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Mixed interval hypergraphs

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Abstract

We investigate the coloring properties of mixed interval hypergraphs having two families of subsets: the edges and the co-edges. In every edge at least two vertices have different colors. The notion of a co-edge was introduced recently in Voloshin (1993, 1995): in every such a subset at least two vertices have the same color. The upper (lower) chromatic number is defined as a maximum (minimum) number of colors for which there exists a coloring of a mixed hypergraph using all the colors.

We find that for colorable mixed interval hypergraph H the lower chromatic number $\chi(H) \leqslant 2$, the upper chromatic number $\bar{\chi}(H) = |X| - s(H)$, where $s(H)$ is introduced as the so-called sieve number. A characterization of uncolorability of a mixed interval hypergraph is found, namely: such a hypergraph is uncolorable if and only if it contains an obviously uncolorable edge.

The co-stability number $\alpha_{\mathscr{A}}(H)$ is the maximum cardinality of a subset of vertices which contains no co-edge. A mixed hypergraph H is called co-perfect if $\bar{\chi}(H') = \alpha_{\mathscr{A}}(H')$ for every subhypergraph H'. Such minimal non-co-perfect hypergraphs as monostars and cycloids C^r_{2r-1} have been found in Voloshin (1995). A new class of non-co-perfect mixed hypergraphs called covered co-bi-stars is found in this paper. It is shown that mixed interval hypergraphs are co-perfect if and only if they do not contain co-monostars and covered co-bi-stars as subhypergraphs.

Linear time algorithms for computing lower and upper chromatic numbers and respective colorings for this class of hypergraphs are suggested.

1. Basic notions

The following problem was described in [4]:

"Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of sources of power supply such that the action time of any source is one quantum of time and all sources acting for any given quantum of time switch on and switch off synchronously.

Consider the following general constraints on their common work:

(1) let $\mathscr{A} = \{A_1, A_2, \dots, A_k\}$, $A_i \subseteq X$, $i = 1, \dots, k$, $k \geqslant 1$, be a family of subsets of X such that at least two sources from every A_i act for the same quantum of time;

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(2) let $\mathscr{E} = \{E_1, E_2, \ldots, E_m\}$, $E_j \subseteq X$, $j = 1, \ldots, m$, $m \geqslant 1$, be a family of subsets of X such that at least two sources from every E_j act for different quanta of time.

Call the set X with such constraints a *system* and denote it by $H = (X, \mathscr{A} \cup \mathscr{E})$. Suppose that system H is active ("working","alive") during any quantum of time if at least one source is active for this time.

We consider the following problem: *how can we schedule the system H in such a way that the time of working (which may be understood also as the lifetime of the whole system) is longest (shortest)*"?

As it was shown in [4] this problem contains the coloring problem in partial case and leads to the notion of the upper chromatic number of a hypergraph. In this paper we give a complete and effective solution of the problem provided that there exists a linear ordering of the set X such that the given subsets induce intervals in this ordering.

Throughout this paper we use the terminology of [4]. The notions not explained here were taken from [1]. Let $X = \{x_1, x_2, \ldots, x_n\}$, $n \geqslant 1$, be a finite set, $\boldsymbol{S} = \{S_1, S_2, \ldots, S_p\}$, $p \geqslant 1$, be a family of subsets of X. The couple $H = (X, \boldsymbol{S})$ is called a hypergraph with the vertex set X and a family of subsets $\boldsymbol{S}$ if $\bigcup_{i=1}^{p} S_i \subseteq X$ (cf. [1]). For any subset $Y \subseteq X$ we call the hypergraph $H/Y = (Y, \boldsymbol{S}')$ the *subhypergraph* of a hypergraph H if $\boldsymbol{S}'$ consists of those subsets of $\boldsymbol{S}$ that are completely contained in Y.

In this paper we consider the hypergraphs $H = (X, \boldsymbol{S}), |X| = n$ such that $\boldsymbol{S} = \mathscr{A} \cup \mathscr{E}$, where $\mathscr{A}$ and $\mathscr{E}$ are two subfamilies of $\boldsymbol{S}$. If $\mathscr{A} \neq \emptyset$ and $\mathscr{E} \neq \emptyset$, then arrange that

$$\mathscr{A} = \{A_1, \ldots, A_k\}, \qquad I = \{1, \ldots, k\}, \qquad \mathscr{E} = \{E_1, \ldots, E_m\}, \qquad J = \{1, \ldots, m\}.$$

We call every E_j, $j \in J$, an *edge*, and every A_i, $i \in I$, a *co-edge*. In special cases if $\mathscr{A} = \emptyset$ then $H = (X, \mathscr{E}) = H_{\mathscr{E}}$ will be called simply a *hypergraph*, if $\mathscr{E} = \emptyset$, then $H = (X, \mathscr{A}) = H_{\mathscr{A}}$ will be called *co-hypergraph*. In general case, if $\mathscr{A} \neq \emptyset$ and/or $\mathscr{E} \neq \emptyset$ then $H = (X, \mathscr{A} \cup \mathscr{E})$ will be called a *mixed hypergraph*. Let us have $t \geqslant 1$ colors.

