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Existence and asymptotics of topological solutions in the self-dual Maxwell–Chern–Simons $O(3)$ sigma model

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We are concerned with the topological vortex equations arising in the self-dual Maxwell–Chern–Simons $O(3)$ sigma model. We prove the existence of solutions for any parameters κ and q by a variational method. We also verify both the Maxwell limit and the Chern–Simons limit for the variational solutions.

1. Introduction

In this paper, we are interested in the following semilinear elliptic system on $\mathbb{R}^2$:

$$\Delta u = 2q\left(-N + s - \frac{1-e^u}{1+e^u}\right) + 4\pi \sum_{j=1}^{l_1} n_j \delta_{p_j} - 4\pi \sum_{j=1}^{l_2} m_j \delta_{q_j},$$

$$\Delta N = -\kappa^2 q^2 \left(-N + s - \frac{1-e^u}{1+e^u}\right) + q\frac{4e^u}{(1+e^u)^2} N. \tag{1.1}$$

Here, $\mathcal{P} = \{p_1, \dots, p_{l_1}\}$ and $\mathcal{Q} = \{q_1, \dots, q_{l_2}\}$ are disjoint sets of distinct points in $\mathbb{R}^2$. The points in $\mathcal{P}$ are called the vortex points, while the points in $\mathcal{Q}$ anti-vortex points. The numbers n_j and m_j

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are positive integers representing multiplicities of vortex or anti-vortex points. The unknowns are $u : \mathbb{R}^2 \backslash (\mathcal{P} \cup \mathcal{Q}) \to \mathbb{R}$ and $N : \mathbb{R}^2 \to \mathbb{R}$. Moreover, κ, q are positive constants and $-1 < s < 1$. The system (1.1) comes from the self-dual equations arising in the Maxwell–Chern–Simons gauged $O(3)$ sigma model (or equivalently $CP(1)$ model) [15]. For the background of this model and the derivation of (1.1), one can refer to [10].

We note that if (u, N) is a solution pair of (1.1) and (1.2) of $0 \leqslant s < 1$, then $(-u, -N)$ is also a solution for $-s$ with the change of roles of p_j's and q_j's. Therefore, throughout this paper, we assume that $0 \leqslant s < 1$. There are two kinds of boundary conditions for (1.1): either

$$u \to \ln \frac{1-s}{1+s}, \quad N \to 0, \tag{1.2}$$

or

$$u \to -\infty, \quad N \to s - 1. \tag{1.3}$$

The former is called topological and the latter nontopological. See [10] for further discussion. In this paper, we consider the topological condition (1.2). For the existence of solutions of (1.1) and (1.2), we have the following result in [10].

Theorem 1.1. *(See [10].) There exists a constant κ_0 satisfying that for each $0 < \kappa < \kappa_0$, there is a constant $q_\kappa > 0$ such that (1.1) and (1.2) admit a solution $(u, N) \in C^\infty(\mathbb{R}^2 \backslash (\mathcal{P} \cup \mathcal{Q})) \times C^\infty(\mathbb{R}^2)$ for all $q > q_\kappa$. Moreover, the functions u^2, N^2, $|\nabla u|^2$, $|\nabla N|^2$ decay exponentially at the infinity.*

The restrictions on the constants κ and q in Theorem 1.1 are due to the method of finding topological solutions. In fact, the authors used the monotone iteration technique with explicit super/subsolutions, and the conditions on κ and q come from the iteration scheme and the construction of appropriate super- and subsolution pairs. It is still open to show the existence of topological solutions for arbitrary κ and q. In this paper, we improve Theorem 1.1 when there appear only vortex points. In other words, for any κ and q, we prove the existence of solutions of the following equation with the topological boundary condition:

$$\begin{aligned}
\Delta u &= 2q\left(-N + s - \frac{1-e^u}{1+e^u}\right) + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}, \\
\Delta N &= -\kappa^2 q^2 \left(-N + s - \frac{1-e^u}{1+e^u}\right) + q \frac{4e^u}{(1+e^u)^2} N.
\end{aligned} \tag{1.4}$$

Here, we put $l = l_1$. The number $d = n_1 + \cdots + n_l$ is called the total vortex number. The first main result is

