Numerically derived boundary conditions on artificial boundaries

A.S. Deakin\textsuperscript{a, *}, J.R. Dryden\textsuperscript{b}

\textsuperscript{a} Department of Applied Mathematics, University of Western Ontario, London, Ont., Canada N6A 5B7
\textsuperscript{b} Faculty of Engineering, University of Western Ontario, London, Ont., Canada N6A 5B7

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Abstract

We consider partial differential equations in an infinite domain in which an artificial boundary $B$ is introduced in order to restrict the computational domain to the region bounded by $B$. The nonlocal boundary condition on $B$ is determined for equations of the type $\nabla^2 u + k^2 u = 0$ in a separable coordinate system, and compared with two methods in which the boundary condition is approximated. One method uses the free space Green's function directly and does not involve the evaluation of surface integrals. The other method, in which the boundary condition is derived from the solution of the Dirichlet problem in the domain exterior to $B$, is considered by several authors in the literature. Using Laplace's equation in two dimensions, numerical results show that Green's function approach is accurate, and that the boundary condition can be computed readily with standard numerical packages.

Keywords: Numerical boundary conditions; Green's function; Finite elements

1. Introduction

We consider partial differential equations in an infinite domain as shown in Fig. 1 where there is a finite region, enclosed by a surface $B_i$, in which the equations may be nonlinear or have variable coefficients. In addition, there may be sources as well as boundaries with appropriate boundary conditions. In the region exterior to $B_i$, the partial differential equation is linear, no sources are present, and we assume that the free space Green's function is known. An artificial boundary $B$ that encloses $B_i$ is introduced, along with a boundary condition on $B$, so that the computational domain in $B$ is sufficiently small for accurate numerical work. In this paper we are concerned with the boundary condition on the artificial boundary $B$, and how this condition can be determined numerically. As with most references in the literature, we consider Laplace's equation and the reduced wave equation in the domain exterior to $B_i$. Our approach appears to apply more generally, and in particular, to Navier's equations in elastostatics.

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\textsuperscript{*} Corresponding author.
Fig. 1. The artificial boundary $B$ encloses a nonlinear region $B_i$ with unit exterior normal $n$ and $n'$, respectively. $D$ is the region exterior to $B$, and $D_i$ the region exterior to $B_i$.

Recently, an exact nonlocal boundary condition has been derived for simple boundaries such as a circle in two dimensions, or a sphere or a cylinder in three dimensions. Since this boundary condition is derived by solving the Dirichlet problem in the domain exterior to $B$, only simple artificial boundaries can be considered owing to the complexities of the boundary condition in other coordinate systems. In this paper we show how to obtain an accurate numerical estimate of this boundary condition in a way that can be extended to other geometrical shapes. The computational aspects are straightforward since only standard numerical packages are required. Except for Laplace’s equation in two dimensions, the partial differential equations that we consider have a nonlocal boundary condition on $B$ of the form

$$\frac{\partial u}{\partial n} = -Mu \equiv - \int_B f(x, x')u(x')dS', \quad x \text{ on } B, \quad (1)$$

where $n$ is the unit exterior normal to $B$ and $f$ is some appropriate function. The boundary condition is nonlocal in the sense that $\partial u/\partial n$ at a point on $B$ is expressed in terms of $u$ at all points on $B$. This form must be modified for Laplace’s equation in two dimensions.

There are three major types of boundary conditions on the artificial surface $B$. Firstly, the exact nonlocal boundary condition is discussed by many authors using different boundaries. Keller and Givoli [8] consider the reduced wave equation where $B$ is a circle in two dimensions and a sphere in three dimensions. The finite element formulation of the problem, using the nonlocal boundary condition, is presented as well as the effect of this boundary condition on the system of equations to be solved for $u$ at the nodes in the computational region. Numerical experiments are discussed showing the improved accuracy of this approach compared with the use of an asymptotic boundary condition. Givoli and Keller [4] determine the nonlocal boundary condition for problems in elastostatics and present some numerical results. For an infinite two-dimensional domain, the boundary condition relates the components of the stress $\sigma_{r\theta}$ and $\sigma_{rr}$, at a point on a circle, in terms of the displacements $u_r$ and $u_\theta$ at all points on the circle. MacCamy and Marin [10] consider the reduced wave equation in two dimensions with two regions and an interface condition. A nonlocal boundary condition is applied to an artificial boundary that encloses the
interface. Fix and Marin [3] discuss some numerical results for the reduced wave equation in a cylindrical region in which a nonlocal boundary condition is applied. Marin [11], Goldstein [5], and Hagstrom and Keller [6] consider a cylindrical domain in which an artificial boundary with a nonlocal boundary condition is introduced. Secondly, the boundary element method (cf. [2]), involves an integral equation over $B$ that relates $u$, $\partial u/\partial n$, and the free space Green's function. For a critique of this approach see [4]. A variation on the boundary element method was proposed in [14]. In this case $u$ on $B$ is expressed in terms of $u$, $\partial u/\partial n$ and the free space Green's function over a surface interior to $B$. The third approach involves an asymptotic boundary condition. Bayliss et al. [1] have derived a sequence of boundary conditions, for the reduced wave equation, that provide progressively more accurate approximations on the artificial boundary.

In Section 2, two forms of the nonlocal boundary condition are determined in a separable coordinate system for Laplace's and the reduced wave equation. With $N$ nodes on $B$, the boundary condition is constructed from the first $N$ eigenfunctions for the exterior problem. The example of Laplace's equation in two dimensions is presented in detail for elliptic coordinates. Also, in Section 2.4, the application of the boundary condition in the finite element formulation illustrates our approach to the boundary condition. In Section 3, the functions on $B$ and $B_i$ are discretized by defining nodes and shape (basis) functions on these surfaces. Then two approximate methods for computing the matrix that relates $\partial u/\partial n$ and $u$ at the nodes are described. Our approach involves the free space Green's function, and the other, as described in [8], requires that the solution of the exterior Dirichlet problem be known. Numerical results are presented in Section 4, for Laplace's equation in two dimensions, to illustrate the accuracy of the boundary condition, generated as outlined above, for unit point sources within $B$. Our main conclusions are given in Section 5.