A strict coloring of a mixed hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$ with t colors is a mapping $c : X \to \{1, 2, \ldots, t\}$ such that the following four conditions hold:

(1) any co-edge $A_i, |A_i| \geqslant 2$, $i \in I$, has at least two vertices of the same color;

(2) any edge $E_j, |E_j| \geqslant 2$, $j \in J$, has at least two vertices colored differently;

(3) the number of used colors is exactly t;

(4) all the vertices are colored.

So, strict colorings exist only for such t, that $1 \leqslant t \leqslant n$. Two strict colorings are called different if there exist such two vertices that have the same color for one coloring and different colors for the other. The maximum (minimum) t for which there exists a strict coloring of a mixed hypergraph H with t colors is called the upper (lower) chromatic number of H and is denoted by $\bar{\chi}(H)(\chi(H))$.

In the problem above, if we denote the sources by vertices of a hypergraph and the given constraints by edges and co-edges, then in any hypergraph coloring every monochromatic subset of vertices represents a set of sources that may be switched on synchronously. Therefore, the initial scheduling problem is equivalent to the problem of finding the lower and upper chromatic numbers and corresponding colorings of a mixed hypergraph.

Let $r_t(H)$ be the number of strict colorings of a mixed hypergraph H with $t \geqslant 1$ colors. For each such hypergraph we associate the vector $R(H) = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$ and call it the chromatic spectrum of H. Hence $R(H) = (0, \ldots, 0, r_{\chi}, \ldots, r_{\bar{\chi}}, 0, \ldots, 0)$.

We will need the following three rules from the splitting-contraction method developed in [3, 4]:

(1) if $|E_i| = 1$, then $R(H) = R(H - E_i)$; if $|E_i| \geqslant 2$ and if $E_i \subseteq E_j$, then $R(H) = R(H - E_j)$, $i, j \in J$ (*clearing*);

(2) if $|A_i| = 1$, then $R(H) = R(H - A_i)$; if $|A_i| \geqslant 2$ and if $A_i \subseteq A_j$, then $R(H) = R(H - A_j)$, $i, j \in I$ (*co-clearing*);

(3) if $A_q = \{x_k, x_l\}$, for some $q \in I$ and $x_k, x_l \in X$, such that $A_q \neq E_s$ for any $s \in J$, then $R(H) = R(H_1)$ where $H_1 = (X_1, \mathscr{A}^1 \cup \mathscr{E}^1)$, $X_1 = (X \setminus \{x_k, x_l\}) \cup \{y\}$, y is a new vertex; if $x_k \in E_j$, or $x_l \in E_j$, $j \in J$, then $E_j^1 = (E_j \setminus \{x_k, x_l\}) \cup \{y\}$, otherwise $E_j^1 = E_j$; if $x_k \in A_i$, or $x_l \in A_i$, $i \in I$, then $A_i^1 = (A_i \setminus \{x_k, x_l\}) \cup \{y\}$, otherwise $A_i^1 = A_i$ (*contraction*).

If for a mixed hypergraph H there exists at least one strict coloring, then it is called colorable. Otherwise H is called uncolorable. For uncolorable mixed hypergraph we suppose that $\chi(H) = \bar{\chi}(H) = 0$.

Note, that if $\mathscr{A} = \emptyset$, then $\bar{\chi}(H) = n$ and we have usual colorings of a hypergraph. If $\mathscr{E} = \emptyset$, then $\chi(H) = 1$.

For each $n \geqslant 2$ one may easily construct some uncolorable hypergraph with $\mathscr{A} \neq \emptyset$ and $\mathscr{E} \neq \emptyset$, for which $\chi(H) = \bar{\chi}(H) = 0$, and this is possible only for mixed hypergraphs.

A set $T \subseteq X$ is called a bi-transversal of a hypergraph $H = (X, \mathscr{E})$ if $|T \cap E_j| \geqslant 2$ for every $j \in J$. The minimum cardinality of a bi-transversal is denoted by $\tau_2(H)$. If $\mathscr{E} = \emptyset$ or $\mathscr{E}$ contains at least one element of cardinality 1, then we put $\tau_2(H) = 0$. Bi-transversal of a co-hypergraph $H_{\mathscr{A}}$ is called *co-bi-transversal*. By $\tau(H)$ we denote the transversal hypergraph number [1].

