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π-Formulas with free parameters

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ABSTRACT

In terms of the hypergeometric method, we establish ten general π-formulas with free parameters which include several known results as special cases.

1. Introduction

For a complex number x and an integer n, define the shifted factorial by

$$(x)_n = \begin{cases} \prod_{k=0}^{n-1}(x+k), & \text{when } n > 0; \\ 1, & \text{when } n = 0; \\ \dfrac{(-1)^n}{\prod_{k=1}^{-n}(k-x)}, & \text{when } n < 0. \end{cases}$$

Recall that the function $\Gamma(x)$ can be given by Euler's integral:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad \text{with } \mathrm{Re}(x) > 0.$$

Then we have the following two relations:

$$\Gamma(x+n) = \Gamma(x)(x)_n, \qquad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

which will frequently be used without indication in this paper.

Following Bailey [1], define the hypergeometric series by

$${}_{r+1}F_s\left[\begin{matrix} a_0, & a_1, & \dots, & a_r \\ & b_1, & \dots, & b_s \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_r)_k}{k!(b_1)_k\cdots(b_s)_k} z^k.$$

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Then a simple ${}_2F_1$-series identity (cf. [2, Eq. (26)]) can be stated as

$$ {}_2F_1\left[\begin{matrix}1, & \frac{1}{2} \\ & \frac{3}{2}\end{matrix}\middle| x^2\right] = \frac{\arcsin(x)}{x\sqrt{1-x^2}} \quad \text{where } |x| < 1. $$

Two beautiful series for π (cf. [2, Eqs. (23) and (27)]) implied by it read as

$$ \frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!}, \tag{1} $$

$$ \frac{2\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \frac{(k!)^2}{(2k+1)!}, \tag{2} $$

where the double factorial has been offered by

$$ (2k+1)!! = \frac{(2k+1)!}{2^k k!}, \qquad (2k)!! = 2^k k!. $$

By means of WZ-method, Guillera [3, p. 221] derive lately the nice series for π^2:

$$ \frac{\pi^2}{4} = \sum_{k=0}^{\infty} \frac{(1)_k^3}{\left(\frac{3}{2}\right)_k^3} \frac{3k+2}{4^k}. \tag{3} $$

Recall the ${}_7F_6$-series identity due to Chu [4, Eq. (5.1e)] and Dougall's ${}_5F_4$-series identity (cf. [1, p. 27]):

$$ {}_7F_6\left[\begin{matrix} a-\frac{1}{2}, & \frac{2a+2}{3}, & 2b-1, & 2c-1, & 2+2a-2b-2c, & a+s, & -s \\ & \frac{2a-1}{3}, & 1+a-b, & 1+a-c, & b+c-\frac{1}{2}, & 2a+2s, & -2s \end{matrix}\middle| 1\right] = \frac{\left(\frac{1}{2}+a\right)_s (b)_s (c)_s \left(a-b-c+\frac{3}{2}\right)_s}{\left(\frac{1}{2}\right)_s (1+a-b)_s (1+a-c)_s \left(b+c-\frac{1}{2}\right)_s} \tag{4} $$

where s is a positive integer,

$$ {}_5F_4\left[\begin{matrix} a, & 1+\frac{a}{2}, & b, & c, & d \\ & \frac{a}{2}, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix}\middle| 1\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} \tag{5} $$

provided that $\mathrm{Re}(1+a-b-c-d) > 0$.

Recently, Chu [5,6] and Liu [7,8] have deduced many surprising π-formulas from some known hypergeometric series identities. Thereinto, Chu [5] showed that (5) implies the Ramanujan-type series for $1/\pi$ with three free parameters:

$$ \frac{2}{\pi} = \frac{\left(\frac{1}{2}\right)_{m-n-p}}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_p} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}\right)_{k+m} \left(\frac{1}{2}\right)_{k+n} \left(\frac{1}{2}\right)_{k+p}}{k!(k+m-n)!(k+m-p)!} (4k+2m+1) \tag{6} $$

where $m, n, p \in \mathbb{Z}$ with $\min\{m-n, m-p, m-2n-2p\} \geq 0$ and the Ramanujan-type series for $1/\pi^2$ with four free parameters:

$$ \frac{2}{\pi^2} = \frac{\left(\frac{1}{2}\right)_{m-n-p} \left(\frac{1}{2}\right)_{m-n-q} \left(\frac{1}{2}\right)_{m-p-q}}{(m-n-p-q-1)! \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_p \left(\frac{1}{2}\right)_q} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}\right)_{k+m} \left(\frac{1}{2}\right)_{k+n} \left(\frac{1}{2}\right)_{k+p} \left(\frac{1}{2}\right)_{k+q}}{k!(k+m-n)!(k+m-p)!(k+m-q)!} (4k+2m+1) \tag{7} $$

where $m, n, p, q \in \mathbb{Z}$ with $\min\{m-n, m-p, m-q, m-n-p-q-1\} \geq 0$. Liu [8] showed that (5) implies the Ramanujan-type series for $1/\pi$ with four free parameters:

$$ \frac{\sqrt{3}}{3\pi} = \frac{\left(\frac{2}{3}\right)_{m-n-p} \left(\frac{1}{3}\right)_{m-n-q} \left(\frac{1}{2}\right)_{m-p-q}}{(m-n-p-q-1)! \left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_p \left(\frac{2}{3}\right)_q} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{k+m} \left(\frac{1}{2}\right)_{k+n} \left(\frac{1}{3}\right)_{k+p} \left(\frac{2}{3}\right)_{k+q}}{k!(k+m-n)! \left(\frac{7}{6}\right)_{k+m-p} \left(\frac{5}{6}\right)_{k+m-q}} (4k+2m+1) \tag{8} $$

where $m, n, p, q \in \mathbb{Z}$ with $\min\{m-n, m-n-p-q-1\} \geq 0$. Eqs. (6)–(8) can create numerous special π-formulas by specifying the free parameters. For example, the case $m = n = p = 0$ of (6) produces the simplest Ramanujan-type formula due to Bauer [9, Section 4] (see also [10]):

$$\frac{2}{\pi} = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3} (4k+1).$$

Inspired by these work just mentioned, we shall explore further the relations of π-formulas and hypergeometric series. The structure of the paper is arranged as follows. Seven general π-formulas with free parameters, which include (1)–(3) as special cases, will be derived from (4) in Section 2. Three general π-formulas with free parameters, which include (6)–(8) as special cases, will be deduced from (5) in Section 3.

2. Summation formulas for π and π^2 with free parameters implied by Chu's ${}_7F_6$-series identity

Letting $s \to \infty$ for (4), we obtain the following equation:

$$\begin{aligned} &{}_5F_4\left[\begin{matrix} a-\frac{1}{2}, & \frac{2a+2}{3}, & 2b-1, & 2c-1, & 2+2a-2b-2c \\ & \frac{2a-1}{3}, & 1+a-b, & 1+a-c, & b+c-\frac{1}{2} \end{matrix} \middle| \frac{1}{4}\right] \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(b+c-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+a\right)\Gamma(b)\Gamma(c)\Gamma\left(a-b-c+\frac{3}{2}\right)}. \end{aligned} \tag{9}$$

Subsequently, one general summation formula for π with two free parameters, five general summation formulas for π with three free parameters and one general summation formula for π^2 with three free parameters will respectively be derived from (9).

Choosing $b = 1+m$, $c = 1+n$ in (9) and then letting $a \to \infty$, we achieve the equation.

Theorem 1. *For $m, n \in \mathbb{N}_0$, there holds the general summation formula for π with two free parameters:*

$$\frac{\pi}{2^{m+n+1}} = \frac{m!n!}{(2m)!(2n)!} \sum_{k=0}^{\infty} \frac{(k+2m)!(k+2n)!}{k!(2k+2m+2n+1)!!}.$$

When $m = n = 0$, Theorem 1 reduces to (1) exactly. Other two examples of the same type are displayed as follows.

Example 1 (*$m = 1$ and $n = 0$ in* Theorem 1).

$$\frac{\pi}{2} = \sum_{k=1}^{\infty} \frac{(k+1)!}{(2k+1)!!}.$$

Example 2 (*$m = 2$ and $n = 0$ in* Theorem 1).

$$\frac{3\pi}{2} = \sum_{k=2}^{\infty} \frac{(k+2)!}{(2k+1)!!}.$$

We point out that the limiting case $a \to \infty$ of (9) is equivalent to Gauss' second summation theorem (cf. [1, p. 11]):

$${}_2F_1\left[\begin{matrix} a, & b \\ & \frac{1+a+b}{2} \end{matrix} \middle| \frac{1}{2}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+a+b}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)}.$$

Therefore, Theorem 1 can also be deduced by fixing $a = 1+2m$ and $b = 1+2n$ in it.

Making $a = \frac{1}{2}+m$, $b = \frac{1}{4}+n$ and $c = \frac{3}{4}+p$ in (9), we attain the equation.

Theorem 2. *For $m, n, p \in \mathbb{Z}$ with $\min\{m, m-n-p\} \geq 0$, there holds the general summation formula for π with three free parameters:*

$$\frac{\pi}{2} = \frac{1}{\left(-\frac{1}{4}\right)_n \left(\frac{1}{4}\right)_p \left(\frac{1}{2}\right)_{m-n-p}} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_{k+2n}\left(\frac{1}{2}\right)_{k+2p}(k+m)!(k+2m-2n-2p)!}{\left(\frac{5}{4}\right)_{k+m-n}\left(\frac{3}{4}\right)_{k+m-p}\left(\frac{1}{2}\right)_{k+n+p}\, k!} \frac{3k+2m}{k+m} \frac{1}{4^{k+m}}.$$

Two examples from Theorem 2 are laid out as follows.