2. Formulation of the problem

The partial differential equations, with the boundary condition at infinity, that we consider are Laplace's equation

$$\nabla^2 u = 0, \quad u \to 0 \quad \text{as} \quad x \to \infty, \quad x \in D_i$$

and the reduced wave equation

$$\nabla^2 u + k^2 u = 0, \quad x \in D_i,$$

$$r^{(d-1)/2}(ur - iku) \to 0 \quad \text{as} \quad r = |x| \to \infty,$$

where $D_i$ is the region exterior to $B_i$; $k$ is a constant; $d$ is the dimension; and the Sommerfeld radiation condition is applied at infinity in (3). The exception that we consider is Laplace's equation in two dimensions in which the condition at infinity can be more general than that given in (2).

In this section, the nonlocal boundary condition in a separable coordinate system is derived. The boundary condition is derived initially for (2),(3) and then it is extended to include Laplace's equation in which the solution is nonzero at infinity. An example, Laplace's equation in two dimensions where $B$ is an ellipse, illustrates the various aspects of the boundary condition.

The differential equation is assumed to be separable in some coordinate system $x = (\rho, \phi)$, where the boundary $B$ is given by $\rho = \rho_0$, and $\phi = (\phi_1, \ldots, \phi_{d-1})$ is a vector representing the tangential
If the metric is defined by \( ds^2 = h_x^2 \, d\rho^2 + \sum_{i=1}^{d-1} h_i^2 \, d\phi_i^2 \), then \( h_\rho^{-1} \partial/\partial \rho = n \cdot \nabla \), where \( n \) is the unit exterior normal to \( B \). The set of eigenfunctions for the exterior problem, satisfying the appropriate boundary conditions at infinity, are taken to be \( \{ v_n(\rho) u_n(\phi) \mid n = 0, 1, 2, \ldots \} \), where \( u_0 \) is a constant independent of \( \phi \). For Laplace's equation, an arbitrary constant can be included in \( v_0(\rho) \), and in two dimensions, an additional unbounded eigenfunction \( v_1(\rho) u_1(\phi) \) \((u_1 = u_0)\) is required.

### 2.1. Boundary condition

It is necessary to discretize the boundary condition (1) by choosing nodes \( x_i \) \((i = 1, \ldots, N)\) on the artificial boundary \( B \). The boundary condition becomes

\[
\frac{\partial u(x_i)}{\partial n} + \sum_{j=1}^{N} M_{ij} u(x_j) = 0
\]

and we now derive the matrix \( (M_{ij}) \) using the set of eigenfunctions for the exterior problem. For problems (2), (3), we construct the solution

\[
u^l(x) = \sum_{n=0}^{N-1} C^n_l u_n(\phi) v_n(\rho)/v_n(\rho_0), \quad \rho \geq \rho_0 \quad (l = 1, \ldots, N),
\]

where the coefficients \( C^n_l \) are such that \( u^l(x) \) is one at the \( l \)th node and zero at the remaining nodes; that is, \( u^l(x^i) = \delta_{il} \) and \( \delta_{il} = \sum_{n=0}^{N-1} C^n_l u_n(\phi^i) \). At this point we assume that, for the chosen nodes \( x_i \) \((i = 1, \ldots, N)\), the coefficients \( C^n_l \) exist, and later, we consider a more appropriate choice of the nodes in order to enhance the symmetry of the boundary condition. In matrix notation, this last equation can be expressed as \( I = C U \), where the \((k, j)\) elements of \( U \) and \( C \) are \( u_{k-1}(\phi^j) \) and \( C^k_j \), respectively, and \( I \) is the identity matrix. Since \( UC = I \) as well, then \( \delta_{kj} = \sum_{n=1}^{N} u_{k-1}(\phi^i) C^k_{j-1} \). With a change in the indices, the equations relating \( u_n(\phi^j) \) and \( C^l_m \) are \((i, j = 1, \ldots, N; \quad m, n = 0, \ldots, N-1)\)

\[
d_{ij} = \sum_{n=0}^{N-1} C^n_l u_n(\phi^i), \quad \delta_{mn} = \sum_{j=1}^{N} C^n_m u_n(\phi^j).
\]

The matrix \( M_{ij} \) in the boundary condition \( \partial u(x_i)/\partial n + \sum_j M_{ij} u(x_j) = 0 \) can now be determined. Since

\[
u^l(x^i) = \sum_{n=0}^{N-1} C^n_l u_n(\phi^i), \quad \frac{\partial u^l(x^i)}{\partial n} = \sum_{n=0}^{N-1} \frac{C^n_l u_n(\phi^i) v_n(\rho_0)}{h_\rho(x^i) v_n(\rho_0)}
\]

then, upon substituting into the boundary conditions, \( M_{ij} = -\partial u^l(x^i)/\partial n \). In the sequel, \( \rho_0 \) has been replaced by \( \rho \) on \( B \) in order to simplify the notation. We now show the following theorem.

**Theorem 1.** The boundary condition, where \( x = (\rho, \phi) \) on \( B \) and \( u \to 0 \) at infinity,

\[
\frac{\partial u(x^i)}{\partial n} + \sum_{j=1}^{N} M_{ij} u(x^j) = 0, \quad M_{ij} = -\sum_{n=0}^{N-1} \frac{C^n_l u_n(\phi^i) v_n(\rho)}{h_\rho(x^i) v_n(\rho)}
\]
is accurate provided $u(x)$ and $\partial u(x)/\partial n$ on $B$ can be adequately approximated by some linear combination of the first $N$ eigenfunctions; namely,

$$
\begin{align*}
  u(x) & \approx \sum_{n=0}^{N-1} D_n u_n(\phi)v_n(\rho), \\
  \frac{\partial u(x)}{\partial n} & \approx \sum_{n=0}^{N-1} \frac{D_n u_n(\phi)v_n'(\rho)}{h_\rho(x)}
\end{align*}
$$

(9)

for some constants $D_n$.

It is straightforward to show, using the identity $\delta_{mn} = \sum_{j=1}^{N} C'_n u_n(\phi^j)$, that these approximations for $u(x^j)$ and $\partial u(x^j)/\partial n$ satisfy the boundary condition. However, the boundary condition does not apply for Laplace’s equation with $u$ nonzero at infinity, and an alternate formulation is required.