A mixed hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$, $\mathscr{A} \neq \emptyset$, is called a co-monostar if the following conditions hold:

(1) $\tau(H_{\mathscr{A}}) = 1$;

(2) $\tau_2(H_{\mathscr{A}}) \geqslant 3$.

A mixed hypergraph is called a co-bi-star if there exists a co-bi-transversal $\{x, y\}$ of cardinality 2, where $\{x, y\}$ is not an edge.

Definition 1.1. A mixed hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$ is called a mixed interval hypergraph, if there exists a linear ordering of the vertex set X such that each edge E_j, $j \in J$ represents an interval, and each co-edge A_i, $i \in I$, represents an interval in this ordering.

The mixed interval hypergraph is illustrated by Fig. 1 where edges are pictured as usually (by circle if cardinality is 1, by segments of line if cardinality is 2 and by

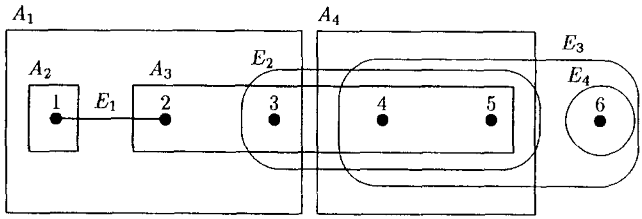

Fig. 1. $H = (X, \mathscr{A} \cup \mathscr{E})$ represents a mixed interval hypergraph, where $X = \{1,2,3,4,5,6\}$ is a lineary ordered set of vertices, $\mathscr{A} = \{A_1, A_2, A_3, A_4\}$; $\mathscr{E} = \{E_1, E_2, E_3, E_4\}$.

ovals if cardinality is $\geqslant 3$). Co-edges are represented by rectangles. Such rules are valid throughout the paper.

2. Coloring properties of mixed interval hypergraphs

Definition 2.1. In a mixed hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$ an edge E_j, $j \in J, |E_j| \geqslant 2$, is called uncolorable if for any pair of vertices $x, y \in E_j$ there exists a sequence $(xA_1z_1A_2z_2 \cdots A_{l-1}z_{l-1}A_ly)$ such that:

(1) $z_1, z_2, \ldots, z_{l-1} \in E_j$,

(2) $A_i \in \mathscr{A}$, $i = 1, \ldots, l$,

(3) $A_1 = \{x, z_1\}, A_2 = \{z_1, z_2\}, \ldots, A_l = \{z_{l-1}, y\}$.

The example of uncolorable edge is shown by the Fig. 2.

Theorem 2.2. *A mixed interval hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$ is colorable if and only if it does not contain uncolorable edges. In this case we have $\chi(H) \leqslant 2$.*

Proof. ($\Rightarrow$) Obvious.

$\Leftarrow$ Let $H = (X, \mathscr{A} \cup \mathscr{E})$ be a mixed interval hypergraph without uncolorable edges and $X = \{x_1, \ldots, x_n\}$ be such an ordering of its vertices on the real line that every A_i, $i \in I$, is an interval and every E_j, $j \in J$ is an interval in this ordering.

Color the vertices $x_1, x_2, \ldots, x_n$ in this order alternatively using the colors $1, 2, 1, 2, \ldots$ until we encounter a co-edge of cardinality 2. At the second vertex of such a co-edge we reverse the order of the coloring. Since H is without uncolorable edges, such procedure gives a coloring of H. □

Remark. Note that in general case a mixed hypergraph without uncolorable edges is not necessarily colorable. Consider as example $H = (X, \mathscr{A} \cup \mathscr{E})$, $X = \{1,2,3,4\}$, $\mathscr{A} = \{(1,2,3); (1,2,4)\}$; $\mathscr{E} = \{(1,2); (2,3); (3,4); (2,4)\}$.

Remark. Theorem 2.2 leads to a linear time algorithm for computing the lower chromatic number of a mixed interval hypergraph.

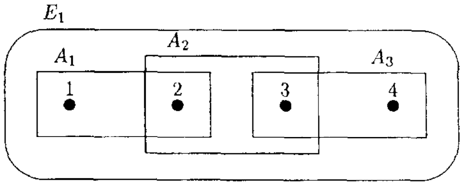

Fig. 2. Example of uncolorable edge E_1 in a mixed interval hypergraph.