Theorem 1.2 *(Existence). For any $\kappa, q > 0$, there exists a solution of (1.2) and (1.4).*

In the rest of this paper, we continue to study Eqs. (1.2) and (1.4). One of the main features of (1.4) is that it unifies both the Maxwell gauged $O(3)$ sigma model and the Chern–Simons gauged $O(3)$ sigma model as suggested in [15]. Indeed, if we set $\kappa = 0$ and $N \equiv 0$, then (1.4) becomes

$$\Delta u = 2q\left(s - \frac{1-e^u}{1+e^u}\right) + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}, \tag{1.5}$$

which corresponds to the self-dual equation for the Maxwell gauged $O(3)$ sigma model [20,22]. Since the right-hand side of (1.5) is monotonically increasing for u, (1.5) allows only one solution. On the other hand, if we take the limit $q \to \infty$, then (1.4) is changed into

$$\Delta u = \frac{8e^u}{\kappa^2(1+e^u)^2}\left(s - \frac{1-e^u}{1+e^u}\right) + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}, \tag{1.6}$$

which corresponds to the self-dual equation for the Chern–Simons gauged $O(3)$ sigma model [5,8,14,20,22]. These limits are called the Maxwell limit and the Chern–Simons limit, respectively. In this paper, we study these limits for (1.2) and (1.4) in a mathematically rigorous way as follows:

Theorem 1.3 *(Maxwell limit). For a fixed $q > 0$, let (u_κ, N_κ) be a solution pair of* (1.2) *and* (1.4) *obtained by Theorem* 1.2 *corresponding to κ. Then, there exists a function u^* such that*

$$\left\| u_\kappa - u^* \right\|_{H^k(\mathbb{R}^2)}, \; \| N_\kappa \|_{H^k(\mathbb{R}^2)} \to 0$$

for any nonnegative integer k as $\kappa \to 0$. Furthermore, u^* is the unique solution of (1.5)*.*

Theorem 1.4 *(Chern–Simons Limit). Let $\kappa > 0$ be fixed. Given $q > 0$, let (u_q, N_q) be a solution pair of* (1.2) *and* (1.4) *obtained by Theorem* 1.2*. Then, as $q \to \infty$, there exist a subsequence, still denoted by (u_q, N_q), and a pair of functions (u_*, N_*) such that*

$$\| u_q - u_* \|_{C^k(K)}, \; \| N_q - N_* \|_{C^k(K)} \to 0$$

for any nonnegative integer k and compact sets $K \subset \mathbb{R}^2$. Moreover, we have

$$N_* = s - \frac{1 - e^{u_*}}{1 + e^{u_*}}$$

and u_* is a solution of (1.6)*.*

Before proceeding further, we make some remarks on Theorems 1.3 and 1.4. First, in $(2+1)$ Maxwell–Chern–Simons gauge field theories, it has been an interesting subject to verify these limits by mathematical arguments. The simplest Maxwell–Chern–Simons model is the $U(1)$ gauged model introduced in [16]. The self-dual equations for this model is given by

$$\begin{aligned} \Delta u &= 2q\big(e^u - 1 + \kappa N\big) + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}, \\ \Delta N &= \kappa q^2\big(e^u - 1 + \kappa N\big) + 2qe^u N. \end{aligned} \tag{1.7}$$

There are three kinds of boundary conditions for (1.7):

topological: $(u, N) \to (0, 0)$ as $|x| \to \infty$,

nontopological: $(u, N) \to (-\infty, 1/\kappa)$ as $|x| \to \infty$,

’t Hooft type: periodic on a lattice cell.

If we take the Maxwell limit, i.e., $\kappa = 0$ and $N \equiv 0$, then we have the classical Abelian–Higgs model [13]

$$\Delta u = 2q\left(e^u - 1\right) + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}.$$

It we take the Chern–Simons limit $q \to \infty$, then (1.7) becomes the Abelian–Chern–Simons–Higgs vortex equation [11,12]

$$\Delta u = \frac{4}{\kappa^2} e^u\left(e^u - 1\right) + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}.$$

Regarding the Maxwell limit and the Chern–Simons limit for solutions of (1.7), one can refer to [2–4] for topological solutions, to [9] for nontopological solutions, to [17,19] for periodic solutions. For the $O(3)$ sigma model treated in this paper, such limits have been studied for only periodic solutions in [6,7,17,18]. So, as far as we know, Theorems 1.3 and 1.4 are the first results for mathematical proof of the Maxwell limit and the Chern–Simons limit for *topological solutions* of (1.1).

