The last subconstituent of the Hemmeter graph

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Abstract

We prove that when \( q \) is any odd prime power, the distance-2 graph on the set of vertices at maximal distance \( D \) from any fixed vertex of the Hemmeter graph \( \text{Hem}_D(q) \) is isomorphic to the graph \( \text{Quad}_{D-1}(q) \) of quadratic forms on \( \mathbb{F}_q^{D-1} \).

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1. Introduction

In this paper we prove that when \( q \) is any odd prime power, the distance-2 graph on the set of vertices at maximal distance \( D \) from any fixed vertex of the Hemmeter graph \( \text{Hem}_D(q) \) is isomorphic to the graph \( \text{Quad}_{D-1}(q) \) of quadratic forms on \( \mathbb{F}_q^{D-1} \). When \( D \) is even, this result follows from a consideration of the halved graph, as described in [2, p. 287]. Our proof establishes the result for arbitrary \( D \). For more about these graphs and our proof techniques, see [6–9].

The paper is organized as follows. In Section 2, we fix notation and recall some basic properties of distance-regular graphs. In Section 3, we motivate our result by recalling some recent work concerning bipartite \( Q \)-polynomial distance-regular graphs. In Sections 4 and 5, we review the dual polar graphs of type \( C_m(q) \), the Hemmeter graphs, the Ustimenko graphs, and the quadratic forms graphs. In Section 6, we prove the main result.

2. Distance-regular graphs

We begin with some basic concepts concerning distance-regular graphs. We refer the reader to [1,2] for a more thorough introduction. Let \( \Gamma = (X, R) \) denote a finite, connected, undirected graph, without loops or multiple edges, with vertex set \( X \), edge set \( R \), path-length distance function \( \delta \), and diameter \( D := \max \{ \delta(x, y) | x, y \in X \} \). For each \( x \in X \) and each integer \( i \), set \( \Gamma_i(x) := \{ y \in X | \delta(x, y) = i \} \). We call \( \Gamma_i(x) \) the \( i \)th subconstituent of \( \Gamma \) with respect to \( x \). We abbreviate \( \Gamma(x) := \Gamma_1(x) \).

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Recall $\Gamma$ is regular, with valency $k$, if $|\Gamma(x)| = k$ for all $x \in X$. We say $\Gamma$ is distance-regular, with intersection numbers $p_i^h$, whenever $|\Gamma_i(x) \cap \Gamma_j(y)| = p_i^h$ for all integers $h$, $i$, $j$ and all $x$, $y$ $\in$ $X$ with $d(x, y) = h$. When $\Gamma$ is distance-regular, we abbreviate $c_i := p_i^{i-1}$, $a_i := p_i^i$, $b_i := p_{i+1}^i$ ($0 \leq i \leq D$). Note that a distance-regular graph $\Gamma$ is regular with valency $k = b_0$. Moreover $c_i + a_i + b_i = k$ for all $i$.

Recall $\Gamma$ is bipartite if there exists a partition of the vertex set $X = X^+ \cup X^-$ such that $X^+$ and $X^-$ contain no edges. A distance-regular graph is bipartite if and only if the intersection numbers satisfy $f(x, y) = f(y, x)$.

Then by [4, Corollary 6.2] there exist integers $A_i$ and $B_i$ for each $i$ such that $\Gamma$ is bipartite dual polar graph $D_D(q)$ whenever there exists a partition of the vertex set $X = X^+ \cup X^-$ such that $X^+$ and $X^-$ contain no edges.

We next review the $Q$-polynomial property. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D$. Let $\text{Mat}_X(C)$ denote the set of matrices with complex entries and whose rows and columns are indexed by $X$. Let $A$ denote the matrix in $\text{Mat}_X(C)$ with $yz$-entry if $d(y, z) = 1$ and 0 otherwise, for all $y, z \in X$. We refer to $A$ as the adjacency matrix for $\Gamma$. Let $0, \theta_1, \ldots, \theta_D$ denote the distinct eigenvalues of $A$, and let $E_0, E_1, \ldots, E_D$ denote the matrices in $\text{Mat}_X(C)$ representing orthogonal projection onto the corresponding maximal eigenspaces of $A$. We refer to $E_0, E_1, \ldots, E_D$ as the primitive idempotents of $\Gamma$. We say $\Gamma$ is $Q$-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ provided that, for each $i$ ($0 \leq i \leq D$), the matrix $E_i = p_i(E_1)$ for some (entrywise) polynomial of degree $i$. We say $\Gamma$ is $Q$-polynomial whenever there exists a $Q$-polynomial ordering of its primitive idempotents.