Example 3 *($m = n = 1$ and $p = 0$ in Theorem 2).*

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(2k)!}{(4k+3)!!}(3k+2).$$

Example 4 *($m = 2$ and $n = p = 1$ in Theorem 2).*

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(2k)!}{(4k+1)!!}(3k+1).$$

A beautiful result should be mentioned. Fixing $a = -\frac{3}{2}$ in the identity due to Chu [4, Eq. (5.3g)]:

$$ {}_6F_5\left[\begin{matrix} 7+4a, & -2-2a, & -\frac{5}{2}-2a, & \frac{2-4a}{5}, & -a+s, & -s \\ & \frac{4}{3}, & \frac{5}{3}, & -\frac{3+4a}{5}, & -2-4s, & -2-4a+4s \end{matrix} \middle| \frac{32}{27}\right]$$
$$= \frac{\left(-\frac{1}{2}-a\right)_s \left(-\frac{1}{4}-a\right)_s \left(\frac{1}{4}-a\right)_s \left(\frac{5}{2}+a\right)_s}{\left(\frac{3}{4}\right)_s \left(\frac{5}{4}\right)_s \left(\frac{3}{2}\right)_s \left(-\frac{3}{2}-2a\right)_s}$$

and then letting $s \to \infty$, we obtain the surprising series for π:

$$\frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!(2k)!}{(3k+2)!}\frac{5k+3}{2^k}.$$

Setting $a = \frac{1}{2}+m$, $b = \frac{1}{6}+n$ and $c = \frac{5}{6}+p$ in (9), we get the equation.

Theorem 3. *For $m, n, p \in \mathbb{Z}$ with $\min\{m, m-n-p\} \geq 0$, there holds the general summation formula for π with three free parameters:*

$$\frac{2\pi}{3\sqrt{3}} = \frac{1}{\left(-\frac{1}{3}\right)_n \left(\frac{1}{3}\right)_p \left(\frac{1}{2}\right)_{m-n-p}} \sum_{k=0}^{\infty} \frac{\left(-\frac{2}{3}\right)_{k+2n} \left(\frac{2}{3}\right)_{k+2p} (k+m)!(k+2m-2n-2p)!}{\left(\frac{4}{3}\right)_{k+m-n} \left(\frac{2}{3}\right)_{k+m-p} \left(\frac{1}{2}\right)_{k+n+p} \, k!} \frac{3k+2m}{k+m}\frac{1}{4^{k+m}}.$$

When $m = n = 1$ and $p = 0$, Theorem 3 specializes to (2) exactly. Other two examples of the same type are displayed as follows.

Example 5 *($m = 2$ and $n = p = 1$ in Theorem 3).*

$$\frac{2\pi}{3\sqrt{3}} = \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k+1)!}(3k+2).$$

Example 6 *($m = 3$, $n = 2$ and $p = 1$ in Theorem 3).*

$$\frac{4\pi}{9\sqrt{3}} = \sum_{k=2}^{\infty} \frac{(k!)^2}{(2k+1)!}(3k+1)k.$$

Taking $a = \frac{1}{2}+m$, $b = \frac{1}{3}+n$ and $c = \frac{2}{3}+p$ in (9), we gain the equation.

Theorem 4. *For $m, n, p \in \mathbb{Z}$ with $\min\{m, m-n-p\} \geq 0$, there holds the general summation formula for π with three free parameters:*

$$\frac{\pi}{\sqrt{3}} = \frac{1}{\left(-\frac{1}{6}\right)_n \left(\frac{1}{6}\right)_p \left(\frac{1}{2}\right)_{m-n-p}} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{3}\right)_{k+2n} \left(\frac{1}{3}\right)_{k+2p} (k+m)!(k+2m-2n-2p)!}{\left(\frac{7}{6}\right)_{k+m-n} \left(\frac{5}{6}\right)_{k+m-p} \left(\frac{1}{2}\right)_{k+n+p} \, k!} \frac{3k+2m}{k+m}\frac{1}{4^{k+m}}.$$

Two examples from Theorem 4 are laid out as follows.

Example 7 (*$m = n = 1$ and $p = 0$ in* Theorem 4).

$$\frac{5\pi}{4\sqrt{3}} = \sum_{k=0}^{\infty} \frac{(1)_k \left(\frac{1}{3}\right)_k \left(\frac{5}{3}\right)_k}{\left(\frac{3}{2}\right)_k \left(\frac{7}{6}\right)_k \left(\frac{11}{6}\right)_k} \frac{3k+2}{4^k}.$$

Example 8 (*$m = 2$ and $n = p = 1$ in* Theorem 4).

$$\frac{\pi}{2\sqrt{3}} = \sum_{k=1}^{\infty} \frac{(1)_k \left(\frac{2}{3}\right)_k \left(\frac{4}{3}\right)_k}{\left(\frac{3}{2}\right)_k \left(\frac{5}{6}\right)_k \left(\frac{7}{6}\right)_k} \frac{3k+1}{4^k}.$$

Choosing $a = \frac{1}{2} + m$, $b = \frac{5}{12} + n$ and $c = \frac{7}{12} + p$ in (9), we achieve the equation.

