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Convex drawings of hierarchical planar graphs and clustered planar graphs

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ABSTRACT

In this paper, we present results on convex drawings of *hierarchical graphs* and *clustered graphs*. A *convex drawing* is a planar straight-line drawing of a plane graph, where every facial cycle is drawn as a convex polygon. Hierarchical graphs and clustered graphs are useful graph models with structured relational information. Hierarchical graphs are graphs with layering structures; clustered graphs are graphs with recursive clustering structures.

We first present the necessary and sufficient conditions for a hierarchical plane graph to admit a convex drawing. More specifically, we show that the necessary and sufficient conditions for a biconnected plane graph due to Thomassen [C. Thomassen, Plane representations of graphs, in: J.A. Bondy, U.S.R. Murty (Eds.), Progress in Graph Theory, Academic Press, 1984, pp. 43–69] remains valid for the case of a hierarchical plane graph. We then prove that every internally triconnected clustered plane graph with a completely connected clustering structure admits a "fully convex drawing," a planar straight-line drawing such that both clusters and facial cycles are drawn as convex polygons. We also present algorithms to construct such convex drawings of hierarchical graphs and clustered graphs.

1. Introduction

Graph drawing has attracted much attention for the last twenty years due to its wide range of applications such as VLSI design, software engineering and bioinformatics. Two or three dimensional drawings of graphs with a variety of aesthetics and edge representations have been extensively studied (see [4,12,16]). One of the most popular drawing conventions is the *straight-line drawing*, where all the edges of a graph are drawn as straight-line segments. Every planar graph is known to have a planar straight-line drawing [8,21].

A straight-line drawing is called a *convex drawing* if every facial cycle is drawn as a convex polygon. Note that not all planar graphs admit a convex drawing. Tutte [19,20] showed that every *triconnected* plane graph which boundary (i.e., the outer facial cycle) is drawn as a convex polygon admits a convex drawing using the polygon, where such a drawing is called an *extension* of the polygon. He also proposed a "barycenter mapping" method which computes a convex drawing of a triconnected plane graph with n vertices by solving a system of $O(n)$ linear equations, which can be implemented in $O(n^{1.5})$ time at best. Thomassen [18] gave a necessary and sufficient condition for a *biconnected* plane graph which boundary is drawn as a convex polygon P to admit a convex drawing as an extension of P. Based on this result, Chiba et al. [2] presented a linear-time algorithm for constructing a convex drawing (if any) for a biconnected plane graph which boundary is drawn as a specified convex polygon.

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In general, the convex drawing problem has been well investigated by the Graph Drawing community. For example, a convex drawing is called a *convex grid drawing* if all the vertices are restricted to be placed on grid points. Chrobak and Kant [1] showed that every triconnected plane graph has a convex grid drawing on an $(n-2) \times (n-2)$ grid, and such a grid drawing can be found in linear time. Miura et al. [14] presented a linear-time algorithm for finding a convex grid drawing of four-connected plane graphs with four or more vertices on the outer face. Miura et al. [13] studied another variation of convex drawing problem with minimum outer apices, and Kamada et al. [11] investigated a problem of convex grid drawings with rectangular contours.

Recently, we introduced a new problem of drawing planar graphs with *non-convex* boundary constraints. A straight-line drawing is called an *inner-convex drawing* if every inner facial cycle is drawn as a convex polygon [9]. We proved that every triconnected plane graph admits an inner-convex drawing if its boundary is fixed with a star-shaped polygon P, i.e., a polygon P whose kernel has a positive area [9]. Note that this is an extension of the classical result by Tutte [19], since any convex polygon is a star-shaped polygon. We also presented a linear-time algorithm for computing an inner-convex drawing of a triconnected plane graph with a star-shaped boundary.

In this paper, we present new results on convex drawings of *hierarchical graphs* and *clustered graphs*. Hierarchical graphs and clustered graphs are useful graph models with structured relational information. Hierarchical graphs are graphs with layering structures; clustered graphs are graphs with recursive clustering structures. Both have applications in VLSI design, CASE tools, software visualisation and visualisation of social networks and biological networks [5].

Hierarchical graphs (sometimes called *level* graphs) are directed graphs with vertices assigned into *layers* (or levels). Hierarchical graphs are drawn with vertices of a layer on the same horizontal line, and edges as curves monotonic in y direction. A hierarchical graph is *hierarchical planar* (*h-planar, in short*) *or level-planar*, if it admits a layered level drawing without edge crossings. Note that the hierarchical layering structure in hierarchical graphs imposes constraints on the y-coordinate. It was known that every hierarchical planar graph admits a *straight-line drawing* by Eades et al. [5], however convex representations of hierarchical graphs has not been studied.

We first present the necessary and sufficient conditions for a hierarchical plane graph whose boundary is drawn as a convex polygon P to admit a convex drawing as an extension of P. Interestingly, we prove that the necessary and sufficient conditions for a biconnected plane graph due to Thomassen [18] remains valid for the case of a hierarchical plane graph. We also present an algorithm which constructs such a convex drawing of a hierarchical plane graph, by finding a suitable x-coordinate of each inner vertex.

We then extend our results to convex representations of clustered planar graphs. A *clustered graph* $C = (G, T)$ consists of an undirected graph (called the *underlying graph*) $G = (V, E)$ and a rooted tree (called the *inclusion tree* of C) $T = (\mathcal{V}, \mathcal{A})$, where the leaves of T are exactly the vertices of G [5]. Each node ν of T represents a *cluster* $V(\nu)$, a subset of the vertices of G that are leaves of the subtree rooted at ν. A clustered graph $C = (G, T)$, originally introduced by Eades et al. [5], is a *connected clustered graph*, if each cluster $V(\nu)$ induces a connected subgraph $G(\nu)$ of G. A clustered graph $C = (G, T)$ is *completely connected* if, for every non-root node ν of T, both subgraphs $G(\nu)$ and $G[V - V(\nu)]$ are connected, as defined by Cornelsen and Wagner [3].

In a *drawing* of a clustered graph $C = (G, T)$, for each node ν of T, the cluster is drawn as a simple closed region $R(\nu)$ enclosed by a simple closed curve such that the drawing of $G(\nu)$ is completely contained in the interior of $R(\nu)$, the regions for all sub-clusters of ν are completely contained in the interior of $R(\nu)$, and the regions for all other clusters are completely contained in the exterior of $R(\nu)$. A clustered graph is *compound planar* (*c-planar*) if it admits a c-planar drawing without edge crossings and *edge-region crossings* (i.e. the drawing of e crosses the boundary of region R more than once). Eades et al. [5] proved that every c-planar clustered graph admits a *straight-line drawing* such that the clusters are drawn as convex polygons. In addition, such a drawing can be chosen so that the boundary of the subgraph induced by each cluster is drawn as a convex polygon, if each cluster induces a biconnected subgraph [15]. However, in these drawings only clusters were drawn as convex polygons [5].

In this paper, we call a planar straight-line drawing of a clustered planar graph a *fully convex drawing* if both clusters and facial cycles are drawn as convex polygons. We first present the necessary and sufficient conditions for a clustered plane graph to admit a fully convex drawing. More specifically, we prove that every *internally triconnected* clustered plane graph with a completely connected clustering structure admits a fully convex drawing. We also present an algorithm to construct a convex drawing of clustered planar graphs.

This paper is organized as follows: Section 2 defines the basic terminology, and reviews the necessary and sufficient conditions for a biconnected plane graph to admit a convex drawing. Section 3 proves our main theorem on convex drawings of hierarchical planar graphs and presents an algorithm for constructing such a drawing. In Section 4, we prove our second theorem on convex drawings of clustered planar graphs and present an algorithm for constructing such a drawing. Section 5 makes some concluding remarks.

2. Preliminaries

2.1. Plane graphs and connectivity

Throughout the paper, a graph stands for a simple undirected graph unless stated otherwise. Let $G = (V, E)$ be a graph. The set of edges incident to a vertex $v \in V$ is denoted by $E(v)$. The degree of a vertex v in G is denoted by $d_G(v)$ (i.e.,

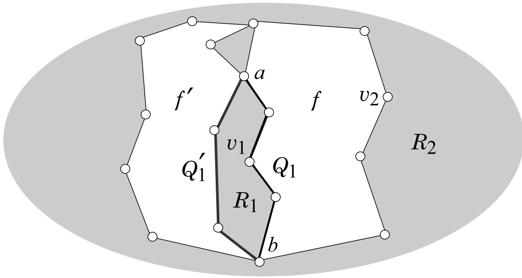

Fig. 1. Illustration of a subpath Q of an inner facial cycle f, where the outer facial cycle may be included in the region R_1.

