\{b(S) : S \subseteq N\}$.

This relation of vectors $b(S)$ and arcs of K_{n+1} provides a labeling of arcs of K_{n+1} by subsets $S \subseteq N$. We call this labeling *feasible* if the corresponding set of vectors $b(S)$ gives a representation of the graphical matroid of K_{n+1}, i.e. (2) holds for each circuit C of K_{n+1}.

Consider a k-circuit $C = \{e_i : 1 \le i \le k\}$, whose arcs have the same directions. Suppose that, for $1 \le i \le k$, S_i is a label of the arc e_i, and that this labeling is feasible. Then the equality $\sum_{i=1}^k b(S_i) = 0$ holds. Since the coordinates of the vectors $b(S)$ take (0, 1)-values, this equality is not possible for a feasible labeling. Hence, the directed graph K_{n+1} with a feasible labeling has no circuit, whose arcs have the same directions. Any finite directed graph with no circuit has at least one *source* vertex (and a *sink* vertex as well).

Now consider a directed 3-circuit $C = \{e_1, e_2, e_3\}$ of a feasible labeled K_{n+1}. Then two arcs of C, say the arcs e_1, e_2, have directions coinciding with the direction of C, and the third arc e_3 has opposite direction. If S_i is a label of e_i, $i = 1, 2, 3$, then we have the equality $b(S_1) + b(S_2) = b(S_3)$. This equality is possible only if $S_1 \bigcap S_2 = \emptyset$ and $S_1 \bigcup S_2 = S_3$. Since any two adjacent arcs of a complete graph belong to a 3-circuit, we obtain the following result:

Lemma 2. *Let two arcs e_i and e_j be adjacent in a feasible labeled graph K_{n+1} and have labels S_i and S_j. Then $S_i \bigcap S_j = \emptyset$ if the directions of these arcs coincide in the 2-path $[e_i, e_j]$. If these directions are opposite, then either $S_i \subset S_j$ or $S_j \subset S_i$.*

Let v be a source vertex of K_{n+1} and $E(v)$ be the set of n arcs incident to v. Since all arcs of $E(v)$ go out from v, any two arcs $e, e' \in E(v)$ have opposite directions in their 2-path $[e, e']$. Let $\mathcal{S}(v) = \{S_e : e \in E(v)\}$. By Lemma 2, the family $\mathcal{S}(v)$ is a nested family of n mutually embedded distinct subsets. This implies that the sets $S \in \mathcal{S}(v)$ and the arcs $e \in E(v)$ can be indexed as S_i, e_i, $1 \le i \le n$, such that S_i is the label of e_i and $|S_i| = i$.

For $2 \le i \le n$, let g_i be the arc of the graph K_{n+1}, which forms a 3-circuit with the arcs $e_{i-1}, e_i \in E(v)$. Lemma 2 implies that the arc g_i has the one-element set $S_i - S_{i-1}$ as label, and the direction of g_i coincides with the direction of e_{i-1} in their 2-path $[e_{i-1}, g_i]$. Now, it is clear that the n arcs g_i for $1 \le i \le n$, where $g_1 = e_1$, form an n-path, whose arcs have the same directions and are labeled by one-element sets. Recall that a non-self-intersecting n-path in a graph with $n + 1$ vertices is called a *Hamiltonian path*. We obtain the following result.

Lemma 3. *A feasible labeled complete directed graph K_{n+1} has a Hamiltonian path such that all its arcs have the same directions and each arc has a one-element labeling set.*

Let $\{0\} \bigcup N$ be the set of vertices of K_{n+1}, where the vertex 0 is the source. Let $0, i_1, i_2, \ldots, i_n$ be the vertices of the Hamiltonian path π in Lemma 3. The path π defines uniquely an orientation and a feasible labeling of K_{n+1} as follows. The arc with end-vertices i_j, i_k, where $0 \le j < k \le n$, is labeled by the set $S_{jk} = \{i_r : j + 1 \le r \le k\} \subseteq N$. If one reverse the above order, then one gets the same family of sets, and the labeled graph K_{n+1} gives the same representation of the unimodular system A_n. We have

Lemma 4. *Any representation of the graphic matroid of the complete graph K_{n+1} by vectors $b(S)$, $S \subseteq N$, $S \neq \emptyset$, is determined by a complete order of the set N. Two opposite orders determine the same representation.*

Since there are $n!$ complete orders on an n-set, as a corollary of Lemma 4, we obtain our main result.