Theorem 5. *For $m, n, p \in \mathbb{Z}$ with $\min\{m, m-n-p\} \geq 0$, there holds the general summation formula for π with three free parameters:*

$$\frac{\pi}{6(2-\sqrt{3})} = \frac{1}{\left(-\frac{1}{12}\right)_n \left(\frac{1}{12}\right)_p \left(\frac{1}{2}\right)_{m-n-p}} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{6}\right)_{k+2n} \left(\frac{1}{6}\right)_{k+2p} (k+m)!(k+2m-2n-2p)!}{\left(\frac{13}{12}\right)_{k+m-n} \left(\frac{11}{12}\right)_{k+m-p} \left(\frac{1}{2}\right)_{k+n+p} k!} \frac{3k+2m}{k+m} \frac{1}{4^{k+m}}.$$

Two examples from Theorem 5 are displayed as follows.

Example 9 (*$m = n = 1$ and $p = 0$ in* Theorem 5).

$$\frac{11\pi}{60(2-\sqrt{3})} = \sum_{k=0}^{\infty} \frac{(1)_k \left(\frac{1}{6}\right)_k \left(\frac{11}{6}\right)_k}{\left(\frac{3}{2}\right)_k \left(\frac{13}{12}\right)_k \left(\frac{23}{12}\right)_k} \frac{3k+2}{4^k}.$$

Example 10 (*$m = 2$ and $n = p = 1$ in* Theorem 5).

$$\frac{\pi}{12(2-\sqrt{3})} = \sum_{k=1}^{\infty} \frac{(1)_k \left(\frac{5}{6}\right)_k \left(\frac{7}{6}\right)_k}{\left(\frac{3}{2}\right)_k \left(\frac{11}{12}\right)_k \left(\frac{13}{12}\right)_k} \frac{3k+1}{4^k}.$$

Making $a = \frac{1}{2} + m$, $b = \frac{1}{12} + n$ and $c = \frac{11}{12} + p$ in (9), we attain the equation.

Theorem 6. *For $m, n, p \in \mathbb{Z}$ with $\min\{n, p, m-n-p\} \geq 0$, there holds the general summation formula for π with three free parameters:*

$$\frac{5\pi}{6(2+\sqrt{3})} = \frac{1}{\left(-\frac{5}{12}\right)_n \left(\frac{5}{12}\right)_p \left(\frac{1}{2}\right)_{m-n-p}} \sum_{k=0}^{\infty} \frac{\left(-\frac{5}{6}\right)_{k+2n} \left(\frac{5}{6}\right)_{k+2p} (k+m)!(k+2m-2n-2p)!}{\left(\frac{17}{12}\right)_{k+m-n} \left(\frac{7}{12}\right)_{k+m-p} \left(\frac{1}{2}\right)_{k+n+p} k!} \frac{3k+2m}{k+m} \frac{1}{4^{k+m}}.$$

Two examples from Theorem 6 are laid out as follows.

Example 11 (*$m = n = 1$ and $p = 0$ in* Theorem 6).

$$\frac{35\pi}{12(2+\sqrt{3})} = \sum_{k=0}^{\infty} \frac{(1)_k \left(\frac{5}{6}\right)_k \left(\frac{7}{6}\right)_k}{\left(\frac{3}{2}\right)_k \left(\frac{17}{12}\right)_k \left(\frac{19}{12}\right)_k} \frac{3k+2}{4^k}.$$

Example 12 (*$m = 2$ and $n = p = 1$ in* Theorem 5).

$$\frac{5\pi}{12(2+\sqrt{3})} = \sum_{k=1}^{\infty} \frac{(1)_k \left(\frac{1}{6}\right)_k \left(\frac{11}{6}\right)_k}{\left(\frac{3}{2}\right)_k \left(\frac{7}{12}\right)_k \left(\frac{17}{12}\right)_k} \frac{3k+1}{4^k}.$$

Setting $a = \frac{3}{2} + m$, $b = 1 + n$ and $c = 1 + p$ in (9), we get the equation.

Theorem 7. *For $m, n, p \in \mathbb{Z}$ with $\min\{n, p, m-n-p\} \geq 0$, there holds the general summation formula for π^2 with three free parameters:*

$$\pi^2 = \frac{1}{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_p \left(\frac{1}{2}\right)_{m-n-p}} \sum_{k=0}^{\infty} \frac{(k+m)!(k+2n)!(k+2p)!(k+2m-2n-2p)!}{\left(\frac{3}{2}\right)_{k+m-n} \left(\frac{3}{2}\right)_{k+m-p} \left(\frac{3}{2}\right)_{k+n+p} k!} \frac{3k+2m+2}{4^{k+m-1}}.$$

When $m = n = p = 0$, Theorem 7 reduces to (3) exactly. Other two examples of the same type are displayed as follows.

