Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 3056–3060
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

# The last subconstituent of the Hemmeter graph

John S. Caughman, IV[a], Elizabeth J. Hart[b], Jianmin Ma[c]

[a]*Department of Mathematics, Portland State University, P.O. Box 751, Portland, OR 97207, USA*
[b]*Department of Mathematics, Penn State University, University Park, State College, PA 16802, USA*
[c]*Department of Mathematics, Oxford College of Emory University, Oxford, GA 30054, USA*

Received 15 June 2006; accepted 9 August 2007
Available online 1 October 2007

**Abstract**

We prove that when $q$ is any odd prime power, the distance-2 graph on the set of vertices at maximal distance $D$ from any fixed vertex of the Hemmeter graph $\mathrm{Hem}_D(q)$ is isomorphic to the graph $\mathrm{Quad}_{D-1}(q)$ of quadratic forms on $\mathbf{F}_q^{D-1}$.


*MSC:* 05E30

*Keywords:* Distance-regular graph; Quadratic form; Hemmeter graph

## 1. Introduction

In this paper we prove that when $q$ is any odd prime power, the distance-2 graph on the set of vertices at maximal distance $D$ from any fixed vertex of the Hemmeter graph $\mathrm{Hem}_D(q)$ is isomorphic to the graph $\mathrm{Quad}_{D-1}(q)$ of quadratic forms on $\mathbf{F}_q^{D-1}$. When $D$ is even, this result follows from a consideration of the halved graph, as described in [2, p. 287]. Our proof establishes the result for arbitrary $D$. For more about these graphs and our proof techniques, see [6–9].

The paper is organized as follows. In Section 2, we fix notation and recall some basic properties of distance-regular graphs. In Section 3, we motivate our result by recalling some recent work concerning bipartite $Q$-polynomial distance-regular graphs. In Sections 4 and 5, we review the dual polar graphs of type $C_m(q)$, the Hemmeter graphs, the Ustimenko graphs, and the quadratic forms graphs. In Section 6, we prove the main result.

## 2. Distance-regular graphs

We begin with some basic concepts concerning distance-regular graphs. We refer the reader to [1,2] for a more thorough introduction. Let $\Gamma = (X, R)$ denote a finite, connected, undirected graph, without loops or multiple edges, with vertex set $X$, edge set $R$, path-length distance function $\partial$, and diameter $D := \max\{\partial(x, y) | x, y \in X\}$. For each $x \in X$ and each integer $i$, set $\Gamma_i(x) := \{y \in X | \partial(x, y) = i\}$. We call $\Gamma_i(x)$ the $i$th *subconstituent* of $\Gamma$ with respect to $x$. We abbreviate $\Gamma(x) := \Gamma_1(x)$.

*E-mail addresses:* caughman@pdx.edu (J.S. Caughman, IV), hart@math.psu.edu (E.J. Hart), jma3@learnlink.emory.edu (J. Ma).

0012-365X/$ - see front matter 
doi:10.1016/j.disc.2007.08.029

Recall $\Gamma$ is *regular*, with *valency* $k$, if $|\Gamma(x)| = k$ for all $x \in X$. We say $\Gamma$ is *distance-regular*, with *intersection numbers* $p^h_{ij}$, whenever $|\Gamma_i(x) \cap \Gamma_j(y)| = p^h_{ij}$ for all integers $h, i, j$ and all $x, y \in X$ with $\partial(x, y) = h$. When $\Gamma$ is distance-regular, we abbreviate $c_i := p^i_{1i-1}$, $a_i := p^i_{1i}$, $b_i := p^i_{1i+1}$ $(0 \leqslant i \leqslant D)$. Note that a distance-regular graph $\Gamma$ is regular with valency $k = b_0$. Moreover $c_i + a_i + b_i = k$ for all $i$.

Recall $\Gamma$ is *bipartite* if there exists a partition of the vertex set $X = X^+ \cup X^-$ such that $X^+$ and $X^-$ contain no edges. A distance-regular graph is bipartite if and only if the intersection numbers satisfy $a_i = 0$ for all $i$.