2.2. Alternate form of the boundary condition

We define, from (8),

$$
M_{ij} = - \frac{C'_0 u_0 v'(\rho)}{h_\rho(x^i)v_0(\rho)} + \frac{K_{ij}}{h_\rho(x^i)}, \quad K_{ij} = - \sum_{n=1}^{N-1} \frac{C'_n u_n(\phi^i)v_n'(\rho)}{v_n(\rho)}.
$$

(10)

With the substitution of these expressions into the boundary condition (8),

$$
\begin{align*}
  h_\rho(x^i) \frac{\partial u(x^i)}{\partial n} - \sum_{j} \frac{C'_0 u_0 v'_0(\rho)u(x^j)}{v_0(\rho)} + \sum_{j} K_{ij} u(x^j) &= 0.
\end{align*}
$$

(11)

The second term in this equation simplifies. Using the expression (9) for $u(x^j)$, this term is equal to $-D_0 u_0 v'(\rho)$, and since $\sum_{i} C'_i h_\rho(x^i) \partial u(x^i)/\partial n = D_0 v'_0(\rho)$ and $\sum_{i} C'_i = \delta_{n0}/u_0$, we can now show the following theorem.

**Theorem 2.** The general nonlocal boundary condition that holds for all cases, and in particular, for Laplace’s equation in any dimension, is

$$
\begin{align*}
  h_\rho(x^i) \frac{\partial u(x^i)}{\partial n} - \left\langle h_\rho \frac{\partial u}{\partial n} \right\rangle + \sum_{j} K_{ij} u(x^j) &= 0, \quad \sum_{j} K_{ij} = 0,
\end{align*}
$$

(12)

where

$$
\left\langle h_\rho \frac{\partial u}{\partial n} \right\rangle = \sum_{i} (u_0 C'_0) h_\rho(x^i) \frac{\partial u(x^i)}{\partial n}, \quad \sum_{i} u_0 C'_i = 1
$$

(13)

represents a weighted average of $h_\rho(x^i) \partial u(x^i)/\partial n$.

Since $\left\langle h_\rho \partial u/\partial n \right\rangle = D_0 u_0 v'_0$ for problems (2), (3), this term has the same physical interpretation as the first term in the expansion (9) of $h_\rho \partial u/\partial n = \partial u/\partial \rho$. For Laplace’s equation in any dimension, the addition of an arbitrary constant to the solution clearly has no effect on the boundary condition. For Laplace’s equation in two dimensions, let $w$ be a bounded solution and let $v_{-1}(\rho)$ be the appropriate unbounded eigenfunction. Then a general solution has the form $u = D_{-1} u_0 v_{-1} + w$ for some constant $D_{-1}$, where $w$ satisfies (12) and (13) and has an expansion of the form (9), where $v_0$ contains an additive constant. Upon substituting for $w$ in the boundary condition (12), we have...
the same boundary condition holding for \( u \). Here, however, \( \langle h_{\rho} \partial u / \partial n \rangle = D_{-1} u_0 v'_{-1} \). As we indicate in our example, the symmetry of the boundary condition depends on the choice of the nodes.

### 2.3. An example

We will use this example, in which the boundary condition can be derived explicitly, as a comparison for our numerical approach. We consider Laplace's equation in two dimensions in which \( B \) is an ellipse. The elliptic coordinate system is defined by

\[
\begin{align*}
x_1 &= c \cosh(\rho) \cos(\phi), \\
x_2 &= c \sinh(\rho) \sin(\phi),
\end{align*}
\]

where \( a = c \cosh(\rho) \) and \( b = c \sinh(\rho) \) are the principal semi-axes and \( c \) is a constant. The general eigenfunctions periodic in \( \phi (\phi = \phi_1) \) are \( \{ \rho, 1, \ldots, \cos(n\phi)e^{-n\rho}, \sin(n\phi)e^{-n\rho}, \ldots \} \) and the metric is

\[
ds^2 = h_2^2 d\rho^2 + h_1^2 d\phi^2,
\]

where \( h_\rho = h_1 = (a^2 - c^2 \cos^2(\phi))^{1/2} \).

In order to enhance the symmetry of the boundary condition, we select the nodes at

\[
x^i = (\rho, \phi^i), \quad \phi^i = (i - 1)2\pi/N \quad (i = 1, \ldots, N; \; N \text{ even}).
\]

An important consequence of this choice of nodes is the orthogonality of the sine and cosine functions. With the definitions

\[
\begin{align*}
&u_0 = \left[ \frac{1}{N} \right]^{1/2}, \quad \ldots, \quad u_{2l-1} = \left[ \frac{2}{N} \right]^{1/2} \cos(l\phi), \\
u_{2l} = \left[ \frac{2}{N} \right]^{1/2} \sin(l\phi), \quad \ldots, \quad u_{N-1} = \left[ \frac{1}{N} \right]^{1/2} \cos \left( \frac{N\phi}{2} \right)
\end{align*}
\]

the functions \( u_n(\phi) \) have the property (cf. [9]) that

\[
\sum_{i=1}^{N} u_n(\phi^i) u_m(\phi^i) = \delta_{nm} \quad (n, m = 0, \ldots, N - 1). \tag{17}
\]

Since \( C^i_m \) in (6) satisfies the property \( \delta_{mm} = \sum_{j=1}^{N} C^j_m u_n(\phi^j) \), clearly \( C^i_m = u_m(\phi^i) \), and \( K_{ij} \) in (12) can be expressed in the simple form, denoted by \( K_{ij}^e \),

\[
K_{ij}^e = \sum_{k=1}^{N/2-1} \frac{2k}{N} \cos k(\phi^j - \phi^i) + \frac{1}{2} \cos \frac{N}{2} (\phi^j - \phi^i). \tag{18}
\]

This series can be summed (cf. [7]) to

\[
K_{ii}^e = \frac{1}{4} N, \quad K_{ij}^e = \frac{(-1)^{\sigma} - 1}{2 N \sin^2(\sigma\pi/N)}, \quad \sigma = |i - j| \neq 0. \tag{19}
\]

The boundary condition in elliptic coordinates is

\[
h_{\rho}(x^i) \frac{\partial u(x^i)}{\partial n} = \left\langle h_{\rho} \frac{\partial u}{\partial n} \right\rangle + \sum_j K_{ij}^e u(x^j) = 0, \tag{20}
\]
where
\[
\left\langle h_p \frac{\partial u}{\partial n} \right\rangle = \frac{1}{N} \sum_{i=1}^{N} h_p(x^i) \frac{\partial u(x^i)}{\partial n}
\]
represents an approximation to the average flux density in \( \phi \) (see Section 2.4). Note that \( K_{ij}^e \) is symmetric and the sum of each row is zero. In addition, the matrix \( (K_{ij}^e) \) is circulant in that the \((i, j)\) element is equal to the \((i + 1, j + 1)\) element and the \((i, N)\)th is equal to the \((i + 1, 1)\)th \((i < N, j < N)\). Thus the first row of \( (K_{ij}^e) \) is
\[
(K_{11}^e, K_{12}^e, \ldots, K_{1N/2}^e, K_{1N/2+1}^e, K_{1N/2}^e, \ldots, K_{12}^e) \quad (N \text{ even})
\]
and the remaining rows can be generated from the first row since the matrix is a circulant. For example, to obtain the second row, shift the elements in the first row to the right by one position where the last element of the first row becomes the first element of the second row. There are only \( \frac{1}{2}(N + 2) \) distinct elements in \( (K_{ij}^e) \), and since the sum of each row is zero, there are only \( \frac{1}{2}N \) unknowns in the matrix.