Proposition 2. *The cone $\xi(\mathrm{HYP}_{n+1})$ contains $\frac{1}{2}n!$ distinct principal L-domains.*

So, each principal domain is determined by an order (and its reverse) of the set N. For the sake of definition, we choose the lexicographically minimal order $\mathcal{O}$ from these two orders. Let $\mathcal{S}(\mathcal{O})$ be the family of sets $S \subseteq N$, $S \neq \emptyset$, such that elements of each set S determine a continuous subchain of the n-chain, corresponding to the order $\mathcal{O}$. A principal domain determined by an order $\mathcal{O}$ of the set N has $\frac{1}{2}n(n + 1)$ extreme rays $p(S)$ for $S \in \mathcal{S}(\mathcal{O})$.

Each face F of a dicing L-domain in $\xi(\mathrm{HYP}_{n+1})$ is uniquely determined by its extreme rays $p(S)$, all of rank 1. Set

$$\mathcal{S}(F) = \{S \subseteq N : p(S) \text{ is an extreme ray of } F\}.$$

Proposition 3. *If $n \ge 4$, then any two principal domains in $\xi(\mathrm{HYP}_{n+1})$ are not contiguous by a facet.*

Proof. If two principal domains $\mathcal{D}(\mathcal{O})$, $\mathcal{D}(\mathcal{O}') \subset \xi(\mathrm{HYP}_{n+1})$ share a facet F, then they have $\frac{1}{2}n(n + 1) - 1$ common extreme rays $p(S)$ for $S \in \mathcal{S}(F)$. This implies that the families $\mathcal{S}(\mathcal{O})$ and $\mathcal{S}(\mathcal{O}')$ should differ by one element only. But, for any two distinct orders $\mathcal{O}$ and $\mathcal{O}'$, the families $\mathcal{S}(\mathcal{O})$ and $\mathcal{S}(\mathcal{O}')$ differ at least by two sets, if $n \ge 4$. For example, suppose $\mathcal{O}$ and $\mathcal{O}'$ differ by a transposition of two elements i and j. Then there is at least one subchain in $\mathcal{O}$ containing i and not containing j, which is not a subchain of $\mathcal{O}'$. The same assertion is true for the order $\mathcal{O}'$. □

5. The decompositions of *HYP*$_3$ and *HYP*$_4$

Since, for $n = 2$ and $n = 3$ there exists only one primitive L-type, namely, the principal L-type, Proposition 2 describes completely the decompositions of the cones $\xi(\mathrm{HYP}_{n+1})$ for $n = 2, 3$.

Recall that each facet F of HYP_{n+1} is described by an inequality (1). If $n \le 4$ and F is a facet of HYP_{n+1}, then $z_i \in \{0, \pm 1\}$. Hence, we can denote triangle and pentagonal facets as $F(ij; k)$ and $F(ijk; lm)$, respectively. Here $z_i = z_j = 1$, $z_k = -1$, $z_l = 0$, $l \in X - \{ijk\}$, for the triangle facet, and $z_i = z_j = z_k = 1$, $z_l = z_m = -1$, $z_r = 0$ if $r \in X - \{ijklm\}$, for the pentagonal facet.

Note that $\xi(\delta_S) = p(S)$. For $S = \{ij, \dots, k\}$, set $S = ij, \dots, k$ and $p(S) = p(ij, \dots, k)$.

n = 2. The cone $\xi(\mathrm{HYP}_3)$ is three-dimensional and simplicial. There is only one order $\mathcal{O} = (12)$ with $\mathcal{S}(\mathcal{O}) = \{1, 2, 12\}$. Its three extreme rays $p(1)$, $p(2)$ and $p(12)$ span a principal domain, i.e. $\xi(\mathrm{HYP}_3)$ coincides with a principal domain.

n = 3. The cone $\xi(\mathrm{HYP}_4)$ is six-dimensional and has seven extreme rays $p(i)$, $i = 1, 2, 3$, $p(ij)$, $ij = 12, 13, 23$, $p(123)$ and 12 facets $\xi(F(ij; k))$ for $i, j, k \in \{0\} \bigcup N$. Note that

$$p(1) + p(2) + p(3) + p(123) = p(12) + p(13) + p(23) = p.$$

Hence, the four-dimensional cone $\mathcal{C}_1 = \mathbb{R}_+ p(1) + \mathbb{R}_+ p(2) + \mathbb{R}_+ p(3) + \mathbb{R}_+ p(123)$ intersects by the ray $\mathbb{R}_+ p$ the three-dimensional cone $\mathcal{C}_2 = \mathbb{R}_+ p(12) + \mathbb{R}_+ p(13) + \mathbb{R}_+ p(23)$. The ray $\mathbb{R}_+ p$ is an interior ray of both.