3. Bipartite $Q$-polynomial distance-regular graphs

In this section, we recall some recent results concerning bipartite $Q$-polynomial distance-regular graphs.

First, suppose $\Gamma$ is bipartite distance-regular with diameter $D$. Fix $x, y \in X$ with $d(x, y) = D$. For each $i$, $j$ define $\Omega_{i,j} := \Gamma_i(x) \cap \Gamma_j(y)$. For each $z \in \Omega_{i,j}$, define $N_{i,j}(z) := |\Gamma(z) \cap \Omega_{i,j}^+|$, $S_{i,j}(z) := |\Gamma(z) \cap \Omega_{i,j}^-|$, $E_{i,j}(z) := |\Gamma(z) \cap \Omega_{i,j}^{+1}|$, and $P_{i,j}(z) := |\Gamma(z) \cap \Omega_{i,j}^{-1}|$. Now suppose further that $\Gamma$ is $Q$-polynomial with diameter $D \geq 3$. Then by [4, Corollary 6.2] there exist integers $n_{i,j}, s_{i,j}, e_{i,j}, w_{i,j}$ such that $N_{i,j}(z) = n_{i,j}$, $S_{i,j}(z) = s_{i,j}$, $E_{i,j}(z) = e_{i,j}$, and $P_{i,j}(z) = w_{i,j}$ for any vertices $x, y, z \in X$ with $d(x, y) = D$ and $z \in \Omega_{i,j}$. As a consequence, the graph $I_{D,x}^2$ is distance-regular for any $x \in X$ [4, Theorem 9.2].

Suppose $\Gamma$ is bipartite and $Q$-polynomial with $D \geq 12$. Then by [5], $\Gamma$ is an even cycle, a Hamming cube, the antipodal quotient of a Hamming cube of even diameter, or there is an integer $q \geq 2$ such that $\Gamma$ has intersection numbers

$$c_i = \frac{q^i - 1}{q - 1} \quad (1 \leq i \leq D), \quad b_i = \frac{q^D - q^i}{q - 1} \quad (0 \leq i \leq D - 1).$$

There are two known families whose parameters satisfy the last condition: the bipartite dual polar graphs $D_D(q)$ (when $q$ is any prime power), and the Hemmeter graphs Hem$_D(q)$ (when $q$ is an odd prime power). It is unknown whether there are other such graphs.

Let us now briefly survey the graphs $I_{D,x}^2$ for some of the graphs mentioned above. If $\Gamma$ is an even cycle or a Hamming cube, then $I_{D,x}^2$ is just an isolated vertex. If $\Gamma$ is the antipodal quotient of a Hamming cube of even diameter, then $I_{D,x}^2$ is the folded Johnson graph $\tilde{J}(4D, 2D)$. If $\Gamma$ is a bipartite dual polar graph $D_D(q)$ where $q$ is a prime power, then $I_{D,x}^2$ is the graph $\text{Alt}_D(q)$ of alternating forms on $F_{q}^{D}$. (For more about these graphs, see [2, Chapter 9]). In Section 6, we show that if $\Gamma$ is the Hemmeter graph Hem$_D(q)$, where $q$ is an odd prime power, then $I_{D,x}^2$ is the graph $\text{Quad}_{D-1}(q)$ of quadratic forms on $F_{q}^{D-1}$.