Second, we point out that the method of the proof for Theorems 1.3 and 1.4 is quite different from the argument used in [2–4] to verify the Maxwell limit and the Chern–Simons limit for topological solutions of (1.7). Eqs. (1.1) and (1.7) share a similar structure. In fact, as shown in the next section, (1.1) can be changed into (2.1) and (2.2) which also corresponds to (1.7) by setting $f(t) = t - 1$ and replacing N by $-\kappa N$. In this point of view, (1.1) can be regarded as a variation of (1.7) with more complicated nonlinearity and it is natural to extend theories for (1.7) to (1.1). However, the nonlinearity arising in (1.1) makes it difficult to use the methods developed for (1.7) in the same way when we study the existence and asymptotic behavior of solutions such as the Maxwell and the Chern–Simons limits. In [3], in order to prove the Maxwell limit for (1.7), the authors used an iteration technique together with a monotonicity of the functional $\mathcal{F}$ about the iteration. As studied in [10], it is required to put some constraints on the parameter κ and q to make the iteration process successful in our model. Indeed, as $\kappa \to 0$, we have $q \to \infty$ and hence such a restriction on κ and q is not suitable for the Maxwell limit. Similarly, the method in [2–4] proving the Chern–Simons limit for (1.7) cannot be applied to (1.1). In [2–4], the authors used a monotonicity property for the maximal solutions, $v_q \leqslant v_{q'}$ for $q \leqslant q'$, to verify the Chern–Simons limit for maximal solutions. As already pointed out before, such a monotonicity comes from an iteration technique which is not valid for more complicated equations as ours. While the argument in this paper does not rely on iteration technique, the argument utilizes various kinds of maximum principle structures. Our method is also suitable for (1.7) and also expected to be applicable for other self-dual Maxwell–Chern–Simons models which have maximum principle structures as used widely in this paper.

Third, in Theorem 1.3 and Theorem 1.4, we prove the Maxwell limit and the Chern–Simons limit for the variational solutions given by Theorem 1.2. So, it remains open to show both limits for *any* solutions of (1.2) and (1.4).

Here is an outline of this paper. In Section 2, we prove Theorem 1.2 by a variational argument. We make two equations of (1.4) into a single equation of fourth order for u, and find an appropriate functional for it. In Section 3, we prove Theorem 1.3. In Section 4, we prove Theorem 1.4. For a regularized solution $v_q := u_q - v_0 + \ln a$ defined in Section 4, the uniform bound of v_q in $H^1(\mathbb{R}^2)$ is obtained by the estimates in the proof of Theorem 1.2. One of the remarkable things in the self-dual Maxwell–Chern–Simons models is that they allow various kinds of maximum principle structures. We apply the maximum principle to some auxiliary equations to get uniform bounds of solutions, from which the convergence follows.

2. Existence of solutions

In this section we prove Theorem 1.2 by a variational method. We make the system (1.4) into a single equation of fourth order for u and find a suitable functional to use a variational argument as in the pure Maxwell–Chern–Simons model [3]. Throughout this paper, we use the notation $\|\cdot\|_2 = \|\cdot\|_{L^2(\mathbb{R}^2)}$. For simplicity, we define some notations as follows:

$$a = \frac{1+s}{1-s} \in [1,\infty), \qquad w = u + \ln a, \qquad f(t) = \frac{t-a}{t+a} + s = \frac{2a(t-1)}{(a+1)(t+a)}.$$

Then, we can rewrite (1.2) and (1.4) as

$$\Delta w = 2q\big(-N + f\big(e^w\big)\big) + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j}, \tag{2.1}$$

$$\Delta N = -\kappa^2 q^2\big(-N + f\big(e^w\big)\big) + 2qf'\big(e^w\big)e^w N \tag{2.2}$$

with

$$w, N \to 0 \quad \text{as } |x| \to \infty. \tag{2.3}$$

To remove the singular terms of (2.1), we introduce a reference function

$$v_{0,\mu}(x) = \sum_{j=1}^{l} n_j \ln\left(\frac{|x-p_j|^2}{\mu + |x-p_j|^2}\right).$$