By Proposition 2, $\xi(\mathrm{HYP}_4)$ contains three principal domains. These three six-dimensional L-domains are determined by the three orders (123), (132) and (213) of $N = \{123\}$. Denote the domain determined by the order (ijk) as $\mathcal{D}_j$, where j is the middle element of the order (ijk). Since the three one-element subsets $\{i\}, i \in N$, and the set N give continuous chains in all the three orders, the four rays $p(1), p(2), p(3)$ and $p(123)$ are common rays of all the three domains $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$. Hence, the cone $\mathcal{C}_1$ is the common four-dimensional face of these three principal domains. The domain $\mathcal{D}_i$ is the cone hull of $\mathcal{C}_1$ and two rays $p(ij)$ and $p(ik)$. The four triangle facets $\xi(F(jk; i))$, $\xi(F(jk; 0))$, $\xi(F(0i; j))$, $\xi(F(0i; k))$ of $\xi(\mathrm{HYP}_4)$ are also facets of $\mathcal{D}_i$. The other two facets of $\mathcal{D}_i$ separating the domain $\mathcal{D}_i$ from $\mathcal{D}_j$ and $\mathcal{D}_k$ are the cone hulls of $\mathcal{C}_1$ with the rays $p(ij)$ and $p(ik)$, respectively.

6. *L*-domains in $\xi(HYP_5)$

A *parallelohedron* is an n-dimensional polytope, whose image under a translation group forms a tiling of $\mathbb{R}^n$. Given a face F of a parallelohedron P, the set of faces of P which are translates of F is called the *zone* of P. For a parallelohedron P the Minkowski sum $P + z(q)$ may not be a parallelohedron. A parallelohedron P is called *free along a vector* q and the vector q is called *free for a parallelohedron* P if the sum $P + z(q)$ is a parallelohedron (see [19] for more details on this notion).

If $P_q = P + z(q)$ is an n-dimensional parallelohedron, then P_q has a *non-zero width* along the line $l(q)$ spanned by q. This means that the intersection of P_q with a line parallel to $l(q)$ is distinct from a point. In this case, the lattice L_q of the parallelohedron P_q has a *lamina* H, i.e. a hyperplane H such that H is transversal to $l(q)$, the intersection $L_q \bigcap H$ is an $(n-1)$-dimensional sublattice of L_q and each Delaunay polytope of L_q lies between two neighboring layers of L_q parallel to $L_q \bigcap H$ (see [20]). If a Voronoi polytope has a non-zero width along a line l, then the lamina H is orthogonal to l.

6.1. *Root lattice* D_4

The lattice D_n is defined as

$$\mathsf{D}_n = \left\{ x \in \mathbb{Z}^n \,\middle|\, \sum_{i=1}^{n} x_i \equiv 0 (\mathrm{mod}\ 2) \right\}.$$

If $\{e_i : 1 \le i \le n\}$ is an orthonormal basis of $\mathbb{Z}^n$, then the set of shortest vectors of D_n is $\pm e_i \pm e_j$ for $1 \le i < j \le n$. It is the set of all facet vectors of $P_V(\mathsf{D}_n)$ and form an irreducible root system, which we also denote by D_n. There are three translation classes of Delaunay polytopes in D_n: the cross polytope β_n whose vertex set is formed by all $e_1 \pm e_i$ for $1 \le i \le n$, the half cube $\frac{1}{2}H_n$ whose vertex set is $\{x \in \{0,1\}^n | \sum_{i=1}^n x_i \equiv 0 (\mathrm{mod}\ 2)\}$ and a second half cube $\frac{1}{2}H'_n$ whose vertex set is $\{x \in \{1,2\} \times \{0,1\}^{n-1} | \sum_{i=1}^n x_i \equiv 0 (\mathrm{mod}\ 2)\}$. The two half cubes are equivalent under the automorphism group $\mathrm{Aut}(\mathsf{D}_n)$ of the root lattice D_n. It is proved in [21,22] that the Voronoi polytope $P_V(\mathsf{D}_n)$ of the n-dimensional root lattice D_n is free only along vectors which are parallel to edges of $P_V(\mathsf{D}_n)$. Thus there are $2^n + 2n$ free vectors.

It turns out that when $n = 4$ all Delaunay polytopes are isometric to the cross-polytope β_4 and equivalent under $\mathrm{Aut}(\mathsf{D}_4)$. Any 2-face of β_4 is contained in three Delaunay polytopes β_4 of the

Delaunay tessellation. This proves that the L-domains of D_4 are not dicing domains. Furthermore, D_4 is a *rigid lattice* (see [23]), i.e. its Delaunay tessellation determines the Gram matrix up to a scalar multiple. This means that D_4 determine a one-dimensional L-type and that any L-domain containing it as an extreme ray is not a dicing domain.