4. Dual polar graphs and the Hemmeter graph

Let $m$ be a positive integer and let $q$ be an odd prime power. Let $F_{q}^{2m}$ denote the vector space of dimension $2m$ over the finite field of order $q$. An alternating form on $F_{q}^{2m}$ is a function $f: F_{q}^{2m} \times F_{q}^{2m} \rightarrow F_{q}$ such that for all $x, \beta \in F_{q}$ and all $x, y, z \in F_{q}^{2m}$,

1. $f(xz + \beta y, z) = xf(x, z) + \beta f(y, z)$,
2. $f(y, x) = -f(x, y)$.

We say $f$ is nondegenerate if for every nonzero $x \in F_{q}^{2m}$, there exists $y \in F_{q}^{2m}$ such that $f(x, y) \neq 0$. Notice that since $q$ is odd, condition (ii) implies $f(x, x) = 0$ for all $x$. 


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Fix a nondegenerate alternating form \( f \) on \( \mathbf{F}_{q}^{2m} \). A subspace \( W \) of \( \mathbf{F}_{q}^{2m} \) is isotropic if \( f(x, y) = 0 \) for all \( x, y \in W \). By Witt’s theorem, the maximal isotropic subspaces all have dimension \( m \). (See [2, p.274] for more information).

The dual polar graph \( C_{m}(q) \) has as vertices the maximal isotropic subspaces of \( \mathbf{F}_{q}^{2m} \). Two such vertices \( \gamma, \delta \) are adjacent if and only if \( \dim(\gamma \cap \delta) = m - 1 \). By [2, Theorem 9.4.3] the graph \( C_{m}(q) \) is distance-regular with diameter \( m \) and intersection numbers

\[
c_i = \frac{q^i - 1}{q - 1} \quad (1 \leq i \leq m), \quad b_i = \frac{q^{m+1} - q^{i+1}}{q - 1} \quad (0 \leq i \leq m - 1).
\]

We now recall the construction of the Hemmeter graph. Given a graph \( \Gamma = (X, R) \), let \( \hat{\Gamma} \) denote the graph with a pair vertices \( x^+, x^- \) for every vertex \( x \in X \), and edges

\[
\{x^+y^-|xy \in R\} \cup \{x^+x^-|x \in X\}.
\]

The graph \( \hat{\Gamma} \) is called the extended bipartite double of \( \Gamma \). By [2, Theorem 1.11.2] the graph \( \hat{\Gamma} \) is distance-regular (with diameter \( D + 1 \)) if and only if \( \Gamma \) is distance-regular (with diameter \( D \)) and \( b_i + c_{i+1} = k + 1 \) (\( 1 \leq i \leq D - 1 \)). So it follows from (1) that the extended bipartite double of \( C_{m}(q) \) is distance-regular, with diameter \( m + 1 \) and intersection numbers

\[
c_i = \frac{q^i - 1}{q - 1} \quad (1 \leq i \leq m + 1), \quad b_i = \frac{q^{m+1} - q^i}{q - 1} \quad (0 \leq i \leq m).
\]

This bipartite graph is known as the Hemmeter graph and we denote it by \( \text{Hem}_{m+1}(q) \).

5. The Ustimenko graph and the quadratic forms graph

Let \( \Gamma = (X, R) \) denote an arbitrary bipartite graph with partition \( X = X^+ \cup X^- \). Define \( \frac{1}{2}\Gamma^+ \) to be the graph with vertex set \( X^+ \) and where \( x, y \) are adjacent whenever \( \partial \Gamma(x, y) = 2 \). Define \( \frac{1}{2}\Gamma^- \) similarly on the vertex set \( X^- \). Then \( \frac{1}{2}\Gamma^+ \) and \( \frac{1}{2}\Gamma^- \) are called the halved graphs of \( \Gamma \).