Here, $\mu \geqslant 1$ is a positive constant to be determined later. A short computation implies that

$$\Delta v_{0,\mu} = -g_\mu + 4\pi \sum_{j=1}^{l} n_j \delta_{p_j},$$

where

$$g_\mu(x) = \sum_{j=1}^{l} \frac{4\mu n_j}{(\mu + |x-p_j|^2)^2}.$$

If we set $v = w - v_{0,\mu}$, then (2.1)–(2.3) are changed into the following system of equations

$$\Delta v = 2q\big(-N + f\big(e^{v_{0,\mu}+v}\big)\big) + g_\mu, \tag{2.4}$$

$$\Delta N = -\kappa^2 q^2\big(-N + f\big(e^{v_{0,\mu}+v}\big)\big) + 2qf'\big(e^{v_{0,\mu}+v}\big)e^{v_{0,\mu}+v} N, \tag{2.5}$$

with the boundary conditions

$$v, N \to 0 \quad \text{as } |x| \to \infty. \tag{2.6}$$

Solving for N from (2.4), we obtain

$$N = -\frac{1}{2q}\Delta v + f\left(e^{v_{0,\mu}+v}\right) + \frac{1}{2q}g_\mu. \tag{2.7}$$

Then substituting (2.7) into (2.5), we have a fourth order equation

$$\begin{cases} \Delta^2 v - \kappa^2 q^2 \Delta v + 4q^2 f\left(e^{v_{0,\mu}+v}\right) f'\left(e^{v_{0,\mu}+v}\right) e^{v_{0,\mu}+v} - 2q\Delta f\left(e^{v_{0,\mu}+v}\right) \\ \quad - 2qf'\left(e^{v_{0,\mu}+v}\right)e^{v_{0,\mu}+v}(\Delta v - g_\mu) + \left(\kappa^2 q^2 g_\mu - \Delta g_\mu\right) = 0, \\ v \to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{2.8}$$

Conversely, if we find a solution of (2.8), then we can construct a solution (v, N) of (2.4)–(2.6) by (2.7). It is easy to verify that Eq. (2.8) is a variational equation of the following functional

$$\begin{aligned} \mathcal{F}_\mu(v) = \int_{\mathbb{R}^2} & \frac{1}{2}|\Delta v|^2 + \frac{1}{2}\kappa^2 q^2 |\nabla v|^2 + 2q^2 f^2\left(e^{v_{0,\mu}+v}\right) \\ & + 2qf'\left(e^{v_{0,\mu}+v}\right)e^{v_{0,\mu}+v}\left|\nabla(v_{0,\mu}+v)\right|^2 + \left(\kappa^2 q^2 g_\mu - \Delta g_\mu\right)v \end{aligned} \tag{2.9}$$

defined on $H^2(\mathbb{R}^2)$ whose norm is given by $\|v\|^2_{H^2(\mathbb{R}^2)} = \|\Delta v\|^2_{L^2(\mathbb{R}^2)} + \|v\|^2_{L^2(\mathbb{R}^2)}$. We note that $\mathcal{F}_\mu$ is well defined on $H^2(\mathbb{R}^2)$ because $f(e^{v_{0,\mu}+v}) \in L^2$ by the argument of Proposition 3.3 (Chapter III) in [13] and

$$f'\left(e^{v_{0,\mu}+v}\right)e^{v_{0,\mu}+v}\left|\nabla(v_{0,\mu}+v)\right|^2 = \frac{2ae^{v_{0,\mu}+v}}{(e^{v_{0,\mu}+v}+a)^2}\left|\nabla(v_{0,\mu}+v)\right|^2 \in L^1\left(\mathbb{R}^2\right).$$

Theorem 1.2 is a consequence of the following proposition.