The free vectors of $P_V(\mathsf{D}_4)$ are parallel and can be identified to the diagonals of the cross polytopes. They are

- $\pm 2e_i$ with $1 \le i \le 4$ for β_4
- $(\pm 1, \pm 1, \pm 1, \pm 1)$ with even plus signs for $\frac{1}{2}H_4 \equiv \beta_4$
- $(\pm 1, \pm 1, \pm 1, \pm 1)$ with odd plus signs for $\frac{1}{2}H'_4 \equiv \beta_4$.

Up to a factor $\sqrt{2}$, those 24 vectors are an isometric copy of the root system D_4, which we denote by $\mathsf{D}_{4,2}$. The union $\mathsf{D}_4 \bigcup \mathsf{D}_{4,2}$ is the irreducible root system F_4 (see [24]).

The Voronoi polytope $P_V(\mathsf{D}_4)$ of the root lattice D_4 is the regular polytope 24-cell, whose automorphism group is the Coxeter group $W(\mathsf{F}_4)$ of Schläfli symbol $\{3, 4, 3\}$ (see [25]). Its number of vertices, 2-faces, 3-faces is 24, 96, 24. Each facet is an octahedron with four pairs of opposite and mutually parallel triangular 2-faces. The facet vectors of the Voronoi polytope $P_V(\mathsf{D}_4)$ are the 24 roots of the root system D_4. The polytope $P_V(\mathsf{D}_4)$ has 12 edge *zones* of mutually parallel edges representing $\mathsf{D}_{4,2}$ and 16 face zones of mutually parallel triangular faces. Each edge zone contains 8 parallel edges, and each face zone contains six parallel faces.

We choose a basis of D_4 and denote by $a(\mathsf{D}_4)$ the Gram matrix of D_4 in this basis. The 24 vertices of $P_V(\mathsf{D}_4)$ are given by $\frac{1}{2}\mathsf{D}_{4,2}$.

6.2. *The L-domains containing* $a(\mathsf{D}_4)$

For $n = 4$, there are three primitive L-types of four-dimensional lattices: the principal type, and L-types called by Delaunay in [26] types II and III. The ten-dimensional L-domains of these L-types are constructed as follows (cf., [22]).

Each k-dimensional face of a principal domain relates to the graphic matroid of a subgraph $G \subseteq \mathsf{K}_5$ on k edges. Hence each facet (of dimension 9) of a principal domain relates to the graphic matroid of $\mathsf{K}_5 - \mathbf{1}$, i.e. the complete graph K_5 without one edge. The Gram matrix $a(\mathsf{D}_4)$ of the root lattice D_4 is an extreme ray of L-domains of type II and III. The cone hull of a facet of a principal domain and of a ray of type $a(\mathsf{D}_4)$ is an L-domain of type II. Hence, any principal domain is contiguous in $\mathcal{S}^4_{>0}$ by facets only with L-domains of type II.

An L-domain of type II has the following three types of facets. One dicing facet by which it is contiguous in $\mathcal{S}^4_{>0}$ to a principal domain relates to the graphic matroid of the graph $\mathsf{K}_5 - \mathbf{1}$. Each of two other types of facets is the cone hull of the ray of type $a(\mathsf{D}_4)$ and a dicing eight-dimensional face related to the graphic matroids of $\mathsf{K}_5 - 2 \times \mathbf{1}$ or of $\mathsf{K}_5 - \mathbf{2}$. Here each of these graphs is the complete graph K_5 without two non-adjacent or two adjacent edges, respectively.

The complete bipartite graph K_{ij} is formed by two blocks S_1, S_2 of vertices with $|S_1| = i$, $|S_2| = j$ and two vertices adjacent if and only if they belong to different blocks. An L-domain of type III is the cone hull of $a(\mathsf{D}_4)$ and a nine-dimensional dicing facet related to the cographic matroid $CoGr(\mathsf{K}_{33})$ of the bipartite graph K_{33}. Each 8-element submatroid of $CoGr(\mathsf{K}_{33})$ is graphic and relates to the graph $\mathsf{K}_5 - 2 \times \mathbf{1}$. Hence, each other facet of an L-domain of type III is the cone hull of $a(\mathsf{D}_4)$ and a dicing eight-dimensional face related to $\mathsf{K}_5 - 2 \times \mathbf{1}$. In $\mathcal{S}^4_{>0}$, this facet is a common facet of L-domains of types II and III.