If \( \Gamma \) is bipartite and distance-regular then the halved graphs are distance-regular (see [2, Proposition 4.2.2]). It follows that the halved graphs of \( \text{Hem}_{m+1}(q) \) are distance-regular. (In fact, they are isomorphic to each other since \( \text{Hem}_{m+1}(q) \) is vertex-transitive [3, Theorem 2.5].) This graph has diameter \( d := \lceil (m + 1)/2 \rceil \) and intersection numbers

\[
c_i = \frac{(q^{2i} - 1)(q^{2i-1} - 1)}{(q^2 - 1)(q - 1)} \quad (1 \leq i \leq d), \quad b_i = \frac{q^{2i+1} - q^{m+1-2i} - 1}{(q^2 - 1)(q - 1)} \quad (0 \leq i \leq d - 1).
\]

This halved graph is known as the Ustimenko graph and we denote it by \( \text{Ust}_{(m+1)/2}(q) \).

Given any graph \( \Gamma = (X, R) \), let \( \Gamma_{1/2} \) denote the graph with vertex set \( X \) and where \( x, y \) are adjacent whenever \( \partial \Gamma(x, y) = 1 \) or 2. The graph \( \Gamma_{1/2} \) is called the distance 1 or 2 graph of \( \Gamma \). It is known [2, Theorem 4.2.18] that if \( \Gamma \) is distance-regular with diameter \( D \), then the graph \( \Gamma_{1/2} \) is distance-regular if and only if \( \Gamma \) satisfies

\[
b_{i-1} + c_{i+1} - a_i = k + c_2 - a_1 \quad (2 \leq i \leq D - 1).
\]

Notice that the dual polar graph \( C_{m}(q) \) satisfies the above condition, so its distance 1 or 2 graph is distance-regular. Indeed, it follows from the construction (or see [2, Theorem 1.11.2(iii)]) that if \( \Gamma = C_{m}(q) \) then the graph \( \Gamma_{1/2} \) is isomorphic to the Ustimenko graph \( \text{Ust}_{(m+1)/2}(q) \).

Now let \( q \) denote an odd prime power. Let \( B \) denote any symmetric \( m \times m \) matrix over \( \mathbf{F}_q \). The quadratic form associated with \( B \) is the function \( f_{B} : \mathbf{F}_q^m \to \mathbf{F}_q \) given by \( f_{B}(x) = x^{\top}Bx \). The quadratic forms graph \( \text{Quad}_{m}(q) \) is defined to be the graph whose vertices are the symmetric \( m \times m \) matrices over \( \mathbf{F}_q \) and where \( B_1, B_2 \) are adjacent whenever \( \text{rank}(B_1 - B_2) = 1 \) or 2. (Indeed, in this case, where \( q \) is odd, the quadratic form graphs are the distance 1 or 2 graphs of the so-called symmetric bilinear form graphs. See [2, p. 290]). By [2, Theorem 9.6.3] the quadratic form graph \( \text{Quad}_{m}(q) \) is distance-regular, with diameter \( d = \lfloor (m + 1)/2 \rfloor \) and intersection numbers

\[
c_i = q^{2i-2} \frac{(q^{2i} - 1)}{(q^2 - 1)} \quad (1 \leq i \leq d), \quad b_i = \frac{q^{4i} (q^{m+1-2i} - 1)(q^{m-2i} - 1)}{(q^2 - 1)} \quad (0 \leq i \leq d - 1).
\]
6. The last subconstituent of the Hemmeter graph

We are now ready to prove our main result.

**Theorem 6.1.** Let \( q \) denote an odd prime power, and let \( \Gamma \) denote the Hemmeter graph \( \text{Hem}_D(q) \) of diameter \( D \). Then for any vertex \( x \), the graph \( \Gamma^2_D(x) \) is isomorphic to the quadratic forms graph \( \text{Quad}_{D-1}(q) \).

**Proof.** First, suppose \( D \) is even. Then by the construction, for any vertex \( x \), the graph \( \Gamma^2_D(x) \) is precisely the subgraph induced on the last subconstituent of the halved graph \( U_{D/2}(q) \). By [2, p. 287] this graph is isomorphic to \( \text{Quad}_{D-1}(q) \), as desired.

