Journal of Computational and Applied Mathematics 137 (2001) 71–76

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

On mean convergence of Hermite–Fejér and Hermite interpolation for Erdős weights

S.B. Damelin[a,*], H.S. Jung[b], K.H. Kwon[b]

[a] Department of Mathematics and Computer Science, Georgia Southern University, Post Office Box 8093, Statesboro, GA 30460-8093, USA

[b] Division of Applied Mathematics, KAIST, Taejon 305-701, South Korea

Received 10 February 2000; received in revised form 10 October 2000

Abstract

We investigate convergence of Hermite–Fejér and Hermite interpolation polynomials in L_p $(0<p<\infty)$ for Erdős weights. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 41A10; 42C05

Keywords: Erdős weight; Hermite–Fejér interpolation; Hermite interpolation; Mean convergence; Weighted approximation

1. Introduction and statement of main result

For a function $f:(a,b)\to\mathbb{R}$, $-\infty\leqslant a<b\leqslant\infty$ and a set

$$\chi_n:=\{x_{1n},x_{2n},\ldots,x_{nn}\},\quad n\geqslant 1$$

of pairwise distinct nodes, let $H_n[\chi_n;f]$ and $\hat{H}_n[\chi_n;f]$ denote the Hermite–Fejér interpolation polynomial and Hermite interpolation polynomial of degrees $\leqslant 2n-1$ to f with respect to χ_n when defined. Formally, $H_n[\chi_n;f]$ and $\hat{H}_n[\chi_n;f]$ are the unique polynomials of degree $\leqslant 2n-1$ satisfying

$$H_n[\chi_n;f](x_{jn})=f(x_{jn}),\quad H_n'[\chi_n;f](x_{jn})=0$$

and

$$\hat{H}_n[\chi_n;f](x_{jn})=f(x_{jn}),\quad \hat{H}_n'[\chi_n;f](x_{jn})=f'(x_{jn})$$

for $j=1,2,\ldots,n$.

* Corresponding author.

E-mail addresses: damelin@gsu.cs.gasou.edu (S.B. Damelin), hsjung@math.kaist.ac.kr (H.S. Jung), khkwon@jacobi.kaist.ac.kr (K.H. Kwon).

PII: S0377-0427(00)00698-1

The subject of convergence of Hermite, Hermite–Fejér and Lagrange interpolation polynomials for various systems of nodes, is a well established and widely studied subject. See [1–12,14,15, 17–25] and the references cited therein. Recently, there has been a resurgence of interest in studying convergence questions for functions which are defined on the real line and may possibly be unbounded there. As polynomials are unbounded on the real line, it has become customary to study convergence questions in suitable weighted function spaces. Until now, the choice of positive even weights

$$w : \mathbb{R} \to (0, \infty)$$

studied has been highly influenced by the following density theorem whose form below is due to Carleson and Dzrabajan (see [16]).

2. Density on the real line

Let $w := \exp(-Q)$, where $Q : \mathbb{R} \to \mathbb{R}$ is even and $Q(e^x)$ is convex in $(0, \infty)$. Then the following are equivalent:

(a) For every continuous function $f : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|x| \to \infty} (fw)(x) = 0$$

and for every $\varepsilon > 0$, there exists a polynomial P such that

$$\|(f - P)w\|_{L_\infty(\mathbb{R})} < \varepsilon.$$

(b)

$$\int_{\mathbb{R}} \frac{Q(x)}{1 + x^2}\, dx = \infty.$$

Thus, in order to study convergence questions on the real line, we are already restricted in that our choice of weight w should decay sufficiently fast near $\pm\infty$. For this reason, it has been helpful to distinguish between those classes of weights that are of polynomial growth at infinity, the so-called *Freud* class and those that are of faster than polynomial growth at infinity, the so-called *Erdős* class. See [13,15] and the references cited therein.