Definition 2.3. A mixed hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$ is called reduced if

$$|A_i| \geqslant 3, \quad i \in I, \qquad |E_j| \geqslant 2, \quad j \in J,$$

and no edge (co-edge) is included in another edge (co-edge).

Such procedures as clearing, co-clearing and contraction being applied as much as possible to an arbitrary colorable mixed hypergraph H, lead to the reduced mixed hypergraph H' with the same chromatic spectrum [4]. This may be done in polynomial time. Therefore, let us consider further in this paper only reduced mixed hypergraphs (if the contrary is not indicated).

Definition 2.4. A mixed hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$ is called a covered co-bi-star if

$$|\mathscr{A}| \geqslant 2, \qquad \bigcup_{i \in I} A_i = X, \qquad \left| \bigcap_{i \in I} A_i \right| \geqslant 2,$$

and every pair $\{x, y\} \subseteq \bigcap_{i \in I} A_i$ is an edge of cardinality 2.

The example of a covered co-bistar is shown by Fig. 3.

Theorem 2.5. *If a mixed interval hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$ represents a covered co-bi-star, then $\bar{\chi}(H) = n - 2$.*

For a mixed hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$ a set $P \subseteq X$ is called co-stable if it does not contain any co-edge A_i, $i \in I$. The co-stability number $\alpha_{\mathscr{A}}(H)$ is the maximum cardinality of a co-stable set of H.

Theorem 2.6 (Voloshin [4]). *For any mixed hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$*

$$\bar{\chi}(H) \leqslant \alpha_{\mathscr{A}}(H).$$

A mixed hypergraph H is called co-perfect hypergraph if for every subhypergraph H' the following equality holds:

$$\bar{\chi}(H') = \alpha_{\mathscr{A}}(H').$$

As it was shown in [4] any co-bi-star is a co-perfect and any co-monostar is not a co-perfect mixed hypergraph.

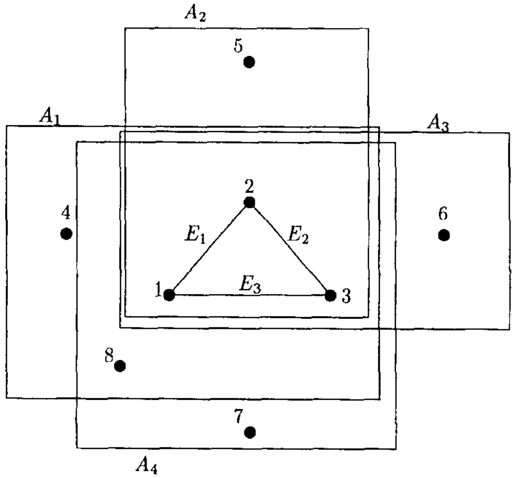

Fig. 3. Mixed hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$ represents a covered co-bistar.

Theorem 2.7. *Every covered co-bi-star* $H = (X, \mathscr{A} \cup \mathscr{E})$ *is not a co-perfect mixed hypergraph.*

Proof. It is clear that for every covered co-bi-star H $\alpha_{\mathscr{A}}(H) = n - 1$.

Let us suppose that $\bar{\chi}(H) = n - 1$ and consider some strict coloring of H with $n - 1$ colors. Hence, there are only two vertices that have the same color. Therefore they belong to all co-edges. On the other hand, the intersection of all co-edges induces a usual complete graph in H and must be colored overall differently. Consequently, $\bar{\chi}(H) \neq n - 1$, and the theorem follows. □

Theorem 2.8. *A mixed interval hypergraph is co-perfect if and only if it does not contain co-monostars and covered co-bi-stars as subhypergraphs.*

Proof. (⇒) It is evident because co-monostars [4] and covered co-bi-stars (Theorem 2.7) are not co-perfect.

⇐ Let $H = (X, \mathscr{A} \cup \mathscr{E})$ be a mixed interval hypergraph without co-monostars and covered co-bi-stars. It is clear that for any $Y \subseteq X$ the subhypergraph

$$H/Y = (Y, \mathscr{A}_1 \cup \mathscr{E}_1)$$

is also a mixed interval hypergraph. So, it is sufficient to prove only that

$$\bar{\chi}(H) = \alpha_{\mathscr{A}}(H).$$

Consider $H_{\mathscr{A}} = (X, \mathscr{A})$ that is an interval co-hypergraph. Since $H_{\mathscr{A}}$ does not contain co-monostars, we have that

$$\bigcap_{i \in I_1} A_i \neq \emptyset$$

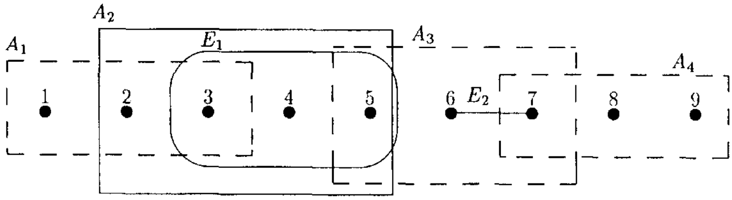

Fig. 4. Set of co-edges $\{A_1, A_3, A_4\}$ forms a sieve in a mixed interval hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$.

implies

$$\left|\bigcap_{i\in I_1} A_i\right| \geqslant 2$$

for any $I_1 \subseteq I$.