Example 13 *($m = 2$ and $n = p = 1$ in Theorem 7).*

$$\frac{\pi^2}{12} = \sum_{k=2}^{\infty} \frac{(1)_k^3}{\left(\frac{1}{2}\right)_k^2 \left(\frac{3}{2}\right)_k} \frac{k}{4^k}.$$

Example 14 *($m = 2$ and $n = p = 1$ in Theorem 7).*

$$\frac{9\pi^2}{4} = \sum_{k=4}^{\infty} \frac{(1)_k^3}{\left(-\frac{1}{2}\right)_k^2 \left(\frac{3}{2}\right)_k} \frac{3k-2}{4^k}.$$

3. Ramanujan-type series for $1/\pi$ with free parameters implied by Dougall's $_5F_4$-series identity

In this section, three general π-formulas with free parameters will be derived from (5). They include not only (6)–(8) but also many other Ramanujan-type series for $1/\pi$ with free parameters as special cases.

Taking $a = x + m$, $b = x + n$, $c = x + p$ in (5) and then letting $d \to -\infty$, we gain the equation.

Theorem 8. *For $x \in \mathbb{C}$ and $m, n, p \in \mathbb{Z}$ with $\min\left\{\mathrm{Re}(1 + \frac{m-3x}{2} - n - p),\ 1 + m - n,\ 1 + m - p\right\} > 0$, there holds the general π-formula with four free parameters:*

$$\frac{\sin(\pi x)}{\pi} = \frac{(1-x)_{m-n-p}}{(x)_n (x)_p} \sum_{k=0}^{\infty} (-1)^k \frac{(x)_{k+m} (x)_{k+n} (x)_{k+p}}{k!(k+m-n)!(k+m-p)!} (2k + m + x).$$

When $x = 1/2$, Theorem 8 specializes to (6) exactly. Other fourteen Ramanujan-type series for $1/\pi$ with three free parameters from this theorem are laid out in Table 1.

Table 1
Series for $1/\pi$ implied by Theorem 8.