Proposition 2.1. *There exists μ_0 such that $\mathcal{F}_\mu$ has a minimizer on $H^2(\mathbb{R}^2)$ for all $\mu > \mu_0$. Here, μ_0 is independent of κ and q.*

Proof. It is not difficult to see that $\mathcal{F}_\mu$ is weakly lower semi-continuous. Thus, in order to show the existence of a global minimizer of $\mathcal{F}_\mu$ in $H^2(\mathbb{R}^2)$, it suffices to prove that $\mathcal{F}_\mu$ is coercive. First, let us define

$$\Omega_{1,\mu} = \left\{x \in \mathbb{R}^2\colon e^{2v_{0,\mu}} < \lambda\right\}, \qquad \Omega_{2,\mu} = \left\{x \in \mathbb{R}^2\colon e^{2v_{0,\mu}} \geqslant \lambda\right\},$$

where $\lambda > 0$ is a constant such that $2\lambda^{1/(2d)} < 1/2$. Since $v_{0,\mu} \to 0$ as $|x| \to \infty$, $\Omega_{1,\mu}$ is bounded. Since $f' > 0$, we have

$$\mathcal{F}_\mu(v) \geqslant \int_{\mathbb{R}^2} \frac{1}{2}|\Delta v|^2 + \frac{1}{2}\kappa^2 q^2|\nabla v|^2 + 2q^2 f^2\left(e^{v_{0,\mu}+v}\right) + \left(\kappa^2 q^2 g_\mu - \Delta g_\mu\right)v. \tag{2.10}$$

Let us consider the term $2\int_{\mathbb{R}^2} f^2(e^{v_0+v})$. We see that

$$2\int_{\mathbb{R}^2}\frac{4a^2}{(a+1)^2}\left(\frac{e^{v_{0,\mu}+v}-1}{e^{v_{0,\mu}+v}+a}\right)^2 \geqslant c_1\int_{\mathbb{R}^2}\frac{e^{2v_{0,\mu}}(e^v-1)^2}{2(e^{v_{0,\mu}+v}+a)^2}-c_1\int_{\mathbb{R}^2}\frac{(e^{v_{0,\mu}}-1)^2}{(e^{v_{0,\mu}+v}+a)^2}$$
$$\geqslant c_1\int_{\mathbb{R}^2}\frac{e^{2v_{0,\mu}}(e^v-1)^2}{2(e^{v_{0,\mu}+v}+a)^2}-c_{0,\mu}, \tag{2.11}$$

where

$$c_1:=\frac{8a^2}{(a+1)^2},\qquad c_{0,\mu}:=\frac{c_1}{a^2}\int_{\mathbb{R}^2}\left(e^{v_{0,\mu}}-1\right)^2.$$

Since $e^{v_{0,\mu}}-1=O(|x|^{-2})$ near ∞, we deduce that $e^{v_{0,\mu}}-1\in L^2(\mathbb{R}^2)$ and hence $c_{0,\mu}$ is well defined. Let $T>1$ be a number such that $t\geqslant(1+a)\ln t+1$ for all $t\geqslant T$. Then, $(t-1)/(t+a)\geqslant \ln t/(1+\ln t)$ for all $t\geqslant T$. Since $e^{v_{0,\mu}}\leqslant 1$, using the inequality $|e^t-1|\geqslant|t|/(1+|t|)$ for $t\in\mathbb{R}$, we derive from (2.11) that

$$2\int_{\mathbb{R}^2}f^2\left(e^{v_{0,\mu}+v}\right)\geqslant\frac{c_1}{2}\int_{\{v\leqslant\ln T\}}\frac{e^{2v_{0,\mu}}(e^v-1)^2}{(T+a)^2}+\frac{c_1}{2}\int_{\{v\geqslant\ln T\}}\frac{e^{2v_{0,\mu}}(e^v-1)^2}{(e^v+a)^2}-c_{0,\mu}$$
$$\geqslant\frac{c_1}{2(T+a)^2}\int_{\{v\leqslant\ln T\}}\frac{v^2e^{2v_{0,\mu}}}{(1+|v|)^2}+\frac{c_1}{2}\int_{\{v\geqslant\ln T\}}\frac{v^2e^{2v_{0,\mu}}}{(1+|v|)^2}-c_{0,\mu}$$
$$\geqslant c_2\int_{\mathbb{R}^2}\frac{v^2e^{2v_{0,\mu}}}{(1+|v|)^2}-c_{0,\mu}$$
$$\geqslant c_2\left(\int_{\Omega_{1,\mu}}\frac{v^2e^{2v_{0,\mu}}}{(1+|v|)^2}+\lambda\int_{\Omega_{2,\mu}}\frac{v^2}{(1+|v|)^2}\right)-c_{0,\mu}, \tag{2.12}$$

where $c_2=c_1/2(T+a)^2$.