In this paper, we study mean convergence of Hermite and Hermite–Fejér interpolation in L_p $(0 < p < \infty)$ for a class of fast decaying even Erdős weights on the real line. The underlying investigation extends earlier work [20], where the case $p = 1$ was covered. For Freud-type weights, results similar to ours appear in [12]. To this end, we limit ourselves to weights and functions that cover natural examples and treat the general L_p theory which is typically more difficult. As a by-product of our investigation, we are, in some cases, able to recover the results of Lubinsky and Rabinowitz [20] under weaker assumptions than those appearing in that paper. The main feature of our weights is that they are even and of faster than smooth polynomial decay at infinity. For example, our results will cover the natural examples (see [13]),

$$w_{k,\alpha}(x) := \exp(-Q_{k,\alpha}(x)),$$

where

$$Q_{k,\alpha}(x) := \exp_k(|x|^\alpha), \quad k \geqslant 1, \ \alpha > 1$$

and

$$w_{A,B}(x) := \exp(-Q_{A,B}(x)),$$

where

$$Q_{A,B}(x) = \exp(\log(A + x^2))^B, \quad B > 1.$$

Here, $\exp_k(;) = \exp(\exp(\cdots(;)))$ denotes the kth iterated exponential and A is a large enough but fixed absolute constant.

More precisely, our results hold for the following subclass of weights w from [13] of which the above weights are natural examples and for which sharp enough estimates for $p_n(w^2)$ (see below) and their zeroes are obtained.

3. Class of weights

Definition 1. Let $w := \exp(-Q)$, where $Q : \mathbb{R} \to \mathbb{R}$ is even and continuous; $Q^{(j)} \geqslant 0$ in $(0, \infty)$ for $j = 0, 1, 2$; and the function

$$T(x) := 1 + xQ''(x)/Q'(x)$$

is increasing in $(0, \infty)$ with

$$\lim_{x \to \infty} T(x) = \infty, \quad T(0+) := \lim_{x \to 0+} T(x) > 1.$$

Assume that for some constants $C_1, C_2, C_3, C_4 > 0$ independent of x,

$$C_1 \leqslant T(x) \Bigg/ \left(\frac{xQ'(x)}{Q(x)} \right) \leqslant C_2, \quad x \geqslant C_3 \tag{1}$$

and

$$T(x) \leqslant C_4(Q(x))^{\varepsilon}, \quad x \to \infty. \tag{2}$$

We shall denote by $\mathscr{E}$, the class of all such weights.

4. Interpolation points

As a suitable set of interpolation points, we recall (see [9]) that given $w \in \mathscr{E}$ as above, there exists a unique system of orthonormal polynomials

$$p_n(w^2, x) = \gamma_n(w^2) x^n + \cdots, \quad \gamma_n(w^2) > 0; \ x \in \mathbb{R}$$

satisfying

$$\int_{\mathbb{R}} p_n(w^2, x) p_m(w^2, x) w^2(x) \, \mathrm{d}x = \delta_{mn}, \quad m, n = 0, 1, 2, \ldots$$

with n real and simple zeroes which we order as

$$x_{n,n}(w^2) < x_{n-1,n}(w^2) < \cdots < x_{2,n}(w^2) < x_{1,n}(w^2).$$

These zeroes will serve as our interpolation points which will again be denoted by χ_n.

Recently in [12], two of us found sufficient conditions for weighted mean convergence of Hermite and Hermite–Fejér interpolation polynomials in L_p $(0<p<\infty)$, for a large class of even Freud weights with respect to χ_n. For notational simplicity, we recall the main result of that paper for

$$w(x):=\exp(-|x|^{\beta}),\quad \beta>1,\ x\in\mathbb{R}.$$

5. Freud weights

Theorem 1. *Let $0<p<\infty$, $\Delta\in\mathbb{R}$, $\alpha>0$, $\hat{\alpha}:=\max\{\alpha,1\}$ and let $H_n[;]$ and $\hat{H}_n[;]$ be the Hermite–Fejér and Hermite interpolation polynomials with respect to the zeroes of $p_n(w^2)$. Assume that for $0<p\leqslant 2$,*