Consider the line graph [1] of $H_{\mathscr{A}}$, that is the graph $G = (\mathscr{A}, V)$, where $(A_i, A_j) \in V$ if and only if $A_i \cap A_j \neq \emptyset$, $i, j \in I$. It is easy to see that in these conditions every clique of graph G induces a co-bi-transversal for the respective set of co-edges in $H_{\mathscr{A}}$. Therefore minimum covering of graph G by cliques corresponds to the minimum co-bi-transversal of H. Let $C_1, C_2, \ldots, C_s$ be the cliques of such minimum covering of G. Consequently, $\alpha_{\mathscr{A}}(H_{\mathscr{A}}) = \alpha_{\mathscr{A}}(H) = n - s$.

Since H does not contain covered co-bi-stars, for any clique $C_i, i = 1, 2, \ldots, s$ there exists a co-bi-transversal, say $\{x_i, y_i\}$, that is not an edge of cardinality 2. All these co-bi-transversals are different because $C_i \neq C_j$ implies that there exist $A_k \in C_i$, and $A_l \in C_j$, with $A_k \cap A_l = \emptyset$. So, we can color the vertices x_1, y_1 with the first color, x_2, y_2 with the second color, ... $, x_s, y_s$ with the s-th color. All the remaining vertices color overall differently with the colors $s+1, s+2, \ldots, n-s$.

Thus we obtain the coloring of $H_{\mathscr{A}}$ with $n-s$ colors, and consequently, the coloring of H with $\alpha_{\mathscr{A}}(H)$ colors, hence the theorem follows. □

Definition 2.9 (*Voloshin* [5]). In a mixed hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$ the set of indices $I_1 \subset I$ and a respective subfamily $\{A_i\}$, $i \in I_1$ of co-edges is called a sieve, if for any $x, y \in X$ and any $j, k \in I_1$ the following implication holds:

$$(x, y) \in A_j \cap A_k \;\Rightarrow\; (x, y) = E_l \in \mathscr{E}, \quad \textit{for some } l \in J.$$

The maximum cardinality of a sieve of a hypergraph H is called the sieve-number of H and is denoted by $s(H)$.

An example of a sieve is illustrated by the Fig. 4.

So, any co-edge, any co-matching are examples of sieves in mixed hypergraphs.

Theorem 2.10 (Voloshin [5]). *If $H = (X, \mathscr{A} \cup \mathscr{E})$ is a mixed interval hypergraph, then*

$$\bar{\chi}(H) = |X| - s(H).$$

Proof. Let the set X be ordered from left to right in order $x_1 < x_2 < \cdots < x_n$. First we show that $\bar{\chi}(H) \leqslant |X| - s(H)$. Let us suppose that $\bar{\chi}(H) \geqslant |X| - s(H) + 1$ and consider some strict coloring of H with $\bar{\chi}(H)$ colors. Let $\mathscr{A}_1 \subset \mathscr{A}$ be a sieve of a maximum cardinality, $|\mathscr{A}_1| = s(H) = s$, and, moreover, the co-edges from $\mathscr{A}_1$ are ordered according to their left extremity from left to right: $\mathscr{A}_1 = \{A_1, A_2, \ldots, A_s\}$. Since H is reduced, this ordering is uniquely determined. Let $x_i, y_i \in A_i$, $x_i < y_i$, $i = 1, \ldots, s$ be the vertices colored with the same color c_i in each $A_i \in \mathscr{A}_1$. We consider the following two possible cases.

Case a: No two consecutive colors c_i, $i = 1, \ldots, s$ coincide. There can be repeated non-consecutive colors. However, if even all c_i, $i = 1, \ldots, s$ are different, the total number of used colors is at most

$$s + |X| - 2s = |X| - s,$$

which contradicts the assumption.

Case b: There exists at least one monochromatic sequence $c_l = c_{l+1} = \cdots = c_{l+t}$ for some $1 \leqslant l \leqslant s - t$, $t \geqslant 1$.