$d_G(v) = |E(v)|$). For a subset $X \subseteq V$, $G[X]$ denotes the subgraph induced by X (i.e., graph $(X, E - \bigcup_{v \in V-X} E(v))$), and $G - X$ denotes subgraph $G[V - X]$. For a subset $E' \subseteq E$, $G - E'$ denotes the graph obtained from G by removing the edges in E'.

A vertex in a connected graph is called a *cut vertex* if its removal from G results in a disconnected graph. A connected graph is called *biconnected* if it is simple and has no cut vertex. Similarly, a pair of vertices in a biconnected graph is called a *cut pair* (or *separation pair*) if its removal from G results in a disconnected graph. A biconnected graph is called *triconnected* if it is simple and has no cut pair. We say that a cut pair $\{u, v\}$ *separates* two vertices s and t if s and t belong to different connected components in $G - \{u, v\}$.

A graph $G = (V, E)$ is called *planar* if its vertices and edges are drawn as points and curves in the plane so that no two curves intersect except at their endpoints, where no two vertices are drawn at the same point and no vertices are on the curve. Such a planar drawing defines a *plane embedding* (embedding, for short) of a planar graph G. An embedding of G can be specified by the circular ordering of the incident edges for each vertex v that appear in the plane, and an outer face. In an embedding of G, the plane is divided into several connected regions, called *faces*, and the set of faces is denoted by F, where we denote the outer face by f_o. A face is characterised by the cycle of G that surrounds the region. Such a cycle is called a *facial cycle*. A *plane* graph $G = (V, E, F)$ is a planar graph $G = (V, E)$ with a fixed embedding of G.

A vertex (respectively, an edge) in the outer facial cycle f_o in a plane graph G is called an *outer vertex* (respectively, an *outer edge*), while a vertex (respectively, an edge) not in f_o is called an *inner vertex* (respectively, an *inner edge*). A path Q between two vertices s and t in G is called *inner* if every vertex in $V(Q) - \{s, t\}$ is an inner vertex. The region enclosed by a facial cycle $f \in F$ may be denoted by f for simplicity. A plane graph is called *inner-triangulated* if all inner facial cycles are triangles. The set of vertices, set of edges and set of facial cycles of a plane graph G may be denoted by $V(G)$, $E(G)$ and $F(G)$, respectively.

A biconnected plane graph G is called *internally triconnected* if, for every cut pair $\{u, v\}$, u and v are outer vertices and each connected component in $G - \{u, v\}$ contains an outer vertex. Note that every inner vertex in an internally triconnected plane graph must be of degree at least 3. For a cut pair $\{u, v\}$ of an internally triconnected plane graph $G = (V, E, F)$ such that u and v are not adjacent, there is an inner facial cycle $f \in F$ such that $\{u, v\} \in V(f)$. We say that f *separates* two vertices s and t if the cut pair $\{u, v\}$ separates them.

2.2. Archfree paths

In this section, we review definitions of *archfree paths*, which were introduced to construct inner convex drawings of triconnected plane graphs [9], and play an important role in our convex drawing algorithm.

We say that a facial cycle f *arches* a path Q in a plane graph if there are two distinct vertices $a, b \in V(Q) \cap V(f)$ such that the subpath $Q_{a,b}$ of Q between a and b is not a subpath of f. A path Q is called *archfree* if no inner facial cycle f arches Q.

Lemma 1. *Let G be an internally triconnected plane graph G, and f be an inner facial cycle of G. Any subpath Q of f with $|E(Q)| \leqslant |E(f)| - 2$ is an archfree path.*

Proof. To derive a contradiction, assume that an inner face f' arches Q. Let Q_1 be a minimal subpath of Q that is arched by f', and let $a, b \in V(Q) \cap V(f')$ be the end vertices of Q_1. Since $|E(Q)| \leqslant |E(f)| - 2$, the subpath $Q_2 = f - (V(Q_1) - \{a, b\})$ of the facial cycle f contains at least one vertex v_2 other than a and b (see Fig. 1).

Consider the region R which consists of the interior of f and f' joined with vertices a and b. Then the exterior of R consists of two regions R_1 and R_2, where we assume without loss of generality that $Q_1 - \{a, b\}$ belongs to R_1 (if $V(Q_1) - \{a, b\} \neq \emptyset$) and $Q_2 - \{a, b\}$ belongs to R_2. By the minimality of Q_1, R_1 contains a cycle C which consists of Q_1 and a subpath Q_1' of f' such that $V(Q_1') \cap V(Q_1) = \{a, b\}$ and $V(C)$ contains at least one vertex other than a and b (since f' arches Q_1). Hence R_1 contains a vertex v_1 other than a and b even if $V(Q_1) - \{a, b\} = \emptyset$.

We see that $\{a, b\}$ is a cut pair in G, which separates the component G_1 containing v_1 from the component G_2 containing v_2 in $G - \{a, b\}$. However, since the outer face belongs to one of regions R_1 and R_2, one of G_1 and G_2 in $G - \{a, b\}$ is contained in the union of the inner faces of G, which contradicts the internal triconnectivity of G. $\square$

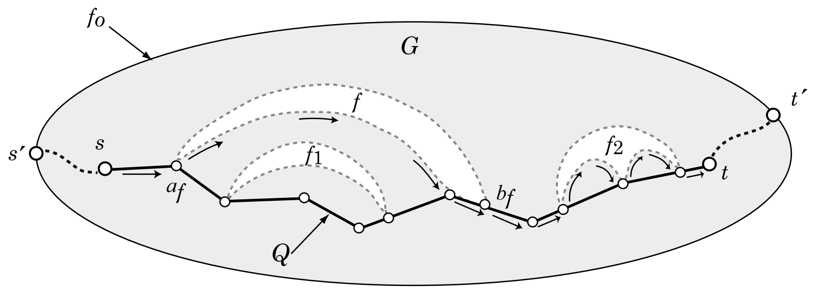

Fig. 2. Construction of the left-aligned path $L(Q)$ from an inner path Q between s and t, where thick lines show Q, and the path following the arrows show $L(Q)$.

Let Q be an inner path that is contained in an inner path Q' between two outer vertices s' and t' in a plane graph $G = (V, E, F)$, and let s and t be the end vertices of Q, where Q and Q' are viewed as directed paths from s' to t', as shown in Fig. 2. The outer facial cycle f_o consists of a subpath Q_o' from s' to t' and a subpath Q_o'' from t' to s' when we walk along f_o in a clockwise direction.

We say that an inner facial cycle $f \in F$ is *on the left side* if f is surrounded by Q_o' and Q', and that f *arches* Q *on the left side* if f arches Q and is on the left side of Q. For example, facial cycles f, f_1 and f_2 in Fig. 2 arch path Q on the left side, where Q is displayed as thick lines. The case of the *right side* is defined symmetrically.

Now we modify Q into a path $L(Q)$ from s to t such that no inner facial cycle arches $L(Q)$ on the left side. Let F_Q be the set of all inner facial cycles $f \in F$ that arch Q on the left side, but are not contained in the region enclosed by Q and any other $f' \in F$. For example, facial cycle f_1 in Fig. 2 is enclosed by Q and f, and thereby $f_1 \notin F_Q$.

The *left-aligned path* $L(Q)$ of Q is defined as an inner path from s to t obtained by replacing subpaths of Q with subpaths of cycles in F_Q as follows. For each $f \in F_Q$, let a_f and b_f be the first and last vertices in $V(f) \cap V(Q)$ when we walk along path Q from s to t, and f_Q be the subpath from a_f to b_f obtained by traversing f in an anticlockwise direction. Let $L(Q)$ be the path obtained by replacing the subpath from a_f to b_f along Q with f_Q for all $f \in F_Q$ (see Fig. 2 for an example of $L(Q)$). The *right-aligned path* $R(Q)$ of Q is defined symmetrically to the left-aligned path.

It is not difficult to observe the next lemma.

Lemma 2. *Given an inner path Q, the left-aligned path $L(Q)$ of Q can be constructed in $O(|E_Q| + |L(Q)|)$ time, where E_Q is the set of all edges incident to a vertex in Q.*

The following results have been previously shown [9].