$$\Delta>-\hat{\alpha}+1/p$$

and for $p>2$

$$-1/\beta(\hat{\alpha}+\Delta+1/p)+1/3(1-2/p)<0.$$

Then,

$$\lim_{n\to\infty}\|(f(x)-H_n[f](x))w^2(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}=0$$

for every continuous function $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$\lim_{|x|\to\infty}|f(x)|w^2(x)(1+|x|^{\alpha})=0.$$

Moreover,

$$\lim_{n\to\infty}\|(f(x)-\hat{H}_n[f](x))w^2(x)(1+|x|)^{-\Delta}\|_{L_p(\mathbb{R})}=0$$

for $f\in C^1(\mathbb{R})$ satisfying the above and

$$\sup_{x\in\mathbb{R}}|f'(x)|w^2(x)(1+|x|^{\alpha})<\infty.$$

Our aim in this paper is to extend the results of Jung and Kwon [12] to the class $\mathscr{E}$. Using the methods of Jung and Kwon [12] and applying the well-known estimates of Levin et al. [13], we are able to prove:

6. Erdős weights

Theorem 2. *Let $w\in\mathscr{E}$, $0<p<\infty$, $\Delta\in\mathbb{R}$, and $\kappa>0$. Assume that*

$$\Delta>\max\left\{0,\frac{2}{3}\left(\frac{1}{2}-\frac{1}{p}\right)\right\}. \tag{3}$$

Then

$$\lim_{n\to\infty}\|(f(x)-H_n[f](x))w^2(x)(1+Q(x))^{-\Delta}\|_{L_p(\mathbb{R})}=0$$

for every continuous function $f:\mathbb{R}\to\mathbb{R}$ *satisfying*

$$\lim_{|x|\to\infty} |f(x)|w^2(x)(\log(|x|^{1+\kappa})T(x)^{1/2}) = 0. \tag{4}$$

Moreover,

$$\lim_{n\to\infty} \|(f(x)-\hat{H}_n[f](x))w^2(x)(1+Q(x))^{-\Delta}\|_{L_p(\mathbb{R})} = 0$$

for $f\in C^1(\mathbb{R})$ *satisfying* (4) *and*

$$\sup_{x\in\mathbb{R}} |f'(x)|w^2(x)(\log(|x|^{1+\kappa}T(x)^{1/2}) < \infty. \tag{5}$$

Our main observation is that for Erdős weights and our given set of interpolation points χ_n, a weighting factor which decays as a negative power of $(1+Q)$, is sufficient for mean convergence of Hermite and Hermite–Fejér interpolation. A similar effect occurs for Lagrange interpolation. See [7,8,3]. This is in sharp contrast to Freud-type weights, where for the given system of nodes, a polynomial decay factor is sufficient. See Theorem 1. Recently, for weighted Lagrange interpolation on the real line and $[-1,1]$, an extended system of interpolation nodes has been found to yield better results for mean and uniform convergence. See [24,1,2,4–6,17–19] and the references cited therein. It is still an open question as to whether this extended system of nodes will yield better mean convergence results for positive operators such as ours. Concerning uniform convergence for Freud weights, the answer is no (see [24]).

Acknowledgements

S.B. Damelin is supported by a faculty grant from Georgia Southern University. H.S. Jung and K.H. Kwon are partially supported by Korea Ministry of Education (BK-21) and KOSEF (99-2-101-001-5).