Since $\mathscr{A}_1$ is a sieve, $|\{x_i, y_i\} \cap \{x_{i+1}, y_{i+1}\}| \leqslant 1$, $i = 1, \ldots, s - 1$, and therefore $x_l < y_l \leqslant x_{l+1} < y_{l+1} \leqslant \cdots \leqslant x_{l+t} < y_{l+t}$.

If even $y_l = x_{l+1}$, $y_{l+1} = x_{l+2}, \ldots, y_{l+t-1} = x_{l+t}$, and thus $t + 2$ vertices are colored with the same color, and if even the remaining $s - t - 1$ colors are all different, then the total number of used colors is

$$1 + (s - t - 1) + (|X| - t - 2 - 2(s - t - 1)) = |X| - s.$$

Hence, we again have a contradiction. Therefore, $\bar{\chi}(H) \leqslant |X| - s(H)$.

We show now that we can color H with $|X| - s$ colors, i.e. $\bar{\chi}(H) \geqslant |X| - s(H)$, by induction on $s(H)$. Let $s(H) = 1$. Then H does not contain co-monostars and covered co-bi-stars. It implies that H is a co-perfect mixed hypergraph (Theorem 2.8), and, moreover, $\alpha_{\mathscr{A}}(H) = |X| - 1$. Hence we have

$$\bar{\chi}(H) = \alpha_{\mathscr{A}}(H) = |X| - 1 = |X| - s.$$

Suppose that our assertion $\bar{\chi}(H) \geqslant |X| - s$ is true for any mixed interval hypergraph H' with $s(H') < s(H)$.

Let $\mathscr{A}_1 = \{A_1, \ldots, A_s\}$, $s = s(H)$ be again a maximum and ordered from left to right sieve of a mixed interval hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$, and $A_1 = (x_1, x_2, \ldots, x_k)$, $k \geqslant 3$ be the first co-edge of this sieve.

If A_1 is not the very left co-edge of H, then we can construct another maximum sieve of H replacing A_1 in $\mathscr{A}_1$ by the very left co-edge of H. Therefore let us assume without loss of generality that A_1 is the very left co-edge of H. Consider the following possible cases.

Case 1: $(x_{k-1}, x_k) \notin \mathscr{E}$. Color $x_1, x_2, \ldots, x_{k-2}$ respectively with the colors $1, 2, \ldots, k-2$ and color x_{k-1}, x_k with the color $k - 1$. If $x_{k-1} \in A_j$ for some $A_j \in \mathscr{A}, A_j \neq A_1$, then $x_k \in A_j$ because no co-edge is included in another one. Therefore any such co-edge A_j is colored correctly.

Consider the subhypergraph

$$H_1 = H/\{x_k, x_{k+1}, \ldots, x_n\} = (X_1, \mathscr{A}^1 \cup \mathscr{E}^1)$$

with

$$s(H_1) = s(H) - 1, \quad |X_1| = |X| - k + 1.$$

By virtue of the induction hypothesis $\bar{\chi}(H_1) = |X_1| - s(H_1)$. Color H_1 with $\bar{\chi}(H_1)$ colors using the colors $k-1, k, \ldots, k + \bar{\chi}(H_1) - 2$ in such a way that the vertex x_k is colored with the color $k-1$. We obtain a coloring of the initial hypergraph H, and

$$\begin{aligned}\bar{\chi}(H) \geqslant k - 1 + \bar{\chi}(H_1) - 1 &= |X_1| - s(H_1) + k - 2 \\ &= |X| - k + 1 - s(H) + 1 + k - 2 = |X| - s(H).\end{aligned}$$

Case 2: $(x_{k-1}, x_k) \in \mathscr{E}$. Color $x_1, x_2, \ldots, x_{k-1}$ with the colors $1, 2, \ldots, k-1$ respectively, and color x_k with the color $k-2$. If $x_{k-2} \in A_j$ for some $A_j \in \mathscr{A}, A_j \neq A_1$, then $x_k \in A_j$ because no co-edge is included in another one. Hence any such co-edge A_j is colored correctly. Consider the subhypergraph

$$H_2 = H/\{x_{k-1}, x_k, \ldots, x_n\} = (X_2, \mathscr{A}^2 \cup \mathscr{E}^2)$$

with

$$s(H_2) = s(H) - 1, \quad |X_2| = |X| - k + 2.$$

By the induction hypothesis $\bar{\chi}(H_2) = |X_2| - s(H_2)$. Color H_2 with $\bar{\chi}(H_2)$ colors using the colors $k-2, k-1, \ldots, k + \bar{\chi}(H_2) - 3$ in such a way that the vertices x_{k-1} and x_k are colored with the colors $k-1$ and $k-2$, respectively. We again obtain a coloring of the initial hypergraph H and thus

$$\begin{aligned}\bar{\chi}(H) \geqslant k - 2 + \bar{\chi}(H_2) - 1 &= |X_2| - s(H_2) + k - 3 \\ &= |X| - k + 2 - s(H) + 1 + k - 3 = |X| - s(H). \qquad \square\end{aligned}$$