Lemma 3. *(See [9].) Let $G = (V, E, F)$ be an internally triconnected plane graph, and Q be an inner path from a vertex s to a vertex t. Then the left-aligned path $L(Q)$ is an inner path from s to t, and no inner facial cycle arches $L(Q)$ on the left side. Moreover, if no inner facial cycle arches Q on the right side, then $L(Q)$ is an archfree path.*

Corollary 4. *(See [9].) For any inner path Q from s to t in an internally triconnected plane graph G, the right-aligned path $R(L(Q))$ of the left-aligned path $L(Q)$ is an archfree path.*

2.3. Convex drawings of plane graphs

In this section, we review the previous results on convex drawings of planar graphs.

For three points a_1, a_2, and a_3 in the plane, the line segment whose end points are a_i and a_j is denoted by (a_i, a_j), and the angle $\angle a_1, a_2, a_3$ is defined by the central angle of a small circle C with center a_2, when we traverse the circumference from the intersection of C and (a_2, a_1) to the intersection of C and (a_2, a_3) in the clockwise order (note that $\angle a_1, a_2, a_3 + \angle a_3, a_2, a_1 = 2\pi$).

A polygon P is given by a sequence $a_1, a_2, \ldots, a_p$ ($p \geqslant 3$) of points, called *apices*, and edges (a_i, a_{i+1}), $i = 1, 2, \ldots, p$ (where $a_{p+1} = a_1$) such that no two line segments (a_i, a_{i+1}) and (a_j, a_{j+1}), $i \neq j$, intersect each other except at apices. We assume that apices $a_1, a_2, \ldots, a_p$ ($p \geqslant 3$) appear in this order, when we traverse P in the clockwise order. Let $V(P)$ denote the set of apices of a polygon P, and $A(P)$ denote such a circular order of apices of a polygon P.

The *inner angle* $\theta(a_i)$ of an apex a_i is the angle $\angle a_{i+1}, a_i, a_{i-1}$, and an apex a_i is called *convex* (respectively, *concave* and *flat*) if $\theta(a_i) < \pi$ (respectively, $\theta(a_i) > \pi$ and $\theta(a_i) = \pi$). A polygon P has no apex a with $\theta(a) = 0$, since no two adjacent edges on the boundary intersect each other. Thus, the interior of a polygon has a positive area.

A polygon P is called *convex* if it has no concave apex. A k-gon is a polygon with exactly k apices, some of which may be flat or concave. A *side* of a polygon is a maximal line segment in its boundary, i.e., a sequence of edges $(a_i, a_{i+1}), (a_{i+1}, a_{i+2}), \ldots, (a_{i+h-1}, a_{i+h})$, such that a_i and a_{i+h} are non-flat apices and the other apices between them are flat. A k-gon has at most k sides.

A *straight-line drawing* of a graph $G = (V, E)$ in the plane is an embedding of G in the two dimensional space $\Re^2$ so that each vertex $v \in V$ is drawn as a point $\psi(v) \in \Re^2$ and each edge $(u, v) \in E$ is drawn as a straight-line segment $(\psi(u), \psi(v))$, where $\Re$ is the set of real numbers. Hence, a straight-line drawing of a graph $G = (V, E)$ is defined by a function $\psi : V \to \Re^2$.

A straight-line drawing ψ of a plane graph $G = (V, E, F)$ is called an *inner-convex drawing* if every inner facial cycle is drawn as a convex polygon. A convex drawing ψ of a plane graph $G = (V, E, F)$ is called a *strictly convex drawing* if it has no flat apex $\psi(v)$ for any vertex $v \in V$ with $d_G(v) \geqslant 3$. We say that a drawing ψ of a graph G is *extended* from a drawing ψ' of a subgraph G' of G if $\psi(v) = \psi'(v)$ for all $v \in V(G')$.

Let $G = (V, E, F)$ be a plane graph with an outer facial cycle f_o, and P be a $|V(f_o)|$-gon. A drawing ϕ of f_o on P is a bijection $\phi : V(f_o) \to V(P)$ so that the vertices in $V(f_o)$ appear along f_o in the same order as the corresponding apices in $A(P)$.

Lemma 5. *(See [2,18].) Let $G = (V, E, F)$ be a biconnected plane graph. Then a drawing ϕ of f_o on a convex polygon P can be extended to a convex drawing of G if and only if the following conditions* (i)–(iii) *hold*:

(i) *For each inner vertex v with $d_G(v) \geqslant 3$, there exist three paths disjoint except v, each connecting v and an outer vertex*;
(ii) *Every cycle of G which has no outer edge has at least three vertices v with $d_G(v) \geqslant 3$; and*
(iii) *Let $Q_1, Q_2, \ldots, Q_k$ be the subpaths of f_o, each corresponding to a side of P. The graph $G - V(f_o)$ has no connected component H such that all the outer vertices adjacent to vertices in H are contained in a single path Q_i, and there is no inner edge (u, v) whose end vertices are contained in a single path Q_i.*

Every inner vertex of degree 2 must be drawn as a point subdividing a line segment in any convex drawing. Hence we can assume without loss of generality that a given plane has no inner vertex of degree 2, since any convex drawing of the plane graph obtained by replacing each maximal path containing inner vertices v with $\deg_G(v) = 2$ with a single edge gives a convex drawing of G by subdividing the replaced edges. Then Lemma 5 can be simplified as follows:

Lemma 6. *(See [9].) Let $G = (V, E, F)$ be a biconnected plane graph with no inner vertices of degree* 2. *Then a drawing ϕ of f_o on a convex polygon P can be extended to a convex drawing of G if and only if the following conditions* (i) *and* (ii) *hold*:

(i) *G is internally triconnected.*
(ii) *Let $Q_1, Q_2, \ldots, Q_k$ be the subpaths of f_o, each corresponding to a side of P. Each Q_i is an archfree path in G.*

3. Convex drawings of hierarchical planar graphs

In this section, we now present one of the main results of this paper. Namely, we prove that every internally triconnected hierarchical plane graph with the outer face fixed with a convex polygon admits a convex drawing. First, however, we define basic terminology related to hierarchical planar graphs.

An edge with a tail u and a head v is denoted by (u, v). A *hierarchical graph* $H = (V, A, \lambda, k)$ consists of a directed graph (V, A), a positive integer k, and, for each vertex u, an integer $\lambda(u) \in 1, 2, \ldots, k$, with the property that if $(u, v) \in A$, then $\lambda(u) < \lambda(v)$. For $1 \leqslant i \leqslant k$, the ith *layer* L_i of G is the set $\{u \mid \lambda(u) = i\}$. The *span* of an edge (u, v) is $\lambda(v) - \lambda(u)$. An edge of span greater than one is *long*, and a hierarchical graph with no long edges is *proper*.

For each vertex v in H, denote $\{u \in V \mid (v, u) \in A\}$ by $V_H^+(v)$ and $\{u \in V \mid (u, v) \in A\}$ by $V_H^-(v)$. A vertex v with $V_H^-(v) = \emptyset$ (respectively $V_H^+(v) = \emptyset$) is called a *source* (respectively *sink*). For a non-sink vertex v, a vertex $w \in V_H^+(v)$ is called an *up-neighbor* of v (see Fig. 3). Further, w is called the *highest up-neighbor* if $\lambda(w) = \max\{\lambda(u) \mid u \in V_H^+(v)\}$. Similarly, for a non-source vertex v, a vertex $w \in V_H^-(v)$ is called a *down-neighbor* of v, and w is called the *lowest down-neighbor* if $\lambda(w) = \min\{\lambda(u) \mid u \in V_H^-(v)\}$.

A hierarchical graph is conventionally drawn with layer L_i on the horizontal line $y = i$, and edges as curves monotonic in y direction. If no pair of non-incident edges intersect in the drawing, then we say that it is a *hierarchical planar* (*h-planar*) drawing. Note that a nonproper hierarchical graph can be transformed into a proper hierarchical graph by adding dummy vertices on long edges. It is easily shown that a nonproper hierarchical graph is h-planar if and only if the corresponding proper hierarchical graph is h-planar.

A *hierarchical planar embedding* of a proper hierarchical graph is defined by the ordering of vertices on each layer of the graph. Note that every such embedding has a unique external face. Also, every proper h-planar graph admits a *straight-line hierarchical drawing*; that is, a drawing where edges are drawn as straight-line segments. However, for nonproper hierarchical graphs, the problem is not trivial, since no bends are allowed on long edges.

We call a plane embedded hierarchical graph a *hierarchical plane graph*. If a hierarchical plane graph has only one source s and one sink t, then we call it a *hierarchical-st plane graph*. Observe that a hierarchical-st plane graph is a connected graph, and its source s and sink t must lie on the bottom layer and the top layer, respectively. Fig. 4(a) shows a hierarchical-st plane graph H.