So, an L-domain of type II is contiguous in $\mathcal{S}^4_{>0}$ to L-domains of all three types. It is useful to note ([27,28]) that if f belongs to the closure $\mathcal{D}$ of an L-domain of type II or III then the Voronoi polytope $P_V(f)$ of $\mathbb{Z}^n$ under the quadratic form f is an affine image of the Minkovski sum

$$\sum_{q \in U} \lambda_q z(q) + \lambda P_V(\mathsf{D}_4), \quad \lambda_q \ge 0, \lambda \ge 0,$$

where U is the unimodular set of vectors related to rank 1 extreme rays of $\mathcal{D}$, and $P_V(\mathsf{D}_4)$ is the Voronoi polytope of the root lattice D_4, whose form $a(\mathsf{D}_4)$ lies also on extreme ray of $\mathcal{D}$.

6.3. Unimodular systems in $\mathsf{D}_{4,2}$

Let $D(4) \subset \mathsf{D}_{4,2}$ be a subset of 12 roots chosen by one from each pair of opposite roots. The vector system $D(4)$ is partitioned into three disjoint *quadruples* Q_i, $i = 1, 2, 3$, of mutually orthogonal roots, given in Section 6.1.

For what follows, we have to consider *triples* (r_1, r_2, r_3) of roots chosen by one from each quadruple Q_i, i.e. $r_i \in Q_i$, $i = 1, 2, 3$. Let $t = (r_1, r_2, r_3)$ be such a triple. Let $\{ijk\} = \{123\}$, i.e. these three indices are distinct. We have $r_i^2 = 2$, $r_i^{\mathrm{T}} r_j \in \{\pm 1\}$. Hence, the vectors

$$r_{ij} = r_i - (r_i^{\mathrm{T}} r_j) r_j \quad \text{for } ij = 12, 23, 31, \tag{3}$$

are roots of $\mathsf{D}_{4,2}$. Since $r_{ij}^{\mathrm{T}} r_i = -r_{ij}^{\mathrm{T}} r_j = 1$, one of two opposite roots $\pm r_{ij}$ belongs to Q_k, say $r_{ij} \in Q_k$. Hence $r_{12} \in Q_3$, $r_{23} \in Q_1$, $r_{31} \in Q_2$. We have two cases:

(i) r_{ij} belongs, up to sign, to the triple t, i.e. $r_{ij} = r_k$, for all pairs ij;
(ii) r_{ij} does not belong to t, i.e. $r_{ij} \neq r_k$.

In case (i), the vectors r_i, $i = 1, 2, 3$, are linearly dependent, and the triple t spans a two-dimensional plane. We say that the triple t is of *rank* 2. Note that any two roots r, r' from distinct quadruples determine uniquely the third root $r'' = r - (r^{\mathrm{T}} r') r'$ and that there are 16 distinct triples of rank 2.

In case (ii), the roots r_i of the triple t are linearly independent. We say that the triple t has rank 3. In this case the roots r_{ij} for $ij = 12, 23, 31$, are distinct and do not coincide with the roots r_i, $i = 1, 2, 3$. Moreover, it is not difficult to verify that the triple (r_{12}, r_{23}, r_{31}) has rank 2.

Triples of rank 2 and 3 are realized in $\sqrt{2} P_V(\mathsf{D}_4)$ as follows. The roots of a triple t of rank 2 are parallel to edges of a 2-face of $\sqrt{2} P_V(\mathsf{D}_4)$. We say that a triple t of rank 2 forms a face of $\sqrt{2} P_V(\mathsf{D}_4)$. Since a triple of rank 2 forms a face of $\sqrt{2} P_V(\mathsf{D}_4)$, the 16 triples of rank 2 relate to the 16 zones of triangular faces of it.

Let $t' = (r'_1, r'_2, r'_3)$ be a triple of rank 3. The 6 vectors $\pm r'_i$, $i = 1, 2, 3$, have end-vertices in vertices of $\sqrt{2} P_V(\mathsf{D}_4)$. From each pair $\pm r'_i$ of opposite roots, we choose a vector r_i such that $r_i^{\mathrm{T}} r_j = 1$ for $ij = 12, 23, 31$. Then end-vertices of the roots r_i are vertices of a face F of $\sqrt{2} P_V(\mathsf{D}_4)$, and the triple $t = (r_{ij} = r_i - r_j : ij = 12, 23, 31)$ has rank 2 and forms the face F.

For what follows, we need subsets U of $D(4)$ which are maximal by inclusion such that $P_V(\mathsf{D}_4) + \sum_{q \in U} \lambda_q z(q)$ is a parallelohedron. Note that each quadruple Q_i is a basis of $\mathbb{R}^4$. Obviously, it is a unimodular set. It is a maximal by inclusion unimodular subset of $D(4)$, since any other vector of $D(4)$ has half-integer coordinates in this basis. However, it is proved in [22], that $P_V(\mathsf{D}_4) + \sum_{q \in U} \lambda_q z(q)$ is not a parallelohedron if $U \subset D(4)$ is a quadruple, and it is a parallelohedron for any other maximal unimodular subsets $U \subseteq D(4)$. Of course, a maximal unimodular set $U \neq Q_i$ does not contain each quadruple Q_i as a subset.