Values of x	Ramanujan-type series for $1/\pi$ with three free parameters
$\frac{1}{6}$	$\frac{3}{\pi} = \frac{(\frac{5}{6})_{m-n-p}}{(\frac{1}{6})_n(\frac{1}{6})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{1}{6})_{k+m}(\frac{1}{6})_{k+n}(\frac{1}{6})_{k+p}}{k!(k+m-n)!(k+m-p)!}(12k+6m+1)$
$\frac{5}{6}$	$\frac{3}{\pi} = \frac{(\frac{1}{6})_{m-n-p}}{(\frac{5}{6})_n(\frac{5}{6})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{5}{6})_{k+m}(\frac{5}{6})_{k+n}(\frac{5}{6})_{k+p}}{k!(k+m-n)!(k+m-p)!}(12k+6m+5)$
$\frac{1}{4}$	$\frac{2\sqrt{2}}{\pi} = \frac{(\frac{3}{4})_{m-n-p}}{(\frac{1}{4})_n(\frac{1}{4})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{1}{4})_{k+m}(\frac{1}{4})_{k+n}(\frac{1}{4})_{k+p}}{k!(k+m-n)!(k+m-p)!}(8k+4m+1)$
$\frac{3}{4}$	$\frac{2\sqrt{2}}{\pi} = \frac{(\frac{1}{4})_{m-n-p}}{(\frac{3}{4})_n(\frac{3}{4})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{3}{4})_{k+m}(\frac{3}{4})_{k+n}(\frac{3}{4})_{k+p}}{k!(k+m-n)!(k+m-p)!}(8k+4m+3)$
$\frac{1}{3}$	$\frac{3\sqrt{3}}{2\pi} = \frac{(\frac{2}{3})_{m-n-p}}{(\frac{1}{3})_n(\frac{1}{3})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{1}{3})_{k+m}(\frac{1}{3})_{k+n}(\frac{1}{3})_{k+p}}{k!(k+m-n)!(k+m-p)!}(6k+3m+1)$
$\frac{2}{3}$	$\frac{3\sqrt{3}}{2\pi} = \frac{(\frac{1}{3})_{m-n-p}}{(\frac{2}{3})_n(\frac{2}{3})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{2}{3})_{k+m}(\frac{2}{3})_{k+n}(\frac{2}{3})_{k+p}}{k!(k+m-n)!(k+m-p)!}(6k+3m+2)$
$\frac{1}{10}$	$\frac{5(\sqrt{5}-1)}{2\pi} = \frac{(\frac{9}{10})_{m-n-p}}{(\frac{1}{10})_n(\frac{1}{10})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{1}{10})_{k+m}(\frac{1}{10})_{k+n}(\frac{1}{10})_{k+p}}{k!(k+m-n)!(k+m-p)!}(20k+10m+1)$
$\frac{3}{10}$	$\frac{5(\sqrt{5}+1)}{2\pi} = \frac{(\frac{7}{10})_{m-n-p}}{(\frac{3}{10})_n(\frac{3}{10})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{3}{10})_{k+m}(\frac{3}{10})_{k+n}(\frac{3}{10})_{k+p}}{k!(k+m-n)!(k+m-p)!}(20k+10m+3)$
$\frac{7}{10}$	$\frac{5(\sqrt{5}+1)}{2\pi} = \frac{(\frac{3}{10})_{m-n-p}}{(\frac{7}{10})_n(\frac{7}{10})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{7}{10})_{k+m}(\frac{7}{10})_{k+n}(\frac{7}{10})_{k+p}}{k!(k+m-n)!(k+m-p)!}(20k+10m+7)$
$\frac{9}{10}$	$\frac{5(\sqrt{5}-1)}{2\pi} = \frac{(\frac{1}{10})_{m-n-p}}{(\frac{9}{10})_n(\frac{9}{10})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{9}{10})_{k+m}(\frac{9}{10})_{k+n}(\frac{9}{10})_{k+p}}{k!(k+m-n)!(k+m-p)!}(20k+10m+9)$
$\frac{1}{12}$	$\frac{3(\sqrt{6}-\sqrt{2})}{\pi} = \frac{(\frac{11}{12})_{m-n-p}}{(\frac{1}{12})_n(\frac{1}{12})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{1}{12})_{k+m}(\frac{1}{12})_{k+n}(\frac{1}{12})_{k+p}}{k!(k+m-n)!(k+m-p)!}(24k+12m+1)$
$\frac{5}{12}$	$\frac{3(\sqrt{6}+\sqrt{2})}{\pi} = \frac{(\frac{7}{12})_{m-n-p}}{(\frac{5}{12})_n(\frac{5}{12})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{5}{12})_{k+m}(\frac{5}{12})_{k+n}(\frac{5}{12})_{k+p}}{k!(k+m-n)!(k+m-p)!}(24k+12m+5)$
$\frac{7}{12}$	$\frac{3(\sqrt{6}+\sqrt{2})}{\pi} = \frac{(\frac{5}{12})_{m-n-p}}{(\frac{7}{12})_n(\frac{7}{12})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{7}{12})_{k+m}(\frac{7}{12})_{k+n}(\frac{7}{12})_{k+p}}{k!(k+m-n)!(k+m-p)!}(24k+12m+7)$
$\frac{11}{12}$	$\frac{3(\sqrt{6}-\sqrt{2})}{\pi} = \frac{(\frac{1}{12})_{m-n-p}}{(\frac{11}{12})_n(\frac{11}{12})_p} \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{11}{12})_{k+m}(\frac{11}{12})_{k+n}(\frac{11}{12})_{k+p}}{k!(k+m-n)!(k+m-p)!}(24k+12m+11)$

Choosing $a = x + m$, $b = x + n$, $c = x + p$ and $d = \frac{1}{2} + q$ in (5), we achieve the equation.

Theorem 9. *For $x \in \mathbb{C}$ and $m, n, p, q \in \mathbb{Z}$ with $\min\left\{\mathrm{Re}(\frac{1}{2} - x + m - n - p - q), 1 + m - n, 1 + m - p\right\} > 0$, there holds the general π-formula with five free parameters:*

$$\frac{\tan(\pi x)}{\pi} = \frac{(1-x)_{m-n-p}\left(\frac{1}{2}\right)_{m-n-q}\left(\frac{1}{2}\right)_{m-p-q}}{\left(\frac{1}{2}-x\right)_{m-n-p-q}(x)_n(x)_p\left(\frac{1}{2}\right)_q}\sum_{k=0}^{\infty}\frac{(x)_{k+m}(x)_{k+n}(x)_{k+p}\left(\frac{1}{2}\right)_{k+q}}{k!(k+m-n)!(k+m-p)!\left(\frac{1}{2}+x\right)_{k+m-q}}(2k+m+x).$$

When $x \to 1/2$, Theorem 9 reduces to (7) exactly. Other ten Ramanujan-type series for $1/\pi$ with four free parameters from this theorem are displayed in Table 2.

Table 2
Series for $1/\pi$ implied by Theorem 9.