Choose δ and R satisfying $0<\delta<\min_{i\neq j}\{|p_i-p_j|/2,1\}$ and $R>\max_j\{|p_j|+1\}$. Define

$$\Omega_\delta=\bigcup_{j=1}^{l}B_\delta(p_j). \tag{2.13}$$

If $x\in\Omega_{1,\mu}$ and $|x|^2>2R^2$, then

$$\lambda>\prod_{j=1}^{l}\left(\frac{|x-p_j|^2}{\mu+|x-p_j|^2}\right)^{2n_j}\geqslant\prod_{j=1}^{l}\left(\frac{\frac12|x|^2-R^2}{\mu+2|x|^2+2R^2}\right)^{2n_j}=\left(\frac{\frac12|x|^2-R^2}{\mu+2|x|^2+2R^2}\right)^{2d},$$

which implies that

$$\left(\frac12-2\lambda^{\frac{1}{2d}}\right)|x|^2\leqslant\lambda^{\frac{1}{2d}}\left(\mu+2R^2\right)+R^2.$$

Hence, by the choice of λ, if we choose $\mu > 2R^2$, then there exists a constant $c_3 > 0$ such that $|x|^2 \leqslant c_3(\mu+1)$ for all $x \in \Omega_{1,\mu}$.

We observe that for $x \in B_\delta(p_j)$,

$$e^{-2v_{0,\mu}} = \prod_{k=1}^{l}\left(\frac{\mu+|x-p_k|^2}{|x-p_k|^2}\right)^{2n_k} \leqslant \frac{(2\mu)^{2d}}{\delta^{4d-4n_j}}\cdot\frac{1}{|x-p_j|^{4n_j}} \leqslant \frac{c_4\mu^{2d}}{|x-p_j|^{4d}},$$

where $c_4 = \max_j\{2^{2d}\delta^{-4d+4n_j}\}$. Hence, if we choose a number $\alpha \in (0, 1/(2d+1))$ such that $4d\alpha/(1-\alpha) < 2$, then

$$\int_{\Omega_\delta} e^{-2v_{0,\mu}\frac{\alpha}{1-\alpha}} \leqslant \sum_{j=1}^{l} c_4^{\frac{\alpha}{1-\alpha}}\mu^{\frac{2d\alpha}{1-\alpha}} \int_{B_\delta(p_j)} |x-p_j|^{-\frac{4d\alpha}{1-\alpha}} \leqslant c_5\mu^{\frac{2d\alpha}{1-\alpha}},$$

where

$$c_5 = lc_4^{\frac{\alpha}{1-\alpha}} \int_{|x|\leqslant\delta} |x|^{-\frac{4d\alpha}{1-\alpha}} < \infty.$$

On the other hand, if $x \in \Omega_{1,\mu}\backslash\Omega_\delta$, then

$$e^{-2v_{0,\mu}} \leqslant \left(\frac{\mu+2c_3(\mu+1)+2R^2}{\delta}\right)^{4d} \leqslant c_6\left(\mu^{4d}+1\right)$$

for some $c_6 > 0$. Therefore,

$$\int_{\Omega_{1,\mu}\backslash\Omega_\delta} e^{-2v_{0,\mu}\frac{\alpha}{1-\alpha}} \leqslant \int_{|x|^2\leqslant c_3(\mu+1)} \left[c_6\left(\mu^{4d}+1\right)\right]^{\frac{\alpha}{1-\alpha}} \leqslant c_7\left(\mu^{1+\frac{4d\alpha}{1-\alpha}}+1\right)$$

for some $c_7 > 0$. As a consequence,

$$\int_{\Omega_{1,\mu}} e^{-2v_{0,\mu}\frac{\alpha}{1-\alpha}} = \int_{\Omega_\delta} e^{-2v_{0,\mu}\frac{\alpha}{1-\alpha}} + \int_{\Omega_{1,\mu}\backslash\Omega_\delta} e^{-2v_{0,\mu}\frac{\alpha}{1-\alpha}} \leqslant c_8\left(\mu^{1+\frac{4d\alpha}{1-\alpha}}+1\right)$$

for some $c_8 > 0$.