We next review the $Q$-polynomial property. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D$. Let $\mathrm{Mat}_X(\mathbf{C})$ denote the set of matrices with complex entries and whose rows and columns are indexed by $X$. Let $A$ denote the matrix in $\mathrm{Mat}_X(\mathbf{C})$ with $yz$-entry 1 if $\partial(y, z) = 1$ and 0 otherwise, for all $y, z \in X$. We refer to $A$ as the *adjacency matrix* for $\Gamma$. Let $\theta_0, \theta_1, \ldots, \theta_D$ denote the distinct eigenvalues of $A$, and let $E_0, E_1, \ldots, E_D$ denote the matrices in $\mathrm{Mat}_X(\mathbf{C})$ representing orthogonal projection onto the corresponding maximal eigenspaces of $A$. We refer to $E_0, E_1, \ldots, E_D$ as the *primitive idempotents* of $\Gamma$. We say $\Gamma$ is *$Q$-polynomial with respect to the ordering* $E_0, E_1, \ldots, E_D$ provided that, for each $i$ $(0 \leqslant i \leqslant D)$, the matrix $E_i = p_i(E_1)$ for some (entrywise) polynomial of degree $i$. We say $\Gamma$ is *$Q$-polynomial* whenever there exists a $Q$-polynomial ordering of its primitive idempotents.

## 3. Bipartite $Q$-polynomial distance-regular graphs

In this section, we recall some recent results concerning bipartite $Q$-polynomial distance-regular graphs.

First, suppose $\Gamma$ is bipartite distance-regular with diameter $D$. Fix $x, y \in X$ with $\partial(x, y) = D$. For each $i, j$ define $\Omega^i_j := \Gamma_i(x) \cap \Gamma_j(y)$. For each $z \in \Omega^i_j$, define $\mathscr{N}_{ij}(z) := |\Gamma(z) \cap \Omega^{i+1}_{j+1}|$, $\mathscr{S}_{ij}(z) := |\Gamma(z) \cap \Omega^{i-1}_{j-1}|$, $\mathscr{E}_{ij}(z) := |\Gamma(z) \cap \Omega^{i+1}_{j-1}|$, and $\mathscr{W}_{ij}(z) := |\Gamma(z) \cap \Omega^{i-1}_{j+1}|$. Now suppose further that $\Gamma$ is $Q$-polynomial with diameter $D \geqslant 3$. Then by [4, Corollary 6.2] there exist integers $\mathfrak{n}_{ij}, \mathfrak{s}_{ij}, \mathfrak{e}_{ij}, \mathfrak{w}_{ij}$ such that $\mathscr{N}_{ij}(z) = \mathfrak{n}_{ij}$, $\mathscr{S}_{ij}(z) = \mathfrak{s}_{ij}$, $\mathscr{E}_{ij}(z) = \mathfrak{e}_{ij}$, and $\mathscr{W}_{ij}(z) = \mathfrak{w}_{ij}$ for any vertices $x, y, z$ with $\partial(x, y) = D$ and $z \in \Omega^i_j$. As a consequence, the graph $\Gamma^2_D(x) := (\Gamma_D(x), \{yz | \partial_\Gamma(y, z) = 2\})$ is distance-regular for any $x \in X$ [4, Theorem 9.2].

Suppose $\Gamma$ is bipartite and $Q$-polynomial with $D \geqslant 12$. Then by [5], $\Gamma$ is an even cycle, a Hamming cube, the antipodal quotient of a Hamming cube of even diameter, or there is an integer $q \geqslant 2$ such that $\Gamma$ has intersection numbers

$$c_i = \frac{q^i - 1}{q - 1} \quad (1 \leqslant i \leqslant D), \qquad b_i = \frac{q^D - q^i}{q - 1} \quad (0 \leqslant i \leqslant D - 1).$$

There are two known families whose parameters satisfy the last condition: the bipartite dual polar graphs $D_D(q)$ (when $q$ is any prime power), and the Hemmeter graphs $\mathrm{Hem}_D(q)$ (when $q$ is an odd prime power). It is unknown whether there are other such graphs.