Values of x	Ramanujan-type series for $1/\pi$ with four free parameters
$\frac{1}{4}$	$\frac{4}{\pi} = \frac{(\frac{3}{4})_{m-n-p}(\frac{1}{2})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(\frac{1}{4})_{m-n-p-q}(\frac{1}{4})_n(\frac{1}{4})_p(\frac{1}{2})_q}\sum_{k=0}^{\infty}\frac{(\frac{1}{4})_{k+m}(\frac{1}{4})_{k+n}(\frac{1}{4})_{k+p}(\frac{1}{2})_{k+q}}{k!(k+m-n)!(k+m-p)!(\frac{3}{4})_{k+m-q}}(8k+4m+1)$
$\frac{3}{4}$	$\frac{1}{\pi} = \frac{(\frac{1}{4})_{m-n-p}(\frac{1}{2})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(\frac{3}{4})_{m-n-p-q-1}(\frac{3}{4})_n(\frac{3}{4})_p(\frac{1}{2})_q}\sum_{k=0}^{\infty}\frac{(\frac{3}{4})_{k+m}(\frac{3}{4})_{k+n}(\frac{3}{4})_{k+p}(\frac{1}{2})_{k+q}}{k!(k+m-n)!(k+m-p)!(\frac{5}{4})_{k+m-q}}(8k+4m+3)$
$\frac{1}{3}$	$\frac{3\sqrt{3}}{\pi} = \frac{(\frac{2}{3})_{m-n-p}(\frac{1}{2})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(\frac{1}{6})_{m-n-p-q}(\frac{1}{3})_n(\frac{1}{3})_p(\frac{1}{2})_q}\sum_{k=0}^{\infty}\frac{(\frac{1}{3})_{k+m}(\frac{1}{3})_{k+n}(\frac{1}{3})_{k+p}(\frac{1}{2})_{k+q}}{k!(k+m-n)!(k+m-p)!(\frac{5}{6})_{k+m-q}}(6k+3m+1)$
$\frac{2}{3}$	$\frac{\sqrt{3}}{2\pi} = \frac{(\frac{1}{3})_{m-n-p}(\frac{1}{2})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(\frac{5}{6})_{m-n-p-q-1}(\frac{2}{3})_n(\frac{2}{3})_p(\frac{1}{2})_q}\sum_{k=0}^{\infty}\frac{(\frac{2}{3})_{k+m}(\frac{2}{3})_{k+n}(\frac{2}{3})_{k+p}(\frac{1}{2})_{k+q}}{k!(k+m-n)!(k+m-p)!(\frac{7}{6})_{k+m-q}}(6k+3m+2)$
$\frac{1}{6}$	$\frac{2\sqrt{3}}{\pi} = \frac{(\frac{5}{6})_{m-n-p}(\frac{1}{2})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(\frac{1}{3})_{m-n-p-q}(\frac{1}{6})_n(\frac{1}{6})_p(\frac{1}{2})_q}\sum_{k=0}^{\infty}\frac{(\frac{1}{6})_{k+m}(\frac{1}{6})_{k+n}(\frac{1}{6})_{k+p}(\frac{1}{2})_{k+q}}{k!(k+m-n)!(k+m-p)!(\frac{2}{3})_{k+m-q}}(12k+6m+1)$
$\frac{5}{6}$	$\frac{2\sqrt{3}}{3\pi} = \frac{(\frac{1}{6})_{m-n-p}(\frac{1}{2})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(\frac{2}{3})_{m-n-p-q-1}(\frac{5}{6})_n(\frac{5}{6})_p(\frac{1}{2})_q}\sum_{k=0}^{\infty}\frac{(\frac{5}{6})_{k+m}(\frac{5}{6})_{k+n}(\frac{5}{6})_{k+p}(\frac{1}{2})_{k+q}}{k!(k+m-n)!(k+m-p)!(\frac{4}{3})_{k+m-q}}(12k+6m+5)$
$\frac{1}{12}$	$\frac{12(2-\sqrt{3})}{\pi} = \frac{(\frac{11}{12})_{m-n-p}(\frac{1}{2})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(\frac{5}{12})_{m-n-p-q}(\frac{1}{12})_n(\frac{1}{12})_p(\frac{1}{2})_q}\sum_{k=0}^{\infty}\frac{(\frac{1}{12})_{k+m}(\frac{1}{12})_{k+n}(\frac{1}{12})_{k+p}(\frac{1}{2})_{k+q}}{k!(k+m-n)!(k+m-p)!(\frac{7}{12})_{k+m-q}}(24k+12m+1)$
$\frac{5}{12}$	$\frac{12(2+\sqrt{3})}{\pi} = \frac{(\frac{7}{12})_{m-n-p}(\frac{1}{2})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(\frac{1}{12})_{m-n-p-q}(\frac{5}{12})_n(\frac{5}{12})_p(\frac{1}{2})_q}\sum_{k=0}^{\infty}\frac{(\frac{5}{12})_{k+m}(\frac{5}{12})_{k+n}(\frac{5}{12})_{k+p}(\frac{1}{2})_{k+q}}{k!(k+m-n)!(k+m-p)!(\frac{11}{12})_{k+m-q}}(24k+12m+5)$
$\frac{7}{12}$	$\frac{2+\sqrt{3}}{\pi} = \frac{(\frac{5}{12})_{m-n-p}(\frac{1}{2})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(\frac{11}{12})_{m-n-p-q-1}(\frac{7}{12})_n(\frac{7}{12})_p(\frac{1}{2})_q}\sum_{k=0}^{\infty}\frac{(\frac{7}{12})_{k+m}(\frac{7}{12})_{k+n}(\frac{7}{12})_{k+p}(\frac{1}{2})_{k+q}}{k!(k+m-n)!(k+m-p)!(\frac{13}{12})_{k+m-q}}(24k+12m+7)$
$\frac{11}{12}$	$\frac{5(2-\sqrt{3})}{\pi} = \frac{(\frac{1}{12})_{m-n-p}(\frac{1}{2})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(\frac{7}{12})_{m-n-p-q-1}(\frac{11}{12})_n(\frac{11}{12})_p(\frac{1}{2})_q}\sum_{k=0}^{\infty}\frac{(\frac{11}{12})_{k+m}(\frac{11}{12})_{k+n}(\frac{11}{12})_{k+p}(\frac{1}{2})_{k+q}}{k!(k+m-n)!(k+m-p)!(\frac{17}{12})_{k+m-q}}(24k+12m+11)$