The symmetries of \( (K_{ij}^e) \) can be seen from a more geometrical and physical view if we consider the boundary condition in polar coordinates, where \( B \) is a circle of radius \( R \) with equally spaced nodes \( x^i = (R, (i - 1)2\pi/N) \). In this case, \( K_{ij} \) in (10) is equal to \( R^{-1}K_{ij}^e \), where \( v_\rho(\rho) = \rho^{-n}, h_p = 1 \) and \( h_1 = R \). Hence, the boundary condition in polar coordinates is
\[
\left\langle h_p \frac{\partial u}{\partial n} \right\rangle = \left\langle \frac{\partial u}{\partial n} \right\rangle + \sum_j \frac{1}{R} K_{ij}^e u(x^i) = 0.
\]
Let \(-\partial u(x^i)/\partial n\) be the flux in the direction of the normal to \( B \) at the node \( x^i \). Then \( K_{ij}^e/R \) is the variation of the flux, about the average flux density, at \( x^i \) owing to the source \( u^j(x) \) in (5). From considerations of symmetry, we would expect \( K_{ij}^e = K_{i+1,j+1}^e \text{ and } K_{i+1,j}^e = K_{i,N}^e \). There is also symmetry with respect to reflection through the radial line that bisects the arc between \( x^i \) and \( x^j \), which implies that \( K_{ij}^e = K_{ji}^e \).

2.4. Finite element formulation

Although the boundary condition (8) or (12) can be combined with the finite difference approach in the computational region, we expect that most users would prefer the finite element approach. In this section we outline the application of the boundary condition to the finite element formulation of the problem as described in [8]. Here, only those details that differ from the results in this reference are provided.

For the problem that we consider in detail, Eqs. (2) and (3) have a nonhomogeneous term in a region bounded by \( B_i \) and a surface \( \Gamma = \Gamma_g \cup \Gamma_h \), interior to \( B_i \). Let \( \Omega \) be the computational domain bounded by the surfaces \( B \) and \( \Gamma \). The equation and boundary conditions are
\[
\nabla^2 u + k^2 u + f = 0 \quad \text{in } \Omega,
\]
\[
u = g \text{ on } \Gamma_g, \quad \partial u/\partial n = h \text{ on } \Gamma_h, \quad \partial u/\partial n = -Mu \text{ on } B.
\]
The approach that we use is identical to that in [8] except for the treatment of the integral over \( B \). Briefly, the weak formulation of this problem leads to a variational problem. Find \( u \):

\[
\begin{align*}
\tag{26}
a(w, u) + b(w, u) &= (w, f) + (w, h)_r, \\
\tag{27}a(w, u) &= \int_\Omega (Vw \cdot Vu - k^2wu) \, dV, \quad (w, f) = \int_\Omega w f \, dS, \\
\tag{28}b(w, u) &= -w \frac{\partial u}{\partial n} \, dS = wMu \, dS, \quad (w, h)_r = \int_{\Gamma_r} wh \, dS,
\end{align*}
\]

where \( u \) and \( w \) are functions in a Sobolev space, and \( w \) is any arbitrary function such that \( w = 0 \) on \( \Gamma_r \). The next step is to define elements and nodes in \( \Omega \) as well as shape (basis) functions on these elements. This leads to a set of equations to solve for \( u \) at the nodes. For this approach, only polar or spherical coordinates—where \( B \) is a circle or a sphere—are employed since \( b(w, u) \) involves the numerical integration of the eigenfunctions over the elements; a task that is more difficult to do in other orthogonal coordinate systems.

We extend the coordinate systems to include elliptic in two dimensions, and to include in three dimensions: oblate and prolate spheroidal, ellipsoidal, where \( B \) and \( B_i \) are surfaces on which \( \rho \) is constant. These coordinate systems are more appropriate if the surface \( \Gamma \) and the region in which \( f \) in (24) is nonzero is more accurately enclosed by an ellipse or an ellipsoid. In addition, we develop a numerical approach that, at most, requires that the eigenfunctions in the tangential variables be evaluated only at the nodes. Only the term \( b(w, u) \) is considered in the sequel; all other details including the nodes, elements, and the shape functions defined over these elements, are the same as in [8].

For these separable coordinate systems (cf. [12]), we have on \( B \)

\[
\tag{29}dS = \prod_{i=1}^{d-1} h_i \, d\phi = h_\rho \sigma(\rho) \mu(\phi) \, d\phi, \quad \frac{\partial u}{\partial n} \, dS = \frac{\partial u}{\partial \rho} \sigma(\rho) \mu(\phi) \, d\phi.
\]

Also, we express \( \frac{\partial u}{\partial \rho} \) and \( w \) on \( B \) in terms of the basis \( u^i(x) \) as defined in (5) so that

\[
\tag{30}w(x) \approx \sum_{i=1}^{N} w(x^i) u^i(x), \quad \frac{\partial u}{\partial \rho}(x) \approx \sum_{k=1}^{N} \frac{\partial u}{\partial \rho}(x^k) u^k(x), \quad u^i(x) = \sum_{n=0}^{N-1} C_n^i u_n(\phi).
\]

The eigenfunctions \( u_n(\phi) \) on \( B \) are surface harmonics (cf. [15] for Laplace’s equation), and they are normalized such that

\[
\tag{31}\int_B u_n(\phi) u_m(\phi) \mu(\phi) \, d\phi = A_\delta_{nm}.
\]

With these details,

\[
\tag{32}b(w, u) = -\int_B w \frac{\partial u}{\partial n} \, dS = \sum_{i, k, n} w(x^i) \frac{\partial u}{\partial \rho}(x^k) C_n^i C_k^i A\sigma(\rho),
\]
and with the boundary condition (8), $b(w, u)$ simplifies further to

$$b(w, u) = \sum_{i,j} w(x^i)u(x^j)b_{ij}, \quad b_{ij} = -A\sigma(\rho) \sum_{n=0}^{N-1} C_n^i C_n^j v_n(\rho)/v_n(\rho).$$  \hspace{1em} (33)

We now show that $P_{ij}$, defined as $h_p(x^i)M_{ij}$, has the following properties.