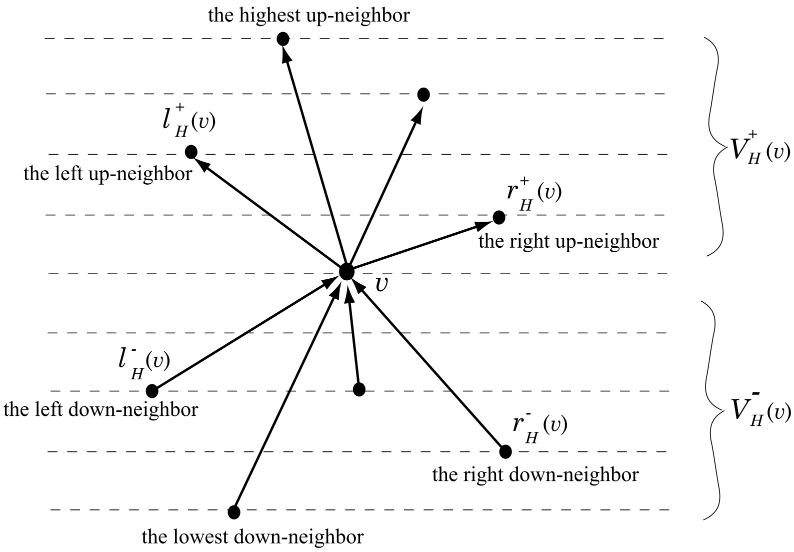

Fig. 3. Definition of left-right relations in $V^-_H(v)$ and $V^+_H(v)$.

The embedding of a hierarchical plane graph H determines, for every vertex v, a left-right relation among up-neighbors of v (see Fig. 3). The head w of the rightmost (respectively leftmost) edge outgoing from v is called the *right up-neighbor* (respectively the *left up-neighbor*) of v, and is denoted by $r^+_H(v)$ (respectively $\ell^+_H(v)$). The *right down-neighbor* $r^-_H(v)$ and the *left down-neighbor* $\ell^-_H(v)$ of v are defined analogously.

Hierarchical graphs are directed graphs and we can therefore borrow much of the standard terminology of graph theory. The terms "path," "cycle," and "biconnectivity," when applied to a directed graph in this paper, refer to the underlying undirected graph. To denote a cycle of a plane graph, we use the sequence of vertices on the cycle in a clockwise direction.

For a cycle or path $\mathcal{P} = (v_1, v_2, \ldots, v_k)$, an edge between two non-consecutive vertices in $\mathcal{P}$ is called a *chord* of $\mathcal{P}$. A cycle or path is called *chordless* if it has no chord. In hierarchical graphs, edges are directed from a lower layer to a higher layer. A path is called *monotonic* if the directions of the edges do not change along the path. In other words, a path is monotonic if the layer increases (or decreases) as we go along the path.

Note that from a vertex v, a monotonic and chordless path from v to a sink can be obtained by traversing the highest up-neighbors one after another. Similarly, a monotonic and chordless path from a source to v can be found by tracing the lowest down-neighbors from v. For straight-line drawings of hierarchical-st plane graphs, the next result is known.

Theorem 7. *(See [5].) Let H be an inner-triangulated hierarchical-st plane graph, and its outer facial cycle f_o be drawn as a convex polygon P such that, for each chord (u, z) of f_o, on each of the two paths of cycle C between u and z, there exists a vertex v which is drawn as a convex apex of polygon P. Then there exists a planar straight-line hierarchical drawing of H with external face P, and such a drawing can be constructed in linear time.*

The theorem implies that every hierarchical-st plane graph H admits a straight-line hierarchical planar drawing, because we can easily augment a given biconnected hierarchical-st plane graph into a triangulated hierarchical-st plane graph H' by triangulating each non-triangle inner face. However, the drawing of H obtained by deleting added edges from the drawing of H' may not be a convex drawing.

In this section, we prove that the necessary and sufficient condition for a biconnected graph G with boundary drawn as a convex polygon P to admit a convex drawing remains valid for the case of hierarchical-st plane graphs. The following theorem summarizes our main result.

Theorem 8. *Let H be a hierarchical-st plane graph such that H is biconnected and no inner vertex is of degree 2. Then a drawing ϕ of the outer facial cycle f_o on a convex polygon P can be extended to a convex drawing of H by choosing an x-coordinate of each inner vertex if and only if the following conditions* (i) *and* (ii) *hold*:

(i) *H is internally triconnected.*
(ii) *Let $Q_1, Q_2, \ldots, Q_k$ be the subpaths of f_o, each corresponding to a side of P. Each Q_i is an archfree path in H.*

Such a convex drawing (*if any*) *can be computed in $O(n^2)$ time.*

We first observe a key lemma to derive the theorem.

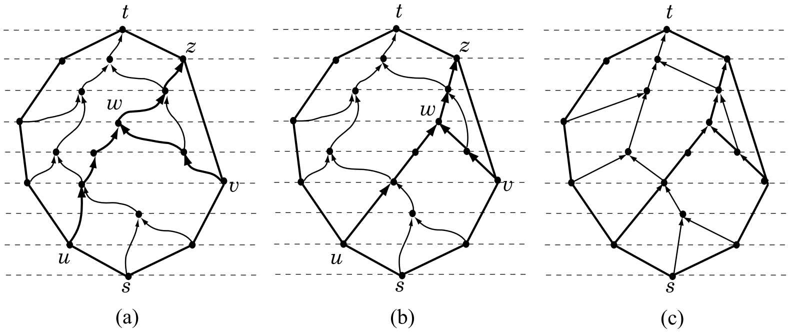

Fig. 4. (a) A hierarchical-st plane graph H with a convex polygon P, where three monotonic paths with thick lines are archfree paths. (b) Three hierarchical-st plane graphs obtained by the archfree paths. (c) A convex drawing of H in (a).

Lemma 9. *Let (H, P) be a pair of a hierarchical-st plane graph and a convex polygon P that satisfies conditions* (i) *and* (ii) *in Theorem 8. For any monotonic inner path Q from a vertex u to a vertex v, $R(L(Q))$ is a monotonic archfree path.*

Proof. By Corollary 4, $R(L(Q))$ is an archfree path. It is sufficient to show that operation L of constructing path $L(Q)$ from a monotonic inner path Q preserves the monotonicity of Q (operation R can be treated symmetrically). For this, we consider an inner facial cycle f on the left side of Q, where a_f and b_f are the first and last vertices in $V(f) \cap V(Q)$ when we walk along path Q from s to t (see Fig. 2), and prove that the subpath Q_f from a_f to b_f obtained by traversing f in an anticlockwise direction is monotonic.

If Q_f is not monotonic, then there are three vertices u_1, u_2 and u_3 which appear in this order on Q_f and whose y-coordinates $y(u_1)$, $y(u_2)$ and $y(u_3)$ satisfy $y(u_1) > y(u_2) < y(u_3)$. This, however, implies that u_2 is another source, since H is a hierarchical-st plane graph, which contradicts the situation where H has no other source than s. This proves the lemma. □

The necessity of Lemma 6 implies the necessity of Theorem 8. We prove the sufficiency of Theorem 8 by induction on the number of inner faces and vertices. The theorem holds if H has only one inner face. Consider an internally triconnected hierarchical-st plane graph H, and assume that the theorem holds for any internally triconnected hierarchical-st plane graph which has a smaller number of inner faces or vertices than H. There must be an outer vertex $v(\neq s, t)$ which is a convex apex in P. We assume that v is on the rightmost path from s to t (the other case can be treated symmetrically). We distinguish two cases:

Case 1: The degree of vertex $v(\neq s, t)$ is 2. Let v' and v'' be the up- and down-neighbors of v. We construct a hierarchical-st plane graph H' from H by removing v and adding a new edge (v'', v') if $v'' \notin V_H^-(v')$. Let P' be the polygon obtained by replacing the segments (v'', v) and (v, v') in P with a new segment (v'', v'), which will form a new side of P' (see Fig. 5(a)).

We claim that H' with boundary P' satisfies conditions (i) and (ii) in Theorem 8. To see condition (i) on H', we consider the graph H'' obtained from H by adding a new edge (v'', v') if $v'' \notin V_H^-(v')$ ($H'' = H$ if $v'' \in V_H^-(v')$). We easily see that H'' with P still satisfies conditions (i) and (ii) in Theorem 8, and that $H' = H'' - v$ remains internally triconnected due to the existence of edge (v'', v'). We next consider condition (ii) on H'. For this, it suffices to show that path $Q = (v'', v')$, which corresponds to the new side of P', is an archfree path in H'.