Making $a = \frac{1}{2} + m$, $b = \frac{1}{2} + n$, $c = x + p$ and $d = 1 - x + q$ in (5), we attain the equation.

Theorem 10. *For $x \in \mathbb{C}$ and $m, n, p, q \in \mathbb{Z}$ with $\min\{m - n, m - n - p - q - 1\} \geq 0$, there holds the general π-formula with five free parameters:*

$$\frac{(1-2x)\tan(\pi x)}{\pi} = \frac{(1-x)_{m-n-p}(x)_{m-n-q}\left(\frac{1}{2}\right)_{m-p-q}}{(m-n-p-q-1)!\left(\frac{1}{2}\right)_n(x)_p(1-x)_q}$$
$$\times \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_{k+m}\left(\frac{1}{2}\right)_{k+n}(x)_{k+p}(1-x)_{k+q}}{k!(k+m-n)!\left(\frac{3}{2}-x\right)_{k+m-p}\left(\frac{1}{2}+x\right)_{k+m-q}}(4k+2m+1).$$

When $x \to 1/2$ and $x = 1/3$, Theorem 10 specializes to (7) and (8) respectively. Other four Ramanujan-type series for $1/\pi$ with four free parameters from this theorem are laid out in Table 3.

Table 3
Series for $1/\pi$ implied by Theorem 10.

Values of x	Ramanujan-type series for $1/\pi$ with four free parameters
$\frac{1}{4}$	$\frac{1}{2\pi} = \frac{(\frac{3}{4})_{m-n-p}(\frac{1}{4})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(m-n-p-q-1)!(\frac{1}{2})_n(\frac{1}{4})_p(\frac{3}{4})_q}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_{k+m}(\frac{1}{2})_{k+n}(\frac{1}{4})_{k+p}(\frac{3}{4})_{k+q}}{k!(k+m-n)!(\frac{5}{4})_{k+m-p}(\frac{3}{4})_{k+m-q}}(4k+2m+1)$
$\frac{1}{6}$	$\frac{2\sqrt{3}}{9\pi} = \frac{(\frac{5}{6})_{m-n-p}(\frac{1}{6})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(m-n-p-q-1)!(\frac{1}{2})_n(\frac{1}{6})_p(\frac{5}{6})_q}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_{k+m}(\frac{1}{2})_{k+n}(\frac{1}{6})_{k+p}(\frac{5}{6})_{k+q}}{k!(k+m-n)!(\frac{4}{3})_{k+m-p}(\frac{2}{3})_{k+m-q}}(4k+2m+1)$
$\frac{1}{12}$	$\frac{5(2-\sqrt{3})}{6\pi} = \frac{(\frac{11}{12})_{m-n-p}(\frac{1}{12})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(m-n-p-q-1)!(\frac{1}{2})_n(\frac{1}{12})_p(\frac{11}{12})_q}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_{k+m}(\frac{1}{2})_{k+n}(\frac{1}{12})_{k+p}(\frac{11}{12})_{k+q}}{k!(k+m-n)!(\frac{17}{12})_{k+m-p}(\frac{7}{12})_{k+m-q}}(4k+2m+1)$
$\frac{5}{12}$	$\frac{2+\sqrt{3}}{6\pi} = \frac{(\frac{7}{12})_{m-n-p}(\frac{5}{12})_{m-n-q}(\frac{1}{2})_{m-p-q}}{(m-n-p-q-1)!(\frac{1}{2})_n(\frac{5}{12})_p(\frac{7}{12})_q}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_{k+m}(\frac{1}{2})_{k+n}(\frac{5}{12})_{k+p}(\frac{7}{12})_{k+q}}{k!(k+m-n)!(\frac{13}{12})_{k+m-p}(\frac{11}{12})_{k+m-q}}(4k+2m+1)$

Besides those formulas displayed in Tables 1–3, Theorems 8–10 can give more Ramanujan-type series for $1/\pi$ with free parameters with the change of x. We shall not lay them out one by one in the paper.

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