**Theorem 3.** The matrix $(P_{ij})$ has eigenvalues $\lambda_m = -v_m'(\rho)/v_m(\rho)$ ($m = 0, \ldots, N - 1$), where $(u_m(\phi^i))$ and $(C_m^i)$ are the corresponding right and left eigenvectors, respectively.

Using the definition of $M_{ij}$, it follows readily that

$$\sum_j C_m^j P_{ij} = \lambda_m C_m^j, \quad \sum_j P_{ij} u_m(\phi^j) = \lambda_m u_m(\phi^i),$$  \hspace{1em} (34)

where the two sets of eigenvectors are related by $\sum_i C_m^i u_m(\phi^i) = \delta_{mn}$, or $CU = I$ in matrix notation. Since

$$UU^t(b_{ij}) = A\sigma(\rho)(P_{ij}),$$  \hspace{1em} (35)

$b_{ij}$ in (33) can be computed once $P_{ij}$ is approximated as described in Section 3. Alternatively, it may be more accurate to compute $U$ directly from the eigenfunctions and then use (35) to determine $(b_{ij})$.

Similarly, using the alternate boundary condition (12),

$$b(w, u) = \sum_{i,j} w(x^i)u(x^j)b'_{ij} - \sum_i w(x^i)d_i, \quad d_i = \left(\frac{\partial u}{\partial n}\right) C_0 A\sigma(\rho)/u_0,$$  \hspace{1em} (36)

where $b'_{ij}$ is the same as $b_{ij}$ except that the sum in $n$ is from 1 to $N - 1$. The physical interpretation of $\langle \partial u/\partial \rho \rangle$ in (36) follows from (32) upon setting $w = -1$. We obtain

$$\langle \partial u/\partial \rho \rangle = \langle h_p \partial u/\partial n \rangle = Q^{-1} \int_B \frac{\partial u}{\partial n} \, dS, \quad Q = \int_B \sigma(\rho) \mu(\phi) \, d\phi,$$  \hspace{1em} (37)

where $\langle \partial u/\partial \rho \rangle$ is proportional to the net flux through the surface $B$, and, in elliptic and polar coordinates, $Q$ is equal to $2\pi$ and $2\pi R$, respectively. In the special case of Laplace’s equation in two dimensions, and this is the only case in which the alternate boundary condition must be used, the expression for $b(w, u)$ simplifies greatly since, in elliptic and polar coordinates as described in Section 2.3, $C_n^i = u_n(\phi^i)$, $\sigma(\rho) = 1$, and $A = 2\pi/N$. Then

$$b(w, u) = \frac{2\pi}{N} \sum_{i,j} w(x^i)u(x^j)K_{ij} - \frac{2\pi}{NQ} \int_B \langle \partial u/\partial n \rangle \, dS \sum_i w(x^i).$$  \hspace{1em} (38)

We show later that $K_{ij}$ can be computed accurately using standard numerical packages.

**3. Two approximations of the boundary condition**

We describe two methods to approximate the matrix $(K_{ij})$ in (12). The first one uses the free space Green’s function to generate the matrix, and this approximation is accurate provided bounds on
a certain expression are sufficiently small. In the other approach, the operator M in (1) is determined by solving the exterior Dirichlet problem for simple geometries. This Dirichlet to Neumann map (DtN) can then be approximated using shape functions to obtain the matrix in (12). These two approximations of $K_{ij}$ for Laplace's equation in two dimensions are compared with the exact value in Section 4.

3.1. The Green's function approach

The free space Green's function satisfies the boundary condition (8) or (12) provided

$$G(x, x') \approx \sum_{n=-1}^{N-1} D_n(\rho', \phi')u_n(\phi)v_n(\rho), \quad h_\rho(x) \frac{\partial G(x, x')}{\partial n} \approx \sum_{n=-1}^{N-1} D_n(\rho', \phi')u_n(\phi)v'_n(\rho)$$

are adequate approximations for $x = (\rho, \phi)$ on $B$ and $x' = (\rho', \phi')$ on $B_i$. In this section, the matrix $(M_{ij})$ is generated directly from the free space Green's function. As we will show in the section on our numerical results, it is a practical and accurate approach.

The initial step in this approach concerns the approximation of $u(x)$ and its derivative on $B$ in terms of shape functions. Once the nodal points $x^j$ and the elements are defined, the shape functions $N_j(x)$ are defined on these elements (cf. [2]). Shape functions have the following properties: $N_j$ is equal to one at the node $x^j$; $N_j$ is equal to zero at all other nodes; and $N_j$ is identically zero on all elements that do not have $x^j$ as a node. Thus the function $u(x)$ and the normal derivative $\frac{\partial u(x)}{\partial n}$ have the approximations

$$u(x) \approx \sum_{j=1}^{N} u(x^j)N_j(x), \quad \frac{\partial u(x)}{\partial n} \approx \sum_{j=1}^{N} \frac{\partial u(x^j)}{\partial n}N_j(x).$$

To approximate $M_{ij}$ numerically, we start with Green's formula

$$u(x) = \int_{B_i} \left( \frac{\partial}{\partial n} G(x, x')u(x') - G(x, x') \frac{\partial}{\partial n} u(x') \right) dS'$$

for $x$ in $D_i$, where $G(x, x')$ is Green's function. We substitute this expression into the nonlocal boundary condition (1), and interchange the order of integration and the operator $\partial/\partial n + M$. Upon defining

$$\frac{\partial}{\partial n} G(x, x') = H(x, x')$$

we have

$$\left( \frac{\partial}{\partial n} + M \right) u = \int_{B_i} \left[ \left( \frac{\partial}{\partial n} + M \right) H(x, x')u(x') - \left( \frac{\partial}{\partial n} + M \right) G(x, x') \frac{\partial}{\partial n} u(x') \right] dS'.$$

The next step is to discretize $H$ and $G$ in both the $x$ and the $x'$ variables by expressing $H$ and $G$ as a sum over the shape functions $N_j(x)$ ($j = 1, \ldots, N$) defined on $B$, and the shape functions $N_k(x')$ ($k = 1, \ldots, N$) defined on $B_i$. Thus

$$G(x, x') \approx \sum_{j=1}^{N} \sum_{k=1}^{N} G(x^j, x'^k)N_j(x)N_k(x')$$

(44)
and there is a similar expression for \( \partial G / \partial n \), \( \partial H / \partial n \), and \( H \). Substituting these expressions into (43), we have

\[
\left( \frac{\partial}{\partial n} + M \right) u \approx \sum_{i=1}^{N} N_i(x) \sum_{k=1}^{N} \left( I_{ik} \int_{B_i} N'_k(x') u(x') dS' - L_{ik} \int_{B_i} N'_k(x') \frac{\partial u(x')}{\partial n} dS' \right),
\]

\[
I_{ik} = \left[ \frac{\partial H(x^i, x'^k)}{\partial n} + \sum_{j=1}^{N} M_{ij} H(x^i, x'^k) \right], \quad L_{ik} = \left[ \frac{\partial G(x^i, x'^k)}{\partial n} + \sum_{j=1}^{N} M_{ij} G(x^i, x'^k) \right].
\]

Our approach in determining \((M_{ij})\) is the following.