Assume indirectly that Q is not an archfree path in H', i.e., H' has a facial cycle f that arches Q. Thus, f contains two vertices v'' and v', but not edge (v'', v'). This, however, means that removal of v'' and v' from H results in at least three connected components, which contradicts the internal triconnectivity of H. This proves the claim.

By the induction hypothesis, H' with P' admits a convex drawing D'. It is not difficult to see that D' can be modified into a convex drawing of H with P by adding segments (v'', v) and (v, v') and deleting segment (v'', v') if $v'' \notin V_H^-(v')$.

Case 2: The degree of vertex $v(\neq s, t)$ is at least 3. We consider the leftmost monotonic path Q_v from v (i.e., a path obtained by traversing the left up-neighbors), and let w be the first vertex in Q_v that has at least one down-neighbor, and Q_1 be the subpath from v to w along Q_v. For example, see vertices v and w in Figs. 4(a) and (b). Hence, Q_1 is a subpath of an inner facial cycle f_v which contains v. Note that $|E(f_v) - E(Q_1)| \geqslant 2$ if w and v are not adjacent or $Q_1 = (v, w)$ otherwise. Hence Q_1 is an archfree path by Lemma 1.

Here we consider the two subcases: (i) w is an outer vertex (see Fig. 5(b)) and (ii) w is an inner vertex (see Fig. 5(c)).

Case 2(i): The outer facial cycle f_o together with path Q_1 splits the interior of P into two regions, say R_1 and R_2. We split H into two hierarchical-st plane graphs H_1 and H_2 such that H_1 and H_2 share Q_1, and each H_i, $i = 1, 2$, contains all faces in R_i (see Fig. 5(b)). We draw Q_1 as a straight-line between two points v and w, by which the x-coordinates of all points on Q_1 are uniquely determined. Let P_1 and P_2 be the two convex polygons for the boundaries of H_1 and H_2, respectively. By choosing the vertex with the lowest (respectively, highest) position on P_i as s (respectively, t) of graph H_i, we can regard H_i as a hierarchical-st plane graph.

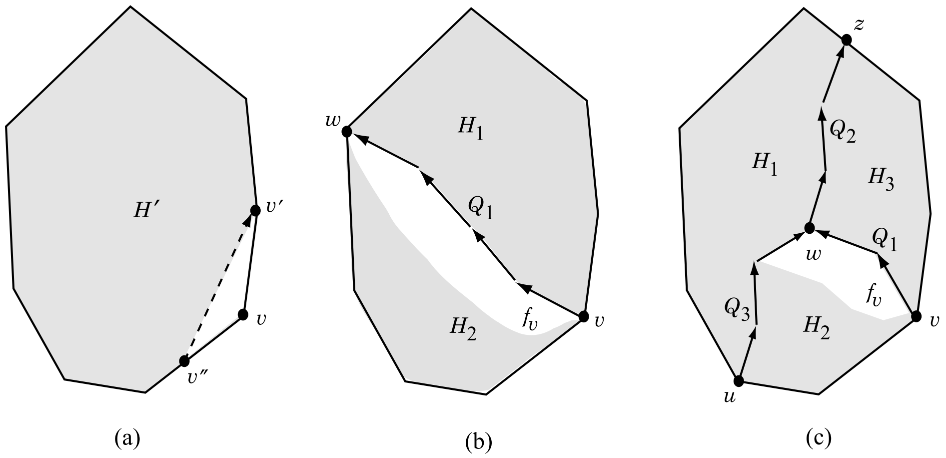

Fig. 5. (a) An outer vertex $v(\neq s, t)$ of degree 2 in H. (b) A monotonic archfree path between two outer vertices v and w. (c) Constructing three monotonic archfree paths starting from v, w and u, respectively.

We claim that each H_i with boundary P_i still satisfies the two conditions in Theorem 8. Since Q_1 is an archfree path and any side of P is an archfree path, H_i with P_i satisfies condition (ii) in Theorem 8. It can be immediately seen that condition (i) in Theorem 8 still holds for each H_i with P_i, since it is a subgraph of H which contains all vertices and edges enclosed by a cycle. This proves the claim.

Therefore, each H_i admits a convex drawing D_i by the induction hypothesis, implying that a convex drawing of H with P can be obtained by placing D_1 and D_2 in the corresponding region inside P.

Case 2(ii): We choose a monotonic path Q_2 that starts from w and ends up with an outer vertex z. We also choose a monotonic path Q_3 to w, from an outer vertex u. By Lemma 9, H has monotonic archfree paths Q_2 from w to z and Q_3 from u to w. The union of paths Q_j and Q_k with $\{j, k\} \subseteq \{1, 2, 3\}$ splits the graph into two plane subgraphs that share Q_j and Q_k, denoted by H_i and H'_i, where H'_i is the one containing the remaining path Q_i, $i \in \{1, 2, 3\} - \{j, k\}$. See Fig. 5(c).

By placing w at a point in the interior of P, we draw each path Q_i, $i = 1, 2, 3$ as a straight-line between its end points, by which the x-coordinates of all points on Q_i are uniquely determined. Note that v, z and w are not on a single straight-line since v is a convex apex in P. Therefore, w can be placed at a point such that the polygons obtained from P by the straight-lines are all convex. Let P_i, $i = 1, 2, 3$ be the convex polygon for the boundary of H_i. By choosing the vertex with the lowest (respectively, highest) position on P_i as s (respectively, t) of graph H_i, we can regard H_i as a hierarchical-st plane graph.

Since each Q_i is an archfree path and each of the resulting subgraphs H_i is internally triconnected, we see that H_i with P_i satisfies the condition in Theorem 8. We then have a convex drawing D_i of H_i with P_i by the inductive hypothesis. We see that a convex drawing of G with P can be obtained by combining D_1, D_2 and D_3 in the corresponding region inside P (see Fig. 4(c)).

The arguments in Cases 1 and 2 complete the induction, proving the existence of a desired convex drawing of H in Theorem 8. It is not difficult to see that the above inductive proof provides a divide-and-conquer algorithm for drawing a convex drawing of H.

The maximum depth of recursive calls from the root call is $O(n)$ since each recursion fixes the drawing of at least one edge. Each of the above reductions can be executed in linear time. Since the total size of graphs that appear at the same depth of recursive calls is $O(n)$, we see that the algorithm runs in $O(n^2)$ time. □

Based on the above proof, given a hierarchical-st plane graph with a convex polygon P that satisfies conditions (i) and (ii) in Theorem 8, a convex drawing of H in Theorem 8 can be obtained by determining the x-coordinates of inner vertices of H by the following recursive algorithm.

Algorithm Hierarchical-Convex-Draw(H, P)

1. **If** H contains no inner vertex, **then** return.
2. Choose an outer vertex $v(\neq s, t)$ of H which is a convex apex in P.
3. **If** v has degree 2 **then**
   (a) Let v' and v'' be the two vertices adjacent to v.
   (b) Let H' be the hierarchical-st plane graph obtained from H by removing v and adding an edge (v'', v'), if it does not exist in H.
   (c) Let P' be the polygon obtained from P by replacing the segments (v'', v) and (v, v') with a new segment (v'', v') (see Fig. 5(a)).
   (d) Hierarchical-Convex-Draw(H', P').