We show below that maximal unimodular subsets in $D(4)$ represent either the graphic matroid of the graph $\mathsf{K}_5 - \mathbf{1}$ or the cographic matroid of the graph K_{33} (see Fig. 2).

The graph $\mathsf{K}_5 - \mathbf{1}$ is planar. Hence, the graphic matroid of $\mathsf{K}_5 - \mathbf{1}$ is isomorphic to the cographic matroid of the graph $(\mathsf{K}_5 - \mathbf{1})^*$. Both graphs $(\mathsf{K}_5 - \mathbf{1})^*$ and K_{33} are cubic graphs on six vertices and nine edges. These graphs have a Hamiltonian 6-circuit C_6 on six vertices v_i, $1 \le i \le 6$. Call C_6 with the six edges (v_i, v_{i+1}) by a *rim*, and the other three edges by *spokes*. The spokes are the following edges: $e_1 = (v_1, v_4)$, $e_2 = (v_2, v_6)$, $e_3 = (v_3, v_5)$ in $(\mathsf{K}_5 - \mathbf{1})^*$, and $e_i = (v_i, v_{i+3})$, $i = 1, 2, 3$, in K_{33} (see Fig. 2). Note that the edges e_2 and e_3 do not intersect in $(\mathsf{K}_5 - \mathbf{1})^*$ and intersect in K_{33}. The described form of K_{33} is the graph Q_4 of [13].

Besides the planarity, the graph $(\mathsf{K}_5 - \mathbf{1})^*$ differs from K_{33} by the number of cuts of cardinality three. All cuts of cardinality 3 of K_{33} are the six one-vertex cuts, i.e. cuts containing three edges incident to a vertex. The graph $(\mathsf{K}_5 - \mathbf{1})^*$, besides the six one-vertex cuts, has a separating cut containing three non-adjacent edges one of which is a spoke. (These are the edges (v_2, v_3), (v_5, v_6) and the spoke (v_1, v_4) of the above description of $(\mathsf{K}_5 - \mathbf{1})^*$.)

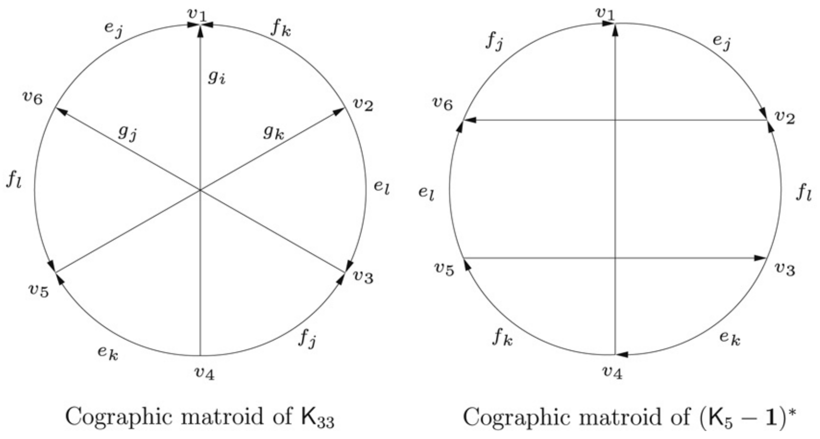

Fig. 2. Two cographic matroids.

Proposition 4. *A maximal by inclusion unimodular subset U of $D(4)$, $U \neq Q_i$, $i = 1, 2, 3$, is obtained by a deletion from $D(4)$ of a triple $t = (r_1, r_2, r_3)$ where $r_i \in Q_i$, $i = 1, 2, 3$. Then*

(i) *if t has rank 2 , then U represents the cographic matroid of the graph K_{33};*
(ii) *if t has rank 3 , then U represents the graphic matroid of the graph $\mathsf{K}_5 - \mathbf{1}$, which is isomorphic to the cographic matroid of the dual graph $(\mathsf{K}_5 - \mathbf{1})^*$.*

Proof. Since $Q_i \not\subseteq U$, for $i = 1, 2, 3$, U does not contain a triple t. We show that a deletion from $D(4)$ of any triple gives a unimodular set.