1. Select the surface \( B \), and to some extent \( B_i \), and the shape functions \( N_j(x) \) and \( N'_k(x') \) such that the nodal approximations in (40) and (44) have an acceptable error.

2. The \((M_{ij})\) in the boundary condition \( \partial u(x^i)/\partial n + \sum_{j=1}^{N} M_{ij} u(x^j) = 0 \) is determined by solving

\[
L_{ik} = \frac{\partial G(x^i, x'^k)}{\partial n} + \sum_{j=1}^{N} M_{ij} G(x^i, x'^k) = 0 \quad (i, k = 1, \ldots, N).
\]

3. Once the bound for \( I_{ik} \) is known, the error bound for \( \partial u/\partial n + Mu \) in (43) can be determined. This last point will not be attempted here since it involves a study of the nodal approximations in (40) and (44).

In order to find \( u(x) \) at a point outside \( B \), Green’s formula (41) can be used. Once \( u(x) \) and \( \partial u(x)/\partial n \) are known on \( B \), the numerical integration of (41) is straightforward since the singularities of the integrand occur on \( B_i \).

The analysis of this section can be repeated if the boundary condition \( \partial u/\partial n = -Mu \) on \( B \) is replaced by the alternate boundary condition, the discrete form of which is given by (12). From (4), (10) and (37), the alternate boundary condition has the form

\[
\frac{h}{\partial n} \frac{\partial u(x)}{\partial n} - Q^{-1} \int_{\partial B} \frac{\partial u}{\partial n} dS + Ku(x) = 0, \quad Q = \int_{\partial B} \sigma(\rho) \mu(\phi) d\phi.
\]

The steps from (40) to (46) can be repeated to approximate the alternate boundary condition by the same expression in (45) as the approximation for \( \partial u/\partial n + Mu \) except that \( I_{ik} \) and \( L_{ik} \) are replaced by

\[
I_{ik} = h(x^i) \frac{\partial H(x^i, x'^k)}{\partial n} - \left( h(x^i) \frac{\partial H}{\partial n} \right) + \sum_{j=1}^{N} K_{ij} H(x^j, x'^k),
\]

\[
L_{ik} = h(x^i) \frac{\partial G(x^i, x'^k)}{\partial n} - \left( h(x^i) \frac{\partial G}{\partial n} \right) + \sum_{j=1}^{N} K_{ij} G(x^j, x'^k).
\]

3.2. The DtN approach

The nonlocal boundary condition (1) for Laplace’s equation in two dimensions where the solution vanishes at infinity can be readily derived following the approach of Keller and Givoli [8], where \( B \) is a circle of radius \( R \). In polar coordinates, the solution of the Dirichlet problem in the
domain exterior to $B$ is

$$u(\rho, \phi) = \frac{1}{\pi} \sum_{n=1}^{\infty} \left( \frac{R}{\rho} \right)^n \int_0^{2\pi} \cos n(\phi - \phi') u(R, \phi') \, d\phi'. \tag{51}$$

The normal derivative on $B$ is

$$u_\rho(R, \phi) = -\sum_{n=1}^{\infty} \frac{n}{\pi R} \int_0^{2\pi} \cos n(\phi - \phi') u(R, \phi') \, d\phi'. \tag{52}$$

The results of Section 2.3 are obtained upon approximating $u(R, \phi)$ by $\sum_{i} u(R, \phi') u'(R, \phi)$. However, we follow the approach of Keller and Givoli [8] and use shape functions to approximate $u(R, \phi)$ as in (40).

To maximize the symmetry we take the $N$ nodes ($N$ even) to be equally spaced on the circle of radius $R$ at $\phi_j = (j-1)2\pi/N$. Substituting the expression (40) for $u$ in terms of the shape functions into (52) and taking the first $\frac{1}{2}N$ terms of the divergent series — the same number of terms as in (18) — the approximation of $K_{ij}$ in the boundary condition is defined as

$$\frac{1}{R} K_{ij} = \sum_{n=1}^{N/2} \frac{n}{\pi R} \int_0^{2\pi} \cos n(\phi - \phi') N_j(R, \phi') \, d\phi'. \tag{53}$$

In order for $K_{ij}$ to have the symmetries as described in Section 2.3, $N_{j+1}(R, \phi)$ must be a translation of $N_j(R, \phi)$. This condition is satisfied for linear shape functions but not for quadratic shape functions as usually defined. We can, however, define a symmetric quadratic shape function in the following way. The quadratic shape functions are defined by $\{1 - \eta^2, \frac{1}{2}\eta(\eta - 1), \frac{1}{2}\eta(\eta + 1)\}$ over each element where the elements are defined by $\phi = (\eta + j - 1)2\pi/N, \ -1 \leq \eta \leq 1$ ($j = 1, 3, \ldots, N - 1$). The same quadratic shape functions are now defined over another set of elements $\phi = (\eta + j)2\pi/N, \ -1 \leq \eta \leq 1$ which is a translation of the first set through half an element. With the addition of these two sets of functions, multiplied by $\frac{1}{2}$, we have a symmetric set of quadratic shape functions $N_j(R, \phi)$. In our numerical results, we consider only linear and quadratic shape functions, although more accurate results are certainly possible with cubic or higher-order shape functions.