Let us now briefly survey the graphs $\Gamma^2_D(x)$ for some of the graphs mentioned above. If $\Gamma$ is an even cycle or a Hamming cube, then $\Gamma^2_D$ is just an isolated vertex. If $\Gamma$ is the antipodal quotient of a Hamming cube of even diameter, then $\Gamma^2_D$ is the folded Johnson graph $\tilde{J}(4D, 2D)$. If $\Gamma$ is a bipartite dual polar graph $D_D(q)$ where $q$ is a prime power, then $\Gamma^2_D$ is the graph $\mathrm{Alt}_D(q)$ of alternating forms on $\mathbf{F}^D_q$. (For more about these graphs, see [2, Chapter 9]). In Section 6, we show that if $\Gamma$ is the Hemmeter graph $\mathrm{Hem}_D(q)$, where $q$ is an odd prime power, then $\Gamma^2_D$ is the graph $\mathrm{Quad}_{D-1}(q)$ of quadratic forms on $\mathbf{F}^{D-1}_q$.

## 4. Dual polar graphs and the Hemmeter graph

Let $m$ be a positive integer and let $q$ be an odd prime power. Let $\mathbf{F}^{2m}_q$ denote the vector space of dimension $2m$ over the finite field of order $q$. An *alternating form* on $\mathbf{F}^{2m}_q$ is a function $f: \mathbf{F}^{2m}_q \times \mathbf{F}^{2m}_q \to \mathbf{F}_q$ such that for all $\alpha, \beta \in \mathbf{F}_q$ and all $x, y, z \in \mathbf{F}^{2m}_q$,

1. $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z)$,
2. $f(y, x) = -f(x, y)$.

We say $f$ is *nondegenerate* if for every nonzero $x \in \mathbf{F}^{2m}_q$, there exists $y \in \mathbf{F}^{2m}_q$ such that $f(x, y) \neq 0$. Notice that since $q$ is odd, condition (ii) implies $f(x, x) = 0$ for all $x$.

Fix a nondegenerate alternating form $f$ on $\mathbf{F}_q^{2m}$. A subspace $W$ of $\mathbf{F}_q^{2m}$ is *isotropic* if $f(x, y)=0$ for all $x, y \in W$. By Witt's theorem, the maximal isotropic subspaces all have dimension $m$. (See [2, p.274] for more information).

The *dual polar graph* $C_m(q)$ has as vertices the maximal isotropic subspaces of $\mathbf{F}_q^{2m}$. Two such vertices $\gamma, \delta$ are adjacent if and only if $\dim(\gamma \cap \delta) = m-1$. By [2, Theorem 9.4.3] the graph $C_m(q)$ is distance-regular with diameter $m$ and intersection numbers

$$c_i = \frac{q^i - 1}{q-1} \quad (1 \leqslant i \leqslant m), \qquad b_i = \frac{q^{m+1} - q^{i+1}}{q-1} \quad (0 \leqslant i \leqslant m-1). \tag{1}$$

We now recall the construction of the Hemmeter graph. Given a graph $\Gamma = (X, R)$, let $\hat{\Gamma}$ denote the graph with a pair vertices $x^+, x^-$ for every vertex $x \in X$, and edges

$$\{x^+ y^- | xy \in R\} \ \cup \ \{x^+ x^- | x \in X\}.$$

The graph $\hat{\Gamma}$ is called the *extended bipartite double* of $\Gamma$. By [2, Theorem 1.11.2] the graph $\hat{\Gamma}$ is distance-regular (with diameter $D+1$) if and only if $\Gamma$ is distance-regular (with diameter $D$) and $b_i + c_{i+1} = k+1$ $(1 \leqslant i \leqslant D-1)$. So it follows from (1) that the extended bipartite double of $C_m(q)$ is distance-regular, with diameter $m+1$ and intersection numbers

$$c_i = \frac{q^i - 1}{q-1} \quad (1 \leqslant i \leqslant m+1), \qquad b_i = \frac{q^{m+1} - q^{i}}{q-1} \quad (0 \leqslant i \leqslant m). \tag{2}$$

This bipartite graph is known as the *Hemmeter graph* and we denote it by $\mathrm{Hem}_{m+1}(q)$.