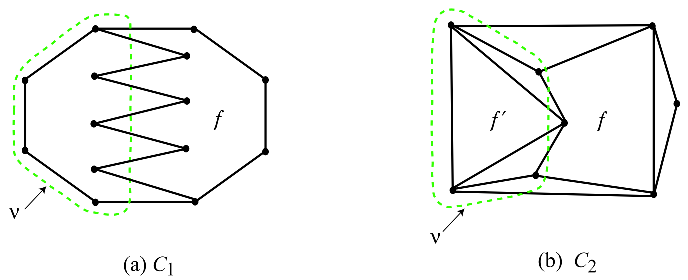

Fig. 6. (a) A c-planar clustered graph C_1, which is not a connected clustered graph. (b) A c-planar and connected clustered graph C_2, which is not completely connected, where only nontrivial clusters are enclosed by dashed curves in (a) and (b).

else
(a) **If** v is on the rightmost path from s to t
  then Compute the leftmost monotonic path Q_v from v, by traversing the left up-neighbors from v.
  else Compute the rightmost monotonic path Q_v from v, by traversing the right up-neighbors from v.
(b) Let w be the first vertex in Q_v that has at least one down-neighbor.
(c) Let Q_1 be the subpath from v to w along Q_v.
(d) **If** w is an outer vertex **then**
  i. Split H into two plane graphs H_1 and H_2 which share Q_1 (see Fig. 5(b)).
  ii. Draw Q_1 as a straight-line between v and w.
  iii. **for** $i = 1, 2$,
    A. Let P_i be the convex polygon for the boundary of H_i.
    B. Choose the vertex with the lowest position on P_i as s of graph H_i.
    C. Choose the vertex with the highest position on P_i as t of graph H_i.
    D. HIERARCHICAL-CONVEX-DRAW(H_i, P_i).
  else
  i. Compute a monotonic path Q_2 from w to an outer vertex z.
  ii. Compute a monotonic path Q_3 from an outer vertex u to w.
  iii. Split H into three plane subgraphs H_1, H_2 and H_3 such that H_i and H_j share Q_k with $k \in \{1, 2, 3\} - \{i, j\}$ (see Fig. 5(c)).
  iv. Choose the x-coordinate of w within the triangle (v, u, z).
  v. **for** $i = 1, 2, 3$, Draw Q_i as a straight-line between its endpoints.
  vi. **for** $i = 1, 2, 3$,
    A. Let P_i be the convex polygon for the boundary of H_i.
    B. Choose the vertex with the lowest position on P_i as s of graph H_i.
    C. Choose the vertex with the highest position on P_i as t of graph H_i.
    D. HIERARCHICAL-CONVEX-DRAW(H_i, P_i).

4. Convex drawings of clustered planar graphs

In this section, we present our second main result on convex drawings of clustered planar graphs. First, we define the terminology related to clustered planar graphs.

A *clustered graph* $C = (G, T)$ consists of an undirected graph $G = (V, E)$ and a rooted tree $T = (\mathcal{V}, \mathcal{A})$, such that the leaves of T are exactly the vertices of G. We call G the *underlying graph* and T the *inclusion tree* of C. To distinguish vertices in T from those in G, vertices in T are called *nodes*. Each node ν of T represents a *cluster* $V(\nu)$, a subset of the vertices of G that are leaves of the subtree rooted at ν. For the root ν of T, $V(\nu) = V$. A node ν in T (or its cluster $V(\nu)$) is called *nontrivial* if ν is neither the root, nor a leaf of T. Let $G(\nu)$ denote the subgraph $G[V(\nu)]$ of G induced by $V(\nu)$. Note that the tree T represents a laminar family of subsets of the vertices in G. If a node ν' is a descendant of a node ν in the tree T, then we say that the cluster of ν' is a *sub-cluster* of ν.

A clustered graph $C = (G, T)$ is a *connected clustered graph* if each cluster $V(\nu)$ induces a connected subgraph $G(\nu)$ of G [5,6]. A clustered graph $C = (G, T)$ is *completely connected* if, for every non-root node ν of T, both subgraphs $G(\nu)$ and $G[V - V(\nu)]$ are connected [3]. Note that G is biconnected if $C = (G, T)$ is completely connected, since $G[V - \{v\}]$ is connected for every leaf cluster $\{v\}$ in T. For example, among clustered graphs C_1 and C_2 in Fig. 6 and C_3 in Fig. 7(a), C_2 and C_3 are connected clustered graphs, and only C_3 is completely connected.

In a *drawing* of a clustered graph $C = (G, T)$, graph G is drawn as points and curves as usual. For each node ν of T, the cluster is drawn as a simple closed region $R(\nu)$ enclosed by a simple closed curve such that:

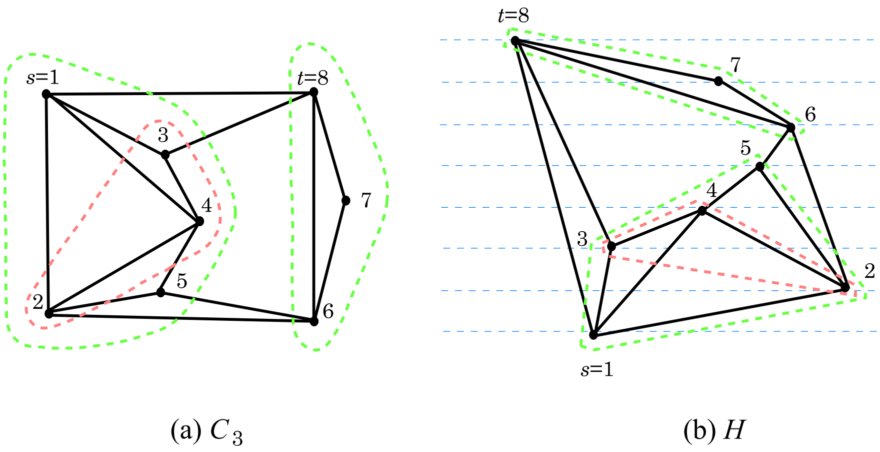

Fig. 7. (a) A c-planar and completely connected clustered graph C_3 together with a *c-st* numbering of vertices, where non-leaf and non-root clusters are enclosed by dashed curves. (b) A convex drawing of the hierarchical-st plane graph H obtained by the *c-st* numbering in (a).

- the drawing of $G(\nu)$ is completely contained in the interior of $R(\nu)$;
- the regions for all sub-clusters of ν are completely contained in the interior of $R(\nu)$;
- the regions for all other clusters are completely contained in the exterior of $R(\nu)$.

We say that the drawing of edge e and region R have an *edge-region crossing* if the drawing of e crosses the boundary of R more than once. A drawing of a clustered graph is *compound planar* (*c-planar*, for short) if there are no edge crossings and edge-region crossings. If a clustered graph C has a c-planar drawing, then we say that it is *c-planar*. Figs. 6(a) and (b) show examples of c-planar drawings of clustered graphs.

The characterisation of c-planar clustered graphs is known only for connected clustered graphs.

Theorem 10. *(See [6].) A connected clustered graph $C = (G, T)$ is c-planar if and only if graph G is planar and there exists a planar drawing of G, such that for each node ν of T, all the vertices and edges of $G[V - V(\nu)]$ are in the external face of the drawing of $G(\nu)$.*

It is shown that a completely connected clustered graph $C = (G, T)$ is c-planar if and only if the underlying graph G is planar [3,10].

A c-planar drawing of a clustered graph is called *a planar straight-line convex cluster drawing* if edges are drawn as straight-line segments and clusters are drawn as convex polygons. The drawings in Figs. 6(a) and (b) are planar straight-line convex cluster drawing.

Theorem 11. *(See [5].) Let $C = (G, T)$ be a c-planar clustered graph with n vertices. A planar straight-line convex cluster drawing of C in which each cluster is a convex hull of points in the cluster can be constructed in $O(n + D)$ time, where D denotes the number of apices of all convex hulls in an output and $D = O(n^2)$ holds.*

In this paper, we define a *fully convex drawing* of a clustered graph as a planar straight-line convex cluster drawing such that facial cycles are also drawn as convex polygons. Among the four c-planar drawings in Figs. 6 and 7, only the drawing in Fig. 7(b) is fully convex. In the following, we only consider c-planar and connected clustered graphs, and present a characterisation of these clustered graphs that have fully convex drawings.

We define a *region-face crossing* as follows. In a c-planar drawing of a clustered graph $C = (G, T)$, the region R of a cluster and a cycle f of G *cross* each other if the boundary of R crosses the drawing of more than two edges (i.e., at least four edges) in f. Thus, for the set $V_R \subseteq V(f)$ of vertices contained inside R, region R and facial cycle f cross each other if and only if each of subsets V_R and $V(f) - V_R$ induces more than one subpath from f. For example in Fig. 6(b), the region of cluster ν and facial cycle f cross each other, while the region of cluster ν and facial cycle f' do not cross each other.

We now prove the following main result of this section.