Now suppose \( D \) is odd and write \( D = m + 1 \). Let \( X = X^+ \cup X^- \) denote the bipartition of the vertex set of \( \Gamma \). Pick any vertex \( \gamma^+ \in X^+ \) and note that since \( D \) is odd, \( \Gamma_D(\gamma^+) \) is contained in \( X^- \). Indeed, it is easily shown that

\[
\Gamma_D(\gamma^+) = \{ \delta^- | \delta = m \text{ in } C_m(q) \}. \tag{3}
\]

Now, without loss of generality, we may take the form \( f \) on \( C_m(q) \) to be

\[
f(x, y) = x^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} y.
\]

And by the transitivity of \( \text{Aut}(C_m(q)) \) (see [2, p. 286]), we may take \( \gamma = \text{cols} \left( I \right) \). We then observe that

\[
\{ \delta | \delta = m \text{ in } C_m(q) \} = \left\{ \text{cols} \left( \begin{array}{c} A \\ I \end{array} \right) | A = A^t \text{ and } A \text{ is } m \times m \right\}. \tag{4}
\]

So by (3) and (4) we have

\[
\Gamma_D(\gamma^+) = \left\{ \delta^- | \delta = \text{cols} \left( \begin{array}{c} A \\ I \end{array} \right), \text{ where } A = A^t \text{ is } m \times m \right\}.
\]

Let \( \theta \) denote the function that associates, to each vertex \( \delta^- \in \Gamma_D(\gamma^+) \), the matrix \( A \), where \( \delta = \text{cols} \left( \begin{array}{c} A \\ I \end{array} \right) \). Then \( \theta \) is clearly a bijection between \( \Gamma_D(\gamma^+) \) and the vertex set of \( \text{Quad}_m(q) \). So it remains to show that for any \( \delta^-, \beta^- \in \Gamma_D(\gamma^+) \),

\[
\tilde{\delta}(\delta^-, \beta^-) = 2 \iff \text{rank} (\theta(\delta^-) - \theta(\beta^-)) = 1 \text{ or } 2.
\]

To see this, let \( \delta = \text{cols} \left( \begin{array}{c} A \\ I \end{array} \right) \) and \( \beta = \text{cols} \left( \begin{array}{c} B \\ I \end{array} \right) \).

Now observe that since \( \Gamma \) is the extended bipartite double of \( C_m(q) \),

\[
\tilde{\delta}(\delta^-, \beta^-) = 2 \iff \tilde{\delta}(\delta, \beta) = 1 \text{ or } 2 \text{ in } C_m(q).
\]

But by the definition of \( C_m(q) \),

\[
\tilde{\delta}(\delta, \beta) = 1 \text{ or } 2 \text{ in } C_m(q) \iff \text{dim (span} (\delta \cup \beta)) = m + 1 \text{ or } m + 2.
\]

But since

\[
\text{span} (\delta \cup \beta) = \text{cols} \left( \begin{array}{c} A \\ I \\ B \\ I \end{array} \right) = \text{cols} \left( \begin{array}{c} A \\ I \\ B - A \\ 0 \end{array} \right),
\]

the above conditions hold if and only if \( \text{rank} (B - A) = 1 \) or 2. Recall this is the criterion for adjacency in \( \text{Quad}_m(q) \). So \( \Gamma^2_D(x) \) is isomorphic to \( \text{Quad}_m(q) \) as desired. \( \Box \)

**Remark.** To put the above result into context, recall that the Hemmeter graphs \( \text{Hem}_D(q) \) have the same intersection numbers as the bipartite dual polar graphs \( D_D(q) \) (see [2, p. 279]). When \( q \) is odd, however, the graphs are not isomorphic. The bipartite dual polar graphs are related to several other families of graphs, and the relationships that
exist among the bipartite dual polar graphs $D_D(q)$, the halved dual polar graphs $\frac{1}{2}D_D(q)$, the dual polar graphs of type $B_{D-1}(q)$, and the alternating form graphs $\text{Alt}_D(q)$ are depicted in Fig. 1. The same relationships exist among the Hemmeter graphs, the Ustimenko graphs, the dual polar graphs of type $C_m(q)$, and the quadratic form graphs, as depicted in Fig. 2. The content of Theorem 6.1 is represented by the top right-most arrow. For more about these graphs, the reader is referred to [2, pp. 274–284].

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