Corollary 2.11. *For any colorable mixed interval hypergraph* $H = (X, \mathscr{A} \cup \mathscr{E})$, $\mathscr{E} \neq \emptyset$, *the following statements are equivalent*:

(1) $\bar{\chi}(H) = \chi(H)$;

(2) $s(H) = |X| - 2$.

The mixed hypergraph in Fig. 5 is an example of a mixed interval hypergraph with $s(H) = |X| - 2$.

Corollary 2.12. *For any interval co-hypergraph with* $|A_i| \geqslant 2$, $i \in I$, *the following statements are equivalent*:

(1) $\chi(H) = \bar{\chi}(H) = 1$;

(2) $s(H) = |X| - 1$;

(3) *H is a co-path.*

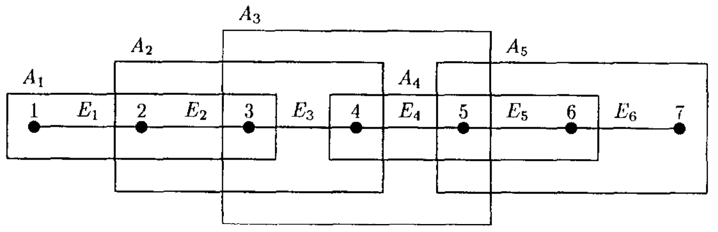

Fig. 5.

Corollary 2.13. *For any colorable mixed interval hypergraph* $H = (X, \mathscr{A} \cup \mathscr{E})$

$$1 = \chi(H_{\mathscr{A}}) \leqslant \chi(H_{\mathscr{E}}) = \chi(H) \leqslant \bar{\chi}(H) = |X| - s(H) \leqslant \bar{\chi}(H_{\mathscr{A}}) \leqslant \bar{\chi}(H_{\mathscr{E}}) = n.$$

Corollary 2.14. *If* $H = (X, \mathscr{A})$ *is a co-perfect interval co-hypergraph,* $v(H)$ *and* $\tau(H)$ *are the maximum cardinality of a matching and the minimum cardinality of a transversal* [1], *respectively, then*

$$\bar{\chi}(H) = |X| - s(H) = |X| - v(H) = |X| - \tau(H) = \alpha_{\mathscr{A}}(H).$$

Proof. It follows from the fact that in this case any maximum sieve is a matching (H is without monostars), and for interval hypergraph $\tau(H) = v(H)$ [1]. □

Corollary 2.15. *For any mixed interval hypergraph* $H = (X, \mathscr{A} \cup \mathscr{E})$ *the following statements are equivalent*:

(1) *H is co-perfect*;

(2) *H does not contain co-monostars and covered co-bi-stars*;

(3) *any maximum sieve in every subhypergraph of H is a co-matching.*

Proof. It follows immediately from Theorems 2.8 and 2.10. □

Remember that we consider reduced mixed hypergraphs.

Definition 2.16. A mixed interval hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$ is called complete, if in linear ordering of X any two consecutive vertices form an edge and any three consecutive vertices form a co-edge.

Hence for complete mixed interval hypergraph $|\mathscr{A}| = n - 2$, $|\mathscr{E}| = n - 1$. Fig. 5 shows an example of such a hypergraph.

Theorem 2.17. *For any mixed interval hypergraph* $H = (X, \mathscr{A} \cup \mathscr{E})$ *with* $|X| \geqslant 2$ *the following statements are equivalent*:

(1) *H is complete*;

(2) $R(H) = (0, 1, 0, \ldots, 0)$.

Proof. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (1). Consider the unique strict coloring of H with two colors, say, 1 and 2. If $(x_l, x_{l+1}) \notin \mathscr{E}$ for some l, $1 \leqslant l \leqslant n-1$, then reverse the color at any vertex x_t, $t \geqslant l+1$. Since $|A_i| \geqslant 3$, $i \in I$, we obtain one more strict coloring of H, which contradicts the assumption.