Denote by $V(r)$ the set of all triples of rank 2 containing a root $r \in D(4)$. If $r \in Q_i$ and $r' \in Q_j$, then $V(r) \bigcap V(r')$ is the unique triple of rank 2 containing r and r' if $i \neq j$, and $|V(r) \bigcap V(r')| = 0$ if $i = j$ and $r \neq r'$.

Let $r_i \in Q_i$, $i = 1, 2, 3$, be the roots of the deleted triple t. Then

$$|V(r_i)| = 4, \qquad \left|V(r_i) \bigcap V(r_j)\right| = 1, \quad \text{and} \quad \left|V(r_1) \bigcap V(r_2) \bigcap V(r_3)\right| = 0 \text{ or } 1.$$

Here, whether 0 or 1 stays in the last equality depends on the triple t has rank 3 or 2, respectively. By the inclusion-exclusion principle, we have

$$\left|\bigcup_{i=1}^{3} V(r_i)\right| = \sum_{i=1}^{3} |V(r_i)| - \sum_{1 \le i < j \le 3} \left|V(r_i) \bigcap V(r_j)\right| + \left|\bigcap_{i=1}^{3} V(r_i)\right|.$$

Hence, we have the following two cases.

(i) If $|\bigcap_{i=1}^3 V(r_i)| = 1$, i.e. if the triple t has rank 2, then $|\bigcup_{i=1}^3 V(r_i)| = 3 \cdot 4 - 3 \cdot 1 + 1 = 10$.
(ii) If $|\bigcap_{i=1}^3 V(r_i)| = 0$, i.e. if the triple t has rank 3, then $|\bigcup_{i=1}^3 V(r_i)| = 9$.

Let $V = \bigcup_{r \in D(4)} V(r) - \bigcup_{i=1}^3 V(r_i)$. Then V is the set of triples of rank 2 that do not contain the deleted roots r_1, r_2 and r_3. Using the set V we construct a cubic directed graph G as follows. We take $R_0 = D(4) - t$ to be the set of 9 arcs of G. As a set of vertices of G we take a subset $V_0 \subseteq V$ of six triples. An arc $r \in R_0$ is incident to a vertex $v \in V_0$ if the root r belongs to the triple v of rank 2. Hence, each vertex v is incident to three edges, and the graph G is cubic. Directions of arcs are chosen as follows. Let the roots r, s, p of a triple v satisfy the equation $r - (r^{\mathrm{T}}s)s - (r^{\mathrm{T}}p)p = 0$. Suppose that the arc r comes in v. Then the arc s goes out or comes in the vertex v if $r^{\mathrm{T}}s = 1$ or $r^{\mathrm{T}}s = -1$, respectively. The same assertion is true for the arc p.

We show that each root $r \in R_0$ belongs to exactly two triples of V_0. Let $r \in Q_k$, then $|V(r) \bigcap V(r_k)| = 0$ and $|V(r) \bigcap V(r_i)| = |V(r) \bigcap V(r_j)| = 1$, where $\{ijk\} = \{123\}$. We have two cases:

$$V(r)\bigcap V(r_i)=V(r)\bigcap V(r_j)\quad\text{or}\quad V(r)\bigcap V(r_i)\neq V(r)\bigcap V(r_j). \tag{4}$$

Suppose that the inequality in (4) holds for r. Since $|V(r)|=4$, only two triples from $V(r)$ belong to V.

Now, if the equality in (4) holds, then it implies that $r=(r^{\mathrm{T}}r_i)r_i+(r^{\mathrm{T}}r_j)r_j$. If t has rank 2, then, up to sign, this gives $r=r_k$, which contradicts to $r\in R_0$. Hence, if t has rank 2, the inequality of (4) holds for all $r\in R_0$. We can set $V_0=V$, since V contains $16-10=6$ triples. It is easy to verify that G is isomorphic to K_{33}.

If t has rank 3, then $r=r_{ij}$, where r_{ij} is defined in Eq. (3). The roots r_{ij}, $ij=12,23,31$, form a triple v_0 of rank 2. Obviously, $v_0\in V$. We set $V_0=V-\{v_0\}$. Since the root r_{ij} belongs to two triples v_0 and (r_{ij},r_i,r_j), the remaining triples of $V(r_{ij})$ belong to V_0. We saw that each root $r\in R_0$, $r\neq r_{ij}$ for $ij=12,23,31$, when the equality holds in (4) belongs to two triples of R_0. Hence, in the case when t has rank 3, the graph G is well defined. It is easy to verify that G is isomorphic to $(\mathsf{K}_5-\mathbf{1})^*$ and v_0 corresponds to a separating cut of cardinality 3.

Arcs of G are labeled naturally by roots from the set R_0. This labeling gives a representation of the cographic matroid of G by vectors of R_0.