## 5. The Ustimenko graph and the quadratic forms graph

Let $\Gamma = (X, R)$ denote an arbitrary bipartite graph with bipartition $X = X^+ \cup X^-$. Define $\frac{1}{2}\Gamma^+$ to be the graph with vertex set $X^+$ and where $x, y$ are adjacent whenever $\partial_\Gamma(x, y) = 2$. Define $\frac{1}{2}\Gamma^-$ similarly on the vertex set $X^-$. Then $\frac{1}{2}\Gamma^+$ and $\frac{1}{2}\Gamma^-$ are called the *halved graphs* of $\Gamma$.

If $\Gamma$ is bipartite and distance-regular then the halved graphs are distance-regular (see [2, Proposition 4.2.2]). It follows that the halved graphs of $\mathrm{Hem}_{m+1}(q)$ are distance-regular. (In fact, they are isomorphic to each other since $\mathrm{Hem}_{m+1}(q)$ is vertex-transitive [3, Theorem 2.5].) This graph has diameter $d := \lfloor (m+1)/2 \rfloor$ and intersection numbers

$$c_i = \frac{(q^{2i} - 1)(q^{2i-1} - 1)}{(q^2-1)(q-1)} \quad (1 \leqslant i \leqslant d), \qquad b_i = q^{4i+1} \frac{(q^{m+1-2i} - 1)(q^{m-2i} - 1)}{(q^2-1)(q-1)} \quad (0 \leqslant i \leqslant d-1).$$

This halved graph is known as the *Ustimenko graph* and we denote it by $\mathrm{Ust}_{\lfloor (m+1)/2 \rfloor}(q)$.

Given any graph $\Gamma = (X, R)$, let $\Gamma_{1\vee 2}$ denote the graph with vertex set $X$ and where $x, y$ are adjacent whenever $\partial_\Gamma(x, y) = 1$ or 2. The graph $\Gamma_{1\vee 2}$ is called the *distance* 1 *or* 2 *graph* of $\Gamma$. It is known [2, Theorem 4.2.18] that if $\Gamma$ is distance-regular with diameter $D$, then the graph $\Gamma_{1\vee 2}$ is distance-regular if and only if $\Gamma$ satisfies

$$b_{i-1} + c_{i+1} - a_i = k + c_2 - a_1 \quad (2 \leqslant i \leqslant D-1).$$

Notice that the dual polar graph $C_m(q)$ satisfies the above condition, so its distance 1 or 2 graph is distance-regular. Indeed, it follows from the construction (or see [2, Theorem 1.11.2(iii)]) that if $\Gamma = C_m(q)$ then the graph $\Gamma_{1\vee 2}$ is isomorphic to the Ustimenko graph $\mathrm{Ust}_{\lfloor (m+1)/2 \rfloor}(q)$.

Now let $q$ denote an *odd* prime power. Let $B$ denote any symmetric $m \times m$ matrix over $\mathbf{F}_q$. The *quadratic form* associated with $B$ is the function $f_B : \mathbf{F}_q^m \to \mathbf{F}_q$ given by $f_B(x) = x^{\mathrm{t}} B x$. The *quadratic forms graph* $\mathrm{Quad}_m(q)$ is defined to be the graph whose vertices are the symmetric $m \times m$ matrices over $\mathbf{F}_q$ and where $B_1, B_2$ are adjacent whenever $\mathrm{rank}(B_1 - B_2) = 1$ or 2. (Indeed, in this case, where $q$ is odd, the quadratic form graphs are the distance 1 or 2 graphs of the so-called symmetric bilinear form graphs. See [2, p. 290]). By [2, Theorem 9.6.3] the quadratic form graph $\mathrm{Quad}_m(q)$ is distance regular, with diameter $d = \lfloor (m+1)/2 \rfloor$ and intersection numbers

$$c_i = q^{2i-2} \frac{(q^{2i} - 1)}{(q^2-1)} \quad (1 \leqslant i \leqslant d), \qquad b_i = q^{4i} \frac{(q^{m+1-2i} - 1)(q^{m-2i} - 1)}{(q^2-1)} \quad (0 \leqslant i \leqslant d-1).$$

## 6. The last subconstituent of the Hemmeter graph

We are now ready to prove our main result.