If $(x_l, x_{l+1}, x_{l+2}) \notin \mathscr{A}$ for some l, $1 \leqslant l \leqslant n-2$, then consider all the vertices x_t, $t \geqslant l+2$ having the same color as x_{l+2}. Re-color these vertices with the color 3. Hence, we again obtain one more strict coloring of H. $\square$

In contrast with $\chi, \bar{\chi}$, the problem to find the chromatic spectrum $R(H)$ for general mixed interval hypergraphs is hard. The only known method is the splitting-contraction algorithm described in [4].

A new approach may be developed using probabilistic methods of coloring. In this case parameters such as the middle chromatic number $\chi_{\rm m} = (\chi + \bar{\chi})/2$ and the breadth of chromatic spectrum $b(H) = \bar{\chi} - \chi + 1$, as introduced in [4], are important. We can give here their exact values:

Corollary 2.18. (1) *For any mixed interval hypergraph*

$$\chi_{\rm m}(H) = \tfrac{1}{2}(|X| - s(H)) + 1, \qquad b(H) = |X| - s(H) - 1.$$

(2) *For any interval co-hypergraph*

$$\chi_{\rm m}(H) = \tfrac{1}{2}(|X| - s(H) + 1), \qquad b(H) = |X| - s(H).$$

(3) *For any interval hypergraph*

$$\chi_{\rm m}(H) = \frac{|X| + 2}{2}, \qquad b(H) = |X| - 1.$$

We now describe an algorithm for finding the upper chromatic number, sieve and optimal coloring for a mixed interval hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$.

As it was shown in [2], for a given hypergraph H one can decide in time proportional to vertex number plus sum of all edges cardinalities whether or not H is an interval hypergraph. We suppose here that the recognition problem is solved and we know the linear ordering of vertices of the mixed interval hypergraph.

So, let $H = (X, \mathscr{A} \cup \mathscr{E})$ be an arbitrary (generally not reduced) mixed interval hypergraph, and $X = \{x_1, x_2, \ldots, x_n\}$ be the linear ordering of its vertex set in such a way that every A_i, $i \in I$, and every E_j, $j \in J$ forms an interval.

Algorithm CHI_UPPER

Input: A mixed interval hypergraph $H = (X, \mathscr{A} \cup \mathscr{E})$, and a linear ordering $X = \{x_1, x_2, \ldots, x_n\}$;

Output: $\bar{\chi}(H)$, optimal coloring, list *SIEVE* of co-edges forming maximum sieve.

1. Do all clearings and co-clearings in $H = (X, \mathscr{A} \cup \mathscr{E})$.

2. Search for an uncolorable edge. If such an edge exists, then output "H is uncolorable, $\bar{\chi} = 0$" and STOP.

3. Find a linear ordering of $\mathscr{A}$ from left to right.

4. Choose the first co-edge A of H and include A in *SIEVE*.

5. If the two last vertices of A constitute an edge of $\mathscr{E}$ then go to step 7. Color all the uncolored vertices before A and all vertices of A, except the last one, with the new different colors. Color the last vertex of A with the color of a preceding vertex.

6. Search for the first co-edge A' beginning not before the last vertex of A. If successful, then include A' in *SIEVE* and go to step 5 with $A = A'$. Otherwise, color all uncolored vertices with different new colors and go to step 9.

7. Color all uncolored vertices before A and all the vertices of A, except the last one, with the new different colors. Color the last vertex of A with the color of a vertex which is preceding to the last but one.

8. Search for the first co-edge A'' beginning not before the last but one vertex of A. If successful, then include A'' in SIEVE and go to step 5 with $A = A''$. Otherwise, color all uncolored vertices with different new colors and go to step 9.

9. Output $\bar{\chi}(H)$, coloring, *SIEVE*. End.

Theorem 2.19. *The algorithm CHI_UPPER correctly finds the upper chromatic number and respective coloring of any mixed interval hypergraph in linear time.*

The algorithm CHI_UPPER correctly finds the maximum sieve of any reduced mixed interval hypergraph in linear time.

Proof. Correctness of algorithm is based on Theorem 2.10. If we use the linked incidence lists as data structures, then clearing, co-clearing, search for the uncolorable edge, linear ordering of $\mathscr{A}$ (steps 1–3) may be implemented in linear time on size of a hypergraph expressed by vertex number plus sum of all edges cardinalities.

Since the steps 4–9 require less time than hypergraph size, we have that the whole algorithm may also be implemented in linear time. □

Remark. The problem to find a maximum sieve for an arbitrary mixed hypergraph is hard. In partial case, when H is a co-hypergraph without monostars, the problem is equivalent to find a maximum matching [1].

We conclude this paper by the following

Open Problem: Characterize all mixed hypergraphs with

$$\bar{\chi}(H) = |X| - s(H).$$

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