Theorem 12. *Let $C = (G, T)$ be a c-planar and connected clustered graph that has a c-planar drawing ψ such that the outer facial cycle does not cross any cluster region, and G be internally triconnected. Then*:

(i) *There exists a fully convex drawing of C if and only if C is completely connected.*
(ii) *A fully convex drawing of C (if any) can be constructed from drawing ψ in $O(n^2)$ time, where each cluster is a convex hull of points in the cluster and n is the number of vertices in G.*

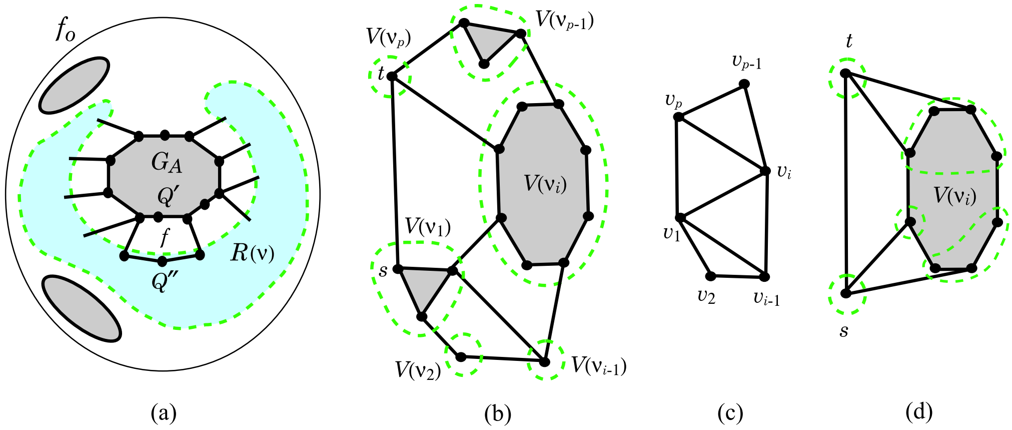

Fig. 8. (a) Component G_A in $G[V - V(\nu)]$ which has no outer vertex. (b) A clustered graph $C = (G, T)$. (c) The graph G^* obtained from G by contracting each vertex set $V(\nu_i)$ into a single vertex v_i. (d) The clustered graph C_i on G_i with clusters $\{s_i\}, \{t_i\}$ and the sub-clusters of ν_i.

Necessity of Theorem 12(i): We prove the necessity of Theorem 12(i) via several lemmas.

Lemma 13. *If a c-planar and connected clustered graph C admits a planar straight-line convex cluster drawing such that the drawing of G is inner-convex, then there is no region-face crossing between regions for clusters and inner facial cycles.*

Proof. We see that if there is a region-face crossing between the region $R(\nu)$ of a cluster ν and an inner facial cycle f in G, then it is impossible to draw both $R(\nu)$ and f as convex polygons in one drawing (note that $G(\nu)$ is connected). Therefore, if C admits a fully convex drawing, then there is no region-face crossing between regions for clusters and inner facial cycles. □

Lemma 14. *Suppose that a connected clustered graph $C = (G, T)$ has a c-planar drawing which has no region-face crossing between regions for clusters and inner facial cycles. Then for every non-root node ν of T, every connected component of the subgraph $G[V - V(\nu)]$ contains an outer vertex.*

Proof. Let ψ be such a c-planar drawing of C. In ψ, G is a plane graph (V, E, F), and we denote the outer facial cycle of G by f_o. Let ν be a non-root node of T. Then $V - V(\nu) \neq \emptyset$. To derive a contradiction, assume that $G[V - V(\nu)]$ has a component G_A that contains no vertex in f_o. In ψ, G_A is a plane graph, and we denote its boundary by B_A and the set of inner faces of G_A by F_A. Since $R(\nu)$ is a simple closed region, G_A is not completely enclosed by $R(\nu)$.

Consider an arbitrary facial cycle $f \in F - F_A$ that shares a vertex with B_A (see Fig. 8(a)). Since component G_A is an inclusionwise maximal induced subgraph in $G[V - V(\nu)]$, a vertex in f is contained inside $R(\nu)$. Let V_R be the set of vertices in $V(f)$ contained inside $R(\nu)$. Since G_A has no outer vertex in f_o, f is an inner face of G. By assumption, f and $R(\nu)$ has no region-face crossing in ψ, and hence f consists of two subpaths Q' and Q'' induced by $V(f) - V_R$ and V_R, respectively.

Note that all the vertices in $V(f) - V_R$ belong to B_A because otherwise an edge in f would join a vertex in B_A and a vertex not in B_A, contradicting that G_A is an inclusionwise maximal induced subgraph. Hence Q' is a subpath of f.

We now consider all such facial cycle $f_1, f_2, \ldots, f_p \in F - F_A$ that share vertices with B_A. Note that the union of regions inside $f_1, f_2, \ldots, f_p$ encloses G_A. Then we see that the subpaths $Q''_1, Q''_2, \ldots, Q''_p$ form a closed curve, which implies that G_A is enclosed by $R(\nu)$, a contradiction. This proves the lemma. □

Lemma 15. *Suppose that a connected clustered graph $C = (G, T)$ has a c-planar drawing which has no region-face crossing. Then C is completely connected.*

Proof. Let ψ be a c-planar drawing of C. In ψ, G is a plane graph (V, E, F), and we denote the outer facial cycle of G by f_o. Since C is a connected clustered graph, it is sufficient to show that, for every non-root node ν of T, subgraph $G[V - V(\nu)]$ is connected.

Let ν be a non-root node of T. To derive a contradiction, assume that $G[V - V(\nu)]$ is not connected, and G_A and G_B are two connected components in $G[V - V(\nu)]$. By Lemma 14, both G_A and G_B contain outer vertices in f_o. This, however, contradicts that region $R(\nu)$ and f_o has no region-face crossing. □

We are ready to prove the necessity of Theorem 12(i).

Let ψ be a fully convex drawing of C. Clearly ψ is a convex drawing on G, and G must be internally triconnected by Lemma 6(i). We show that C is completely connected. By Lemma 13, there is no region-face crossing between regions for clusters and inner facial cycles. By assumption on C, there is no region-face crossing between regions for clusters and the outer facial cycle, either. By Lemma 15, C is completely connected. This proves the necessity in Theorem 12(i).

Sufficiency of Theorem 12(i):

To prove the sufficiency of Theorem 12(i), we follow an approach by Eades et al. [5] which was used to derive Theorem 11. They used the *c-st numbering* of vertices in G, which is an extension of *st* numberings. Suppose that (s,t) is an edge of a biconnected graph G with n vertices. In an *st* numbering, the vertices of G are numbered from 1 to n so that vertex s receives number 1, vertex t receives number n, and any vertex except s and t is adjacent both to a lower-numbered vertex and a higher-numbered vertex. An *st* numbering of a biconnected graph can be computed in linear time [7].

A *c-st numbering* for a clustered graph $C=(G,T)$ is an *st* numbering of the vertices in G such that the vertices that belong to the same cluster are numbered consecutively. This numbering provides us a layer assignment of the vertices of G. Hence, a c-planar clustered graph C is transformed to a hierarchical-st plane graph H, and each cluster has consecutive layers.

Based on this property, Eades et al. [5] proved that a planar straight-line convex cluster drawing can be constructed from the straight-line hierarchical drawing. In this method, a given underlying graph is augmented to a triangulated graph to ensure the existence of *c-st* numberings in the clustered graph, and then all added edges are removed from a drawing of the triangulated graph to obtain a desired straight-line drawing of C.

However, the resulting drawing may not be convex. Since we aim to construct a convex drawing of G, we cannot use triangulation to find *c-st* numberings. Fortunately, complete connectedness ensures the existence of *c-st* numberings instead. An edge $e=(u,v)$ in a clustered graph is called a *root edge* if no other cluster than the root cluster contains the end vertices u and v at the same time.

Lemma 16. *Let $C=(G,T)$ be a connected clustered graph (where G is not necessarily planar). Then, C has a c-st numbering for each root edge $e=(s,t)$ if and only if C is completely connected. For a given root edge (s,t), a c-st numbering of C can be computed in $O(k|E(G)|)$ time, where $k=O(|V(G)|)$ is the height of T.*

Proof. Necessity: Assume that a connected clustered graph C is not completely connected; i.e., C has a cluster ν such that $G[V-V(\nu)]$ is not connected. We choose an arbitrary root edge (s,t), and prove that C cannot admit a *c-st* numbering. Then at least one of the components in $G[V-V(\nu)]$, say G', contains neither s or t since (s,t) is a root edge. Note that all vertices adjacent to G' are contained in $V(\nu)$. If the vertices in $V(\nu)$ receive consecutive numbers, then the numbers assigned to the vertices in G' must be all larger or less than those in $V(\nu)$. Thus it is impossible for any vertex in G' to be adjacent both to a lower-numbered vertex and a higher-numbered vertex in any *c-st* numbering.