**Theorem 6.1.** *Let $q$ denote an odd prime power, and let $\Gamma$ denote the Hemmeter graph* $\mathrm{Hem}_D(q)$ *of diameter $D$. Then for any vertex $x$, the graph $\Gamma^2_D(x)$ is isomorphic to the quadratic forms graph* $\mathrm{Quad}_{D-1}(q)$.

**Proof.** First, suppose $D$ is even. Then by the construction, for any vertex $x$, the graph $\Gamma^2_D(x)$ is precisely the subgraph induced on the last subconstituent of the halved graph $\mathrm{Ust}_{D/2}(q)$. By [2, p. 287] this graph is isomorphic to $\mathrm{Quad}_{D-1}(q)$, as desired.

Now suppose $D$ is odd and write $D = m + 1$. Let $X = X^+ \cup X^-$ denote the bipartition of the vertex set of $\Gamma$. Pick any vertex $\gamma^+ \in X^+$ and note that since $D$ is odd, $\Gamma_D(\gamma^+)$ is contained in $X^-$. Indeed, it is easily shown that

$$\Gamma_D(\gamma^+) = \{\delta^- | \partial(\gamma, \delta) = m \text{ in } C_m(q)\}. \tag{3}$$

Now, without loss of generality, we may take the form $f$ on $C_m(q)$ to be

$$f(x, y) = x^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} y.$$

And by the transitivity of $\mathrm{Aut}(C_m(q))$ (see [2, p. 286]), we may take $\gamma = \mathrm{cols}\begin{pmatrix} I \\ 0 \end{pmatrix}$. We then observe that

$$\{\delta | \partial(\gamma, \delta) = m \text{ in } C_m(q)\} = \left\{\mathrm{cols}\begin{pmatrix} A \\ I \end{pmatrix} | A = A^{\mathrm{t}} \text{ and } A \text{ is } m \times m\right\}. \tag{4}$$

So by (3) and (4) we have

$$\Gamma_D(\gamma^+) = \left\{\delta^- | \delta = \mathrm{cols}\begin{pmatrix} A \\ I \end{pmatrix}, \text{ where } A = A^{\mathrm{t}} \text{ is } m \times m\right\}.$$

Let $\theta$ denote the function that associates, to each vertex $\delta^- \in \Gamma_D(\gamma^+)$, the matrix $A$, where $\delta = \mathrm{cols}\begin{pmatrix} A \\ I \end{pmatrix}$. Then $\theta$ is clearly a bijection between $\Gamma_D(\gamma^+)$ and the vertex set of $\mathrm{Quad}_m(q)$. So it remains to show that for any $\delta^-, \beta^- \in \Gamma_D(\gamma^+)$,

$$\partial_\Gamma(\delta^-, \beta^-) = 2 \quad \text{iff} \quad \mathrm{rank}(\theta(\delta^-) - \theta(\beta^-)) = 1 \text{ or } 2.$$

To see this, let $\delta = \mathrm{cols}\begin{pmatrix} A \\ I \end{pmatrix}$ and $\beta = \mathrm{cols}\begin{pmatrix} B \\ I \end{pmatrix}$.

Now observe that since $\Gamma$ is the extended bipartite double of $C_m(q)$,

$$\partial_\Gamma(\delta^-, \beta^-) = 2 \quad \text{iff} \quad \partial(\delta, \beta) = 1 \text{ or } 2 \quad \text{in } C_m(q).$$

But by the definition of $C_m(q)$,

$$\partial(\delta, \beta) = 1 \text{ or } 2 \quad \text{in } C_m(q) \ \text{ iff } \ \dim\,(\mathrm{span}(\delta \cup \beta)) = m + 1 \text{ or } m + 2.$$