Note that triples of rank 2 are equivalent under action of the automorphism group of $P_V(\mathsf{D}_4)$. Similarly, all triples of rank 3 are equivalent under the automorphism group of $P_V(\mathsf{D}_4)$ extended by changing signs of roots. Hence, any explicit representations of the cographic matroids of the graphs K_{33} and $(\mathsf{K}_5-\mathbf{1})^*$ for fixed triples of rank 2 and 3 prove that the above labeling gives representations for all pairs of triples of rank 2 and 3. □

6.4. *L-types in $\xi(HYP_5)$*

From Proposition 4 we deduce that each L-domain corresponding to D_4 is contained in $64=4^3$ different L-types. 48 of them are of type II and 16 are of type III.

Recall that $P_V(\mathsf{D}_4)$ is free along lines spanned by roots of the root system $\mathsf{D}_{4,2}$, vectors of which are parallel to diagonals of the cross-polytopes β_4 of the Delaunay partition of the lattice D_4. Note that bases related to the forms $a(\mathsf{D}_4)\in\xi(\mathrm{HYP}_5)$ contain a diagonal of a β_4.

Lemma 5. *Let the basis related to $a(\mathsf{D}_4)$ contains a diagonal $q\in\mathsf{D}_{4,2}$ of a cross-polytope β_4. Then the L-domain of the parallelohedron $P_q=P_V(\mathsf{D}_4)+z(q)$ related to this basis, i.e. the L-domain of the form $a(\mathsf{D}_4)+\lambda f_q$, does not belong to $\xi(\mathrm{HYP}_5)$.*

Proof. The parallelohedron P_q has a non-zero width along the line $l(q)$ parallel to q. The lattice L_q of P_q has a lamina H which is orthogonal to q. The lamina H separates the cross-polytope β_4 with a diagonal q into two Delaunay polytopes, each being a pyramid with a base β_3 orthogonal to q and lying in H. These two pyramids have the end-points of the diagonal q as apexes. Hence vertices of the basic simplex of $a(\mathsf{D}_4)+\lambda f_q$ belong to two distinct Delaunay polytopes. This implies that the L-domain of L_q does not belong to $\xi(\mathrm{HYP}_5)$. □

Proposition 5. *The cone $\xi(\mathrm{HYP}_5)$ contains 12 principal L-domains, 120 L-domains of type II and 40 L-domains of type III, total 172 L-domains.*

Proof. By Proposition 2, $\xi(\mathrm{HYP}_{4+1})$ contains $\frac{1}{2}4!=12$ principal L-domains.

The closure of an L-domain of types II and III is the convex hull of $a(\mathsf{D}_4)$ and a dicing facet $F(U)$ related to a unimodular subset $U\subset\mathsf{D}_{4,2}$ of 9 vectors that are free for $P_V(\mathsf{D}_4)$.

Each tile of the Delaunay tiling of the root lattice D_4 is the regular four-dimensional cross polytope β_4. Any affine base of β_4 contains exactly the two vertices w, w' of a diagonal and 3 vertices v_1,v_2,v_3 chosen from the other diagonals. We have $d_{\beta_4}(w,w')=4$ and 2 for all other pairs. There are $\binom{5}{2}=10$ ways to choose a pair $\{w,w'\}$ in a 5-elements set so there are exactly 10 rays $a_i(\mathsf{D}_4)$ representing D_4 in HYP_5.

Every L-domain $a_i(\mathsf{D}_4)$ is contained in the closure of 64 primitive L-domains $\mathcal{D}(U)$, but not all of them are included in $\xi(\mathrm{HYP}_5)$. Each L-domain $\mathcal{D}(U)$ has the form $\mathcal{D}(U)=\mathrm{conv}(a_i(\mathsf{D}_4)+F(U))$, where the subset $U\subset\mathsf{D}_{4,2}$ is obtained by a deletion of a triple from $\mathsf{D}_{4,2}$. By Lemma 5, the inclusion

$\mathcal{D}(U) \subset \xi(\mathrm{HYP}_5)$ implies U does not contain the diagonal q which is a basic vector of $a_i(\mathsf{D}_4)$. Let $q = r_1 \in Q_1$, then we have 4 choices for the root r_2 in Q_2. For the third root either we have a rank 2 triple and r_3 is completely determined or we have a rank 3 triple in which case there are 3 choices. This means that a ray $a_i(\mathsf{D}_4)$ is contained in 4 L-domains of type II included in $\xi(\mathrm{HYP}_5)$ and 12 L-domains of type III included in $\xi(\mathrm{HYP}_5)$. This means that there are 120, respectively 40 L-domains of type II, respectively III in $\xi(\mathrm{HYP}_5)$. $\square$

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