4. Numerical results

There are two sets of numerical results that we would like to present in this section using our example in Section 2.3 of Laplace’s equation in two dimensions as a comparison. The two methods, described in the previous section, generate estimates of the $K_{ij}$ in the boundary condition (12), and we compare these estimates with the exact values in (20) and (23). Our results are presented in Tables 1–4. In Table 1, the first few elements of the first row of $(K_{ij})$ in (19) are given for 10 and 40 nodes. The remaining elements of the matrix can be generated using the symmetry properties of the matrix. In the other tables various bounds are given to show how accurate the two approximate methods are. Our conclusions are presented in the next section: the rest of this section deals with the details of the bounds where all numerical calculations were carried out to approximately 16 digits of accuracy.
Table 1
Elements $K^e_{i,j}$ to $K^e_{s,s}$, where $s = 6$ for $N = 10$ and $s = 21$ for $N = 40$. See (22) for the remaining elements of $(K^e_I)$

<table>
<thead>
<tr>
<th>$N = 10$</th>
<th>2.500</th>
<th>-1.047</th>
<th>0.000</th>
<th>-0.153</th>
<th>0.000</th>
<th>-0.100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 40$</td>
<td>10.000</td>
<td>-4.061</td>
<td>0.000</td>
<td>-0.459</td>
<td>0.000</td>
<td>-0.171</td>
</tr>
<tr>
<td></td>
<td>-0.092</td>
<td>0.000</td>
<td>-0.059</td>
<td>0.000</td>
<td>-0.043</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>0.000</td>
<td>-0.029</td>
<td>0.000</td>
<td>-0.026</td>
<td>0.000</td>
<td>-0.025</td>
</tr>
</tbody>
</table>

Table 2
Error bounds for circle $B$ of radius 10 with $N$ nodes. In boundary condition (57), $z = e$ or $z = n$ in $K^e_{i,j}$. $(K^e_I)$ is given in Table 1 and $(K^e_{i,j})$ is generated with sources on $B_i$ of radius $R'$. The remaining entries in each row give the errors for these matrices. Max $|I_{ik}|$ is equal to maximum value of the residual $|I_{ik}|$ in (49) and $\Delta K^e_{ij}$ is equal to Max $|K^e_{ij} - K^e_{ij}|$ for all $i, j$ and $k$. MREN: maximum relative error in the normal derivative from (57) where the unit point sources are placed on a circle with radius $R' = 2, 4, 6, 8$. *: Best results using a program with 16 digits of accuracy

| $z$ | $R'$ | Max $|I_{ik}|$ | $\Delta K^e_{ij}$ | MREN |
|-----|-----|--------------|------------------|------|
|     |     | $R' = 2$ | $R' = 4$ | $R' = 6$ | $R' = 8$ |
| $N = 10$ |     |             |             |             |             |
| $e$ | na  | na          | 0             | 2.5E - 5    | 1.9E - 3    | 2.7E - 2    | 2.5E - 1    |
| $n$ | 0.2*| 3.6E - 12   | 1.1E - 4      | 2.5E - 5    | 1.9E - 3    | 2.7E - 2    | 2.5E - 1    |
|     | 3   | 7.5E - 6    | 2.6E - 2      | 8.6E - 5    | 2.4E - 3    | 2.7E - 2    | 2.5E - 1    |
|     | 4   | 4.6E - 5    | 4.9E - 2      | 3.4E - 4    | 3.3E - 3    | 3.0E - 2    | 2.6E - 1    |
| $N = 40$ |     |             |             |             |             |             |             |
| $e$ | na  | na          | 0             | 5.7E - 14   | 6.6E - 10   | 4.6E - 6    | 3.4E - 3    |
| $n$ | 3*  | 3.7E - 14   | 9.1E - 3      | 1.5E - 13   | 7.8E - 10   | 4.3E - 6    | 3.4E - 3    |
|     | 5   | 2.9E - 9    | 3.3E - 2      | 2.7E - 12   | 2.1E - 9    | 5.6E - 6    | 3.5E - 3    |
|     | 7   | 1.1E - 5    | 1.2E - 1      | 4.6E - 7    | 1.6E - 6    | 1.9E - 5    | 4.1E - 3    |

The free space Green's function $G(x, x')$, and its expansion in elliptic coordinates, are given by the following expressions:

$$G(x, x') = -\frac{1}{2\pi} \ln r, \quad r^2 = \sum_{i=1}^{N} (x_i - x'_i)^2,$$

$$G(x, x') = -\frac{1}{2\pi} (\rho + \ln(c/2)) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} (e^{-n(\rho - \rho')} \cos n(\phi - \phi') + e^{-n(\rho + \rho')} \cos n(\phi + \phi')).$$

Let the semi-major and semi-minor axes of $B$ and $B_i$ be $[a, b]$ and $[a', b']$, respectively, then $a^2 - b^2 = a'^2 - b'^2 = c^2$ and $e^\rho = (a + b)/c$ and $e^{\rho'} = (a' + b')/c$. In the example of Section 2.3, we
Table 3
Error bounds for ellipse $B$ with semi-major axis 10 and semi-minor axis 5. To generate $(K_{ij}^n)$, unit sources are placed on the ellipse $B_i$ with semi-major axis $a'$ and semi-minor axis $b'$. The remaining details are the same as those in Table 2 except that in MREND the unit sources are placed on an ellipse with semi-axes [8.7, 1.0], etc.

<table>
<thead>
<tr>
<th>$a'$, $b'$</th>
<th>$[8.7, 1.0]$</th>
<th>$[8.9, 1.9]$</th>
<th>$[9.1, 2.9]$</th>
<th>$[9.5, 3.9]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$ na na</td>
<td>2.6E-5 3.1E-4 4.1E-3 6.8E-2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n$ [8.7, 0.5]*</td>
<td>5.3E-5 6.7E-2 4.2E-5 4.1E-4 5.0E-3 7.7E-2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n$ [9.0, 2.4]</td>
<td>3.3E-3 1.9E-1 1.9E-4 7.4E-4 6.2E-3 8.6E-2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4
Error bounds for circle of radius 10 where linear and quadratic shape functions (SF) are used to generate $(K_{ij}^n)$ in (57). See Table 2 for other details.

<table>
<thead>
<tr>
<th>SF</th>
<th>$\Delta K_{ij}^n$</th>
<th>MREND</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R' = 1$</td>
<td>$R' = 2$</td>
</tr>
<tr>
<td>$N = 10$</td>
<td>lin 0.65</td>
<td>4.3E-3</td>
</tr>
<tr>
<td></td>
<td>quad 0.48</td>
<td>7.1E-4</td>
</tr>
<tr>
<td>$N = 40$</td>
<td>lin 3.12</td>
<td>2.8E-4</td>
</tr>
<tr>
<td></td>
<td>quad 2.40</td>
<td>3.4E-6</td>
</tr>
</tbody>
</table>

take $B$ to be an ellipse with major semi-axis $a = 10$. The nodes on $B$ are as described in Section 2.3, namely, $x^i = (\rho^i, \phi^i)$, $\phi^i = (i-1)2\pi/N$ ($i = 1, \ldots, N$) and the source points on $B_i$ for Green's function are $x' = (\rho', \phi')$.