Sufficiency: We prove the sufficiency by induction on the height of inclusion trees. If the height is 1, i.e., C contains only trivial clusters, then G is biconnected by the complete connectivity and hence admits an *st* numbering, which is a *c-st* numbering to C. Assume that for an integer $k \geqslant 1$ any completely connected clustered graph such that the height of the inclusion tree is at most k admits a *c-st* numbering.

Let C be a completely connected clustered graph such that the height of the inclusion tree is $k+1$. Let $\nu_1,\nu_2,\ldots,\nu_p$ be the child nodes of the root node in T, where, for the root edge $e=(s,t)$, $s\in V(\nu_1)$ and $t\in V(\nu_p)$ are assumed without loss of generality (see Fig. 8(b)). Let G^* denote the graph obtained from G by contracting each vertex set $V(\nu_i)$ into a single vertex v_i (see Fig. 8(c)). We see that G^* is biconnected and the edge e remains to join the two vertices v_1 and v_p. Hence G^* has a *st* numbering for $st=v_1v_p$, where we assume without loss of generality that each vertex v_i receives number i.

For each $i=1,2,\ldots,p$, let G_i be the graph obtained from G by contracting vertex set $V(\nu_1)\cup V(\nu_2)\cup\cdots\cup V(\nu_{i-1})$ (respectively, $V(\nu_{i+1})\cup V(\nu_{i+1})\cup\cdots\cup V(\nu_p)$) into a single vertex s (respectively, t), and let C_i be the clustered graph on G_i with clusters $\{s\},\{t\}$ and the sub-clusters of ν_i (excluding the cluster ν_i). See Fig. 8(d). We easily see that C_i remains completely connected and the height of the inclusion tree is at most k. Therefore, by the induction hypothesis, C_i has a *c-st* numbering $\ell_i: V(\nu_i)\to\{1,2,\ldots,|V(\nu_i)|\}$ with $\ell_i(s)=1$ and $\ell_i(t)=|V(\nu_i)|$.

A *c-st* numbering $\ell: V\to\{1,2,\ldots,|V|\}$ of C can be obtained by setting $\ell(v):=\ell_i(v)-1+\sum_{1\leqslant j\leqslant i}|V(\nu_j)|$ for all vertices $v\in V(\nu_i)$, $1\leqslant i\leqslant p$. This proves the sufficiency.

The above constructive proof provides a divide-and-conquer algorithm for computing a *c-st* numbering of a given completely connected clustered graph C. Note that an edge e will be duplicated only when it joins two different clusters $V(\nu_i)$ and $V(\nu_{i'})$, since one copy of e becomes incident to s and the other copy becomes incident to t. Thus the total number of edges in the generated cluster graphs at the same depth of recursive calls is at most $2|E(G)|$. Since each clustered graph can be processed in linear time, the total time complexity for computing st-numberings of all cluster graphs at the ith depth of recursive calls is $O(|E(G)|)$. The maximum depth of calls is the height k of the inclusion tree T of an input clustered graph C. Therefore the entire time complexity is $O(k|E(G)|)$. $\square$

We are ready to prove the sufficiency of Theorem 12(i) and Theorem 12(ii). Given a c-planar and completely connected clustered graph $C=(G,T)$ with a c-planar drawing ψ such that the outer facial cycle f_o does not cross any cluster region, a fully convex drawing can be constructed by the following algorithm.

Algorithm CLUSTERED-CONVEX-DRAW

Step 1 Compute a *c-st* numbering of $C = (G, T)$ such that s and t are chosen as adjacent vertices in the outer facial cycle f_o (Fig. 7(a) illustrates a c-planar and completely connected clustered graph C_3 and its *c-st* numbering).

Step 2 Transform the plane graph G to a hierarchical-st plane graph by assigning the layer of each vertex with its *c-st* number (Fig. 7(b) illustrates the hierarchical-st plane graph H obtained from clustered graph C_3 by its *c-st* numbering).

Step 3 Compute a convex drawing ψ^* of H using HIERARCHICAL-CONVEX-DRAW.

Step 4 For each cluster $V(\nu)$, let the convex hull of the vertices in $V(\nu)$ in ψ^* be the region $R(\nu)$ of the cluster.

Step 1 can be done in $O(n^2)$ time by Lemma 16. A drawing ψ^* in Step 3 can be obtained in $O(n^2)$ time by Theorem 8. At Step 4, we use the fact that a convex hull of a given simple polygon with m apices can be constructed in $O(m)$ time [17]. Let $\mathcal{V}_i$ denote the set of clusters in $\mathcal{V}$ with depth i in T, where $\sum_{\nu \in \mathcal{V}_i} |V(\nu)| = O(n)$ holds for each depth i. Hence the total time of computing all convex hulls in $C = (G, T)$ is $O(\sum_{\text{all depths } i} \sum_{\nu \in \mathcal{V}_i} |V(\nu)|) = O(n^2)$.

Overall, the entire running time of algorithm CLUSTERED-CONVEX-DRAW is $O(n^2)$.

To prove the sufficiency of Theorem 12(i) and Theorem 12(ii), we only need to prove that the resulting drawing $(\psi^*, \{R(\nu) \mid \nu \in \mathcal{V}\})$ is a fully convex drawing. By Theorem 8, ψ^* is a convex drawing. We show that $(\psi^*, \{R(\nu) \mid \nu \in \mathcal{V}\})$ is c-planar.

Since $R(\nu)$ is a convex hull of the vertices in $V(\nu)$ in ψ^* and edges are drawn as line-segments, it contains the drawing of $G(\nu)$ (and hence the regions of all sub-clusters of ν). The *c-st* numbers within a cluster are consecutive; therefore, if the convex hulls of two clusters overlap in y-coordinate, then one is a sub-cluster of the other. This keeps the disjoint clusters apart; for two clusters ν and ν' with $V(\nu) \cap V(\nu') = \emptyset$, the convex hulls of ν and ν' are disjoint.

We finally see that there are no edge crossings and no edge-region crossings. Since ψ^* is a plane drawing of G, it has no edge crossings. Assume that the drawing of an edge e intersects region $R(\nu)$ of a cluster ν twice (i.e., the end vertices of e are outside $R(\nu)$). This, however, contradicts that ψ^* is a plane drawing of G, since $G(\nu)$ is connected and the line-segment for e must create an edge crossing with some edge in $G(\nu)$. Therefore, $(\psi^*, \{R(\nu) \mid \nu \in \mathcal{V}\})$ is a fully convex drawing of C. This completes the proof of Theorem 12.

From the argument for establishing Theorem 12, we easily derive the following corollary.

Corollary 17. *Let $C = (G, T)$ be a c-planar and connected clustered graph, and G be internally triconnected. Then*:

(i) *There exists a planar straight-line convex cluster drawing of C such that the drawing of G is inner-convex if and only if, for every non-root node ν of T, every connected component of the subgraph $G[V - V(\nu)]$ contains an outer vertex.*

(ii) *Such a drawing of C (if any) can be constructed from drawing ψ in $O(n^2)$ time, where each cluster is a convex hull of points in the cluster and n is the number of vertices in G.*

Proof. (i): By Lemmas 13 and 14, we see the necessity of this corollary. To show the sufficiency, we augment the clustered graph C as follows. Let V_o be the set of outer vertices in G. We create a new vertex s^* in the exterior of G, join G and s^* with new edges (s^*, v), $v \in V_o$ to obtain a new plane graph G^*, where its boundary f_o^* is a triangle, and add new clusters $\nu^* = V \cup \{s^*\}$, $\nu_v = \{v\}$, $v \in V_o$ to T.

We see that the resulting clustered graph $C^* = (G^*, T^*)$ remains c-planar and connected. Additionally, we easily see that C^* has no region-face crossing between regions for clusters and facial cycles. Therefore, by applying Theorem 12 to C^*, there is a fully convex drawing ψ of C^*. After removing the drawing of s^* and edges (s^*, v), $v \in V_o$ from ψ, we obtain a desired drawing of C.

(ii): Immediate from (i) and Theorem 12(ii). □

5. Conclusion

In this paper, we studied convex drawings of hierarchical planar graphs and clustered planar graphs. We first proved that every internally triconnected hierarchical plane graph with the outer facial cycle drawn as a convex polygon admits a convex drawing. Then we proved that every internally triconnected clustered plane graph with a completely connected clustering structure admits a convex drawing. We also presented algorithms to construct convex drawings of hierarchical planar graphs and clustered planar graphs.

It would be interesting to apply the notion of archfree paths to convex drawings or inner convex drawings of other types of graphs. Further, adding more constraints such as *grid* constraints to convex representations may be an interesting research direction in the future.

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