References

[1] S.B. Damelin, The Lebesgue function and Lebesgue constant of Lagrange interpolation for Erdős weights, J. Approx. Theory 94 (2) (1998) 235–262.
[2] S.B. Damelin, The weighted Lebesgue constant of Lagrange interpolation for exponential weights on $[-1,1]$, Acta Math. Hungar. 81 (3) (1998) 211–228.
[3] S.B. Damelin, H.S. Jung, K.H. Kwon, On mean convergence of Lagrange interpolation in $L_p\,(0<p<1)$, postscript.
[4] S.B. Damelin, H.S. Jung, K.H. Kwon, Converse Marcinkiewicz–Zygmund inequalities on the real line with applications to mean convergence of Lagrange interpolation, postscript.
[5] S.B. Damelin, H.S. Jung, K.H. Kwon, A note on mean convergence of extended Lagrange interpolation for Erdős weights, postscript.
[6] S.B. Damelin, H.S. Jung, K.H. Kwon, Necessary conditions for weighted mean convergence of Lagrange interpolation for exponential weights, J. Comput. Appl. Math. 132 (2001) 357–369.
[7] S.B. Damelin, D.S. Lubinsky, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdős weights, Canad. J. Math. 48 (4) (1996) 710–736.
[8] S.B. Damelin, D.S Lubinsky, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Erdős weights II, Canad. J. Math. 48 (4) (1996) 737–757.
[9] G. Freud, Orthogonal Polynomials, Pergamon Press, Oxford, 1971.

[10] G. Freud, On Hermite–Fejér interpolation process, Studia Sci. Math. Hungar. 7 (1972) 307–316.
[11] G. Freud, On Hermite–Fejér interpolation sequences, Acta Math. Acad. Sci. Hungar. 23 (1972) 175–178.
[12] H.S. Jung, K.H. Kwon, Mean convergence of Hermite–Fejér and Hermite interpolation for Freud weights, J. Comput. Appl. Math. 99 (1–2) (1998) 219–238.
[13] A.L. Levin, D.S. Lubinsky, T.Z. Mthembu, Christoffel functions and orthogonal polynomials for Erdős weights on $(-\infty,\infty)$, Rend. Mat. Appl. 14 (7) (1994) 199–289.
[14] D.S. Lubinsky, Hermite and Hermite–Fejér interpolation and associated product integration rules on the real line: L_∞ theory, J. Approx. Theory 70 (1992) 284–334.
[15] D.S. Lubinsky, An update on orthogonal polynomials and weighted approximation on the real line, Acta Appl. Math. 33 (1993) 121–164.
[16] D.S. Lubinsky, Weierstrass' theorem in the twentieth century: A selection, Quastiones Math. 18 (1995) 91–130.
[17] D.S. Lubinsky, Mean convergence of Lagrange interpolation for exponential weights on $[-1,1]$, Canad. J. Math. 50 (6) (1998) 1273–1297.
[18] D.S. Lubinsky, On converse Marcinkiewicz–Zygmund inequalities in L_p, $p>1$, Constr. Approx. 15 (1999) 577–610.
[19] D.S. Lubinsky, G. Mastroianni, Mean convergence of extended Lagrange interpolation with Freud weights, Acta Sci. Math. 84 (1–2) (1999) 47–63.
[20] D.S. Lubinsky, P. Rabinowitz, Hermite and Hermite–Fejér interpolation and associated product integration rules on the real line: L_1 theory, Canad. J. Math. 44 (1992) 561–590.
[21] H.N. Mhaskar, Introduction to the theory of weighted polynomial approximation, Series in Approximations and Decompositions, Vol. 7, World Scientific, Singapore, 1996.
[22] P. Nevai, Geza Freud, orthogonal polynomials and Christoffel functions: a case study, J. Approx. Theory 48 (1986) 3–167.
[23] P. Nevai, P. Vertesi, Mean convergence of Hermite–Fejér interpolation, J. Math. Anal. Appl. 105 (1985) 26–58.
[24] J. Szabados, Weighted Lagrange interpolation and Hermite–Fejér interpolation on the real line, J. Inequ. Appl. 1 (1997) 99–123.
[25] J. Szabados, P. Vértesi, Interpolation of Functions, World Scientific, Singapore, 1991.