In using Green's function approach to estimate $(K_{ij})$ numerically, we solve $L_{ik} = 0$ in (50). Note that the form of $K_{ij}$ in the two coordinate systems is indicated in (20) and (23). In matrix notation, we solve for $K_{ij}$, where $(F_{ik} + (K_{ij}^n)(G_{jk}) = 0$ in elliptic coordinates and $(F_{ik} + R^{-1}(K_{ij}^n))(G_{jk}) = 0$ in polar coordinates. In these equations, $G_{jk} = G(x^j, x'^k)$ and $F_{ik} = h_p(x^i)\partial G(x^i, x'^k)/\partial n - <h_p\partial G/\partial n>$. The matrix $G_{jk}$ is inverted readily using the numerical recipes in [13] for the singular value decomposition of the matrix.

We illustrate the accuracy of the boundary condition for $(K_{ij}^n)$ ($\alpha = e, n, s$) using point sources within $B$, by computing the maximum relative error in the normal derivative (MREND). The details are as follows. Replace $u(x^j)$ by $G(x^j, x^k)$, $K_{ij}$ by $K_{ij}^n$ in elliptic coordinates or $R^{-1}K_{ij}^n$ in polar coordinates, and $\partial u(x^j)/\partial n$ by $(\partial G/\partial n)^a(x^i, x'^k)$ in (12). Since $G(x^j, x^k)$ represents a unit positive source at $x^k$, $\langle h_p \partial G/\partial n \rangle$ is approximated by $-1/(2\pi)$ in elliptic coordinates and by
\(-1/(2\pi R)\) in polar coordinates. The boundary conditions in elliptic and polar coordinates are, respectively,

\[
\begin{align*}
 h_\rho \left(\frac{\partial G}{\partial n}\right)^a (x^i, x^k) + \frac{1}{2\pi} + \sum_j K_{ij}^a G(x^i, x^k) &= 0 \quad (x = e, n), \\
 h_\rho \left(\frac{\partial G}{\partial n}\right)^a (x^i, x^k) + \frac{1}{2\pi R} + \sum_j \frac{1}{R} K_{ij}^a G(x^i, x^k) &= 0 \quad (x = e, n, s).
\end{align*}
\]

(56)

(57)

We define \(\partial G/\partial n = (\partial G/\partial n)^a + \Delta G^a\), and define the relative error in the normal derivative as \(|\Delta G^a(x^i, x^k)/(\partial G(x^i, x^k)/\partial n)|\). MREND, the maximum value of this error for the \(N\) points \(x^i\) on \(B\) and \(4N\) points \(x^k\) on some \(B_i\) is given in Tables 2–4. For ellipses in Table 3, the \(4N\) points are \(x^k = (\rho^k, \phi^k)\), \(\phi^k = (k - 1)\pi/(2N)\) \((k = 1, \ldots, 4N)\), and the semi-major and semi-minor axes of \(B_i\) appear directly under MREND. For circles in Tables 2 and 4, the \(4N\) points are \(x^k = (R^k, \phi^k)\) and the radius \(R^*\) appears under MREND.

There are two details we would like to justify in the above calculations. Firstly, the semi-major axis of the ellipse or the radius of the circle \(B\) was chosen to be 10 for the following reason. If the radius of \(B\) is 1, then \((G_{jk})\) is singular since \(\sum_j G(x^i, x^k) \approx 0\). In order to remove this singularity in the matrix, the elements in one of the columns of \((G_{jk})\) are replaced by 1, and the elements in the corresponding column of \((F_{ik})\) by 0. This is possible since a constant is a solution of Laplace’s equation. However, there is no advantage in doing this, and the accuracy of \((K_{nj}^a)\) is not improved. Also, the errors in Table 2 are only slightly altered by these changes.

Secondly, the best results are obtained by defining \(B_i\) as \(\rho^i\) equal to a constant: a conclusion that is suggested by the form of Green’s function (50) in elliptic coordinates. In this case the numerical calculations for \((K_{nj}^a)(G_{jk}) = 0\) provide good results. However, if \(B\) is an ellipse and \(B_i\) is a circle with a radius much less than \(c\), then the singular values of \((G_{jk})\) in this case are much smaller for a fixed \(N\), and hence the results are inaccurate.

5. Conclusions

The free space Green’s function is an accurate way to generate the nonlocal boundary condition in which the artificial boundary \(B\) is an ellipse or a circle. In a separable coordinate system \((\rho, \phi)\), our results suggest that both the artificial surface \(B\) and the surface on which the sources \(B_i\) are located should be given by \(\rho\) equal to a constant. As \(B_i\) approaches \(B\), the errors increase; while, as \(B_i\) moves away from \(B\), the errors decrease until the errors associated with the inversion of the appropriate matrix become significant. Thus, there is a \(B_i\) at which the best results are obtained as indicated in Tables 2 and 3. Moreover, the errors can be reduced further by increasing the number of significant figures in the computer program. Also, as the eccentricity of the ellipses \(B\) and \(B_i\) increases, more nodes are required to keep the errors at a prescribed level. Surprisingly, the accuracy of the boundary condition, generated by either Green’s function or the eigenfunctions of the exterior problem, are approximately the same provided \(B_i\) is not too close to \(B\). The accuracy was determined by computing the relative error in the normal derivative using unit positive sources within \(B\).
The alternate approach of using the solution of the exterior Dirichlet problem along with a shape function is not as accurate. Although the matrix in the boundary condition has the same symmetries as the actual matrix, their entries are quite different. With shape functions, the solution of the exterior Dirichlet problem has a discontinuity in the tangential derivative on the boundary, and this accounts for the much larger errors in this approach.

There are several advantages to generate the boundary conditions numerically using a Green's function. Firstly, only standard numerical packages are required. Secondly, in a separable coordinate system, the region $B_1$, which encloses the domain of complexity of the partial differential equation, can be chosen more appropriately than the case where $B_1$ is a circle or sphere. Thirdly, the eigenfunctions in an elliptic — for the reduced wave equation — or ellipsoidal coordinate system are more difficult to handle numerically, and our approach reduces the computations involving these functions.

References