But since

$$\mathrm{span}(\delta \cup \beta) = \mathrm{cols}\begin{pmatrix} A & B \\ I & I \end{pmatrix} = \mathrm{cols}\begin{pmatrix} A & B - A \\ I & 0 \end{pmatrix},$$

the above conditions hold if and only if $\mathrm{rank}\,(B - A) = 1$ or 2. Recall this is the criterion for adjacency in $\mathrm{Quad}_m(q)$. So $\Gamma^2_D(x)$ is isomorphic to $\mathrm{Quad}_m(q)$ as desired. □

**Remark.** To put the above result into context, recall that the Hemmeter graphs $\mathrm{Hem}_D(q)$ have the same intersection numbers as the bipartite dual polar graphs $D_D(q)$ (see [2, p. 279,3]). When $q$ is odd, however, the graphs are not isomorphic. The bipartite dual polar graphs are related to several other families of graphs, and the relationships that

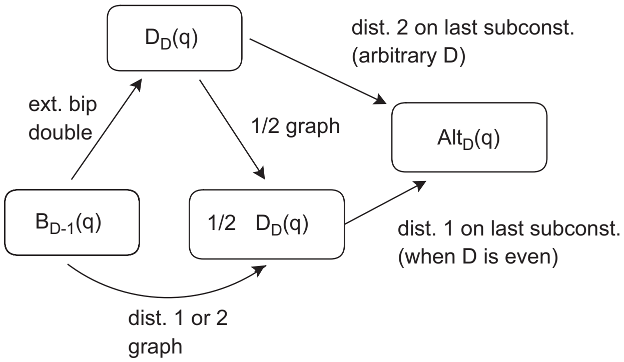


Fig. 1.

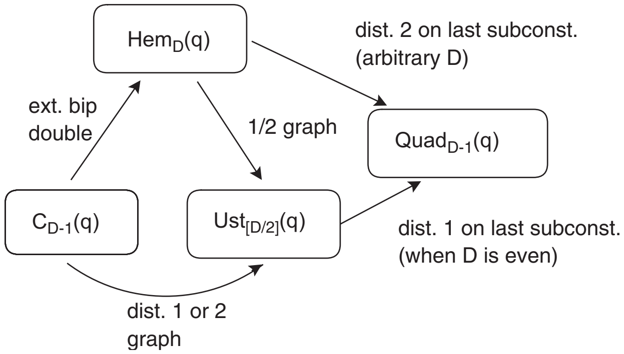


Fig. 2.

exist among the bipartite dual polar graphs $D_D(q)$, the halved dual polar graphs $\frac{1}{2}D_D(q)$, the dual polar graphs of type $B_{D-1}(q)$, and the alternating form graphs $\mathrm{Alt}_D(q)$ are depicted in Fig. 1. The same relationships exist among the Hemmeter graphs, the Ustimenko graphs, the dual polar graphs of type $C_m(q)$, and the quadratic form graphs, as depicted in Fig. 2. The content of Theorem 6.1 is represented by the top right-most arrow. For more about these graphs, the reader is referred to [2, pp. 274–284].

## Acknowledgements

The authors would like to thank the referees for their helpful comments and suggestions.

## References

[1] E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin, Cummings, London, 1984.
[2] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer, Berlin, 1989.
[3] A.E. Brouwer, J. Hemmeter, A new family of distance-regular graphs and the 0,1,2-cliques in dual polar graphs, European J. Combin. 13 (1992) 71–79.
[4] J.S. Caughman, The last subconstituent of a bipartite $Q$-polynomial distance-regular graph, European J. Combin. 24 (2003) 459–470.
[5] J.S. Caughman, IV, Bipartite $Q$-polynomial distance-regular graphs, Graphs Combin. 20 (2004) 47–57.
[6] Y. Egawa, Association schemes of quadratic forms, J. Combin. Theory Ser. A 38 (1985) 1–14.
[7] J. Hemmeter, Distance-regular graphs and halved graphs, European J. Combin. 7 (1986) 119–129.
[8] A.A. Ivanov, M.E. Muzichuk, V.A. Ustimenko, On a new family of ($P$ and $Q$)-polynomial schemes, European J. Combin. 10 (1989) 337–345.
[9] P. Terwilliger, Balanced sets and $Q$-polynomial association schemes, Graphs Combin. 4 (1988) 87–94.