NOTE

THE BIPARTITE TOURNAMENT ASSOCIATED WITH A FABRIC

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For a two-way periodic fabric, a bipartite tournament is defined; and some results for fabrics are derived from known results about bipartite tournaments.

A fabric with two sets of strands, the horizontal strands being the weft and the vertical the warp, is periodic if its design can be obtained from a fundamental block by translations through multiples of $n$ and $m$ units in the horizontal and vertical directions. This fundamental block may be denoted by an $m$ by $n$ matrix with the entry equal to 0 if the corresponding weft strand passes over the warp strand and equal to 1 if the weft passes under the warp. The geometry of fabrics has been described in [4] and the more recent work on isonemal fabrics may be found in [5].

Suppose now that the set of weft strands is represented by a set $X$ of vertices and the warp by $Y$ and that a directed arc is drawn from the vertex $x$ to the vertex $y$, where $x \in X$ and $y \in Y$, if the weft strand represented by $x$ passes under the warp strand represented by $y$ and from $y$ to $x$ if the weft passes over the warp. Then the vertices and arcs form what we call the bipartite tournament associated with the given fabric. Conversely, any bipartite tournament can be interpreted as that associated with a fabric.

From the matrix of the fabric can be calculated the row-sums $r_1, \ldots, r_m$, with $r_1 \leq r_2 \leq \cdots \leq r_m$ and the column-sums $c_1, \ldots, c_n$, with $c_1 \geq c_2 \geq \cdots \geq c_n$. It was shown in [2] that the fabric hangs together unless $E(s, t) = 0$ for some values of $s$ and $t$ (the two cases $s = 0$, $t = 0$ and $s = m$, $t = n$ being excluded), where

$$E(s, t) = r_1 + \cdots + r_s + (m - c_1) + \cdots + (m - c_t) - st.$$ 

A procedure for checking this was described.

Now one vertex of a tournament is said to dominate another if there is an arc directed from the first to the second and the score of a vertex is the number of
vertices that it dominates. It is clear that \( r_1, \ldots, r_m \) are the scores of the vertices in \( X \) and \( m - c_1, \ldots, m - c_n \) are the scores of the vertices in \( Y \).

A bipartite tournament is reducible if there is a non-empty proper subset of the set of vertices with no arc directed from the other vertices to the vertices in this subset. It is an elementary result that the property of being irreducible is equivalent to that of being strongly connected. When the bipartite tournament is interpreted in the language of fabrics, this result is clear since being irreducible means there being no non-empty proper subset of the strands that 'lifts off' the remainder, whereas being strongly connected means that the lifting of any one strand causes every other strand to be lifted. Both mean that the fabric hangs together. The time complexity of an algorithm to determine this is obtained in this way in [3].

Some results in Beineke and Moon [1] now have some interesting implications for fabrics.

**Theorem (Moon).** The lists \( a_1, \ldots, a_m \) and \( b_1, \ldots, b_n \) of non-negative integers, each in non-decreasing order, are the score lists of the two sets of vertices of a bipartite tournament if and only if

\[
\sum_{i=1}^{s} a_i + \sum_{j=1}^{t} b_j \geq st,
\]

for all \( s \) and \( t \), with equality when \( s = m, t = n \).

**Furthermore,** the bipartite tournament is irreducible if and only if the inequality is strict except when \( s = 0, t = 0 \) and \( s = m, t = n \).

The final remark here is exactly the condition for a fabric to hang together already described above. The earlier part of the theorem gives this:

**Theorem 1.** The lists \( r_1, \ldots, r_m \) and \( c_1, \ldots, c_n \) of non-negative integers with \( r_1 \leq r_2 \leq \cdots \leq r_m \) and \( c_1 \geq c_2 \geq \cdots \geq c_n \) are the row-sums and column-sums associated with a fabric if and only if

\[
\sum_{i=1}^{s} r_i + \sum_{j=1}^{t} (m - c_j) \geq st,
\]

for all \( s \) and \( t \), with equality when \( s = m, t = n \).

Beineke and Moon [1] also give a proof of the following:

**Theorem (Moon).** If a bipartite tournament has score lists \( a_1, \ldots, a_m \) and \( b_1, \ldots, b_n \) satisfying \( \frac{1}{4}n < a_i < \frac{3}{4}n \), for all \( i \) and \( \frac{1}{4}m < b_j < \frac{3}{4}m \), for all \( j \), then it is irreducible.

This gives the following result about fabrics:
**Theorem 2.** If every weft strand of a periodic fabric passes over more than and under more than a quarter of the warp strands and every warp strand passes over more than and under more than a quarter of the weft strands, then the fabric hangs together.

This is best possible in the sense that the strict 'more than' cannot be weakened: if \( m \) and \( n \) are divisible by 4 then the fabric with half its row-sums equal to \( \frac{1}{4}n \) and half equal to \( \frac{3}{4}n \), and half its column-sums equal to \( \frac{3}{4}m \) and half equal to \( \frac{1}{4}m \), does not hang together.

**References**