Now it follows from the Hölder inequality that

$$\begin{aligned}\int_{\Omega_{1,\mu}} \frac{v^2}{(1+|v|)^2} &\leqslant \int_{\Omega_{1,\mu}} \left(\frac{v^2}{(1+|v|)^2}\right)^{\alpha} \\ &\leqslant \left(\int_{\Omega_{1,\mu}} e^{-2v_{0,\mu}\frac{\alpha}{1-\alpha}}\right)^{1-\alpha}\left(\int_{\Omega_{1,\mu}} \frac{e^{2v_{0,\mu}}v^2}{(1+|v|)^2}\right)^{\alpha} \\ &\leqslant c_8\left(\mu^{1+\frac{4d\alpha}{1-\alpha}}+1\right) + \int_{\Omega_{1,\mu}} \frac{e^{2v_{0,\mu}}v^2}{(1+|v|)^2}.\end{aligned}$$

Therefore, (2.12) becomes

$$2\int_{\mathbb{R}^2} f^2\big(e^{v_{0,\mu}+v}\big) \geqslant c_2\bigg(\int_{\Omega_{1,\mu}} \frac{v^2}{(1+|v|)^2} + \lambda \int_{\Omega_{2,\mu}} \frac{v^2}{(1+|v|)^2}\bigg) - c_{0,\mu} - c_2c_8\big(\mu^{1+\frac{4d\alpha}{1-\alpha}}+1\big)$$
$$\geqslant c_2\lambda \int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2} - c_{0,\mu} - c_2c_8\big(\mu^{1+\frac{4d\alpha}{1-\alpha}}+1\big). \tag{2.14}$$

Now let us consider other terms in $\mathcal{F}_\mu(v)$. First, a direct computation yields that

$$\|g_\mu\|_2^2 \leqslant l\int_{\mathbb{R}^2} \frac{(4\mu d)^2}{(\mu+|x|^2)^4}\,dx = \frac{16\pi l d^2}{3\mu} =: \frac{c_9^2}{\mu}.$$

Hence,

$$\int_{\mathbb{R}^2} \big(\kappa^2 q^2 g_\mu - \Delta g_\mu\big)v \geqslant -\kappa^2q^2\|v\|_2\|g_\mu\|_2 - \|\Delta v\|_2\|g_\mu\|_2$$
$$\geqslant -\frac{c_9\kappa^2q^2}{\sqrt{\mu}}\|v\|_2 - \frac{1}{4}\|\Delta v\|_2^2 - \frac{c_9^2}{\mu}. \tag{2.15}$$

We also recall the following inequality from [21];

$$\|v\|_2 \leqslant 2 + \int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2} + 2\int_{\mathbb{R}^2} |\nabla v|^2. \tag{2.16}$$

Now using (2.14) and (2.15) together with (2.16), we deduce from (2.10) that

$$\mathcal{F}_\mu(v) \geqslant \frac{1}{4}\|\Delta v\|_2^2 + \kappa^2q^2\bigg(\frac{1}{2} - \frac{2c_9}{\sqrt{\mu}}\bigg)\|\nabla v\|_2^2 + q^2\bigg(\lambda c_2 - \frac{c_9\kappa^2}{\sqrt{\mu}}\bigg)\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2}$$
$$- q^2\bigg(c_{0,\mu} + c_2c_8\big(\mu^{1+\frac{4d\alpha}{1-\alpha}}+1\big) + \frac{2c_9\kappa^2}{\sqrt{\mu}}\bigg) - \frac{c_9^2}{\mu}. \tag{2.17}$$

We take $\mu > 0$ large enough such that

$$\sqrt{\mu} > \sqrt{\mu_0} := \max\bigg\{4c_9, \frac{\kappa^2 c_9}{\lambda c_2}, \sqrt{2}R\bigg\}. \tag{2.18}$$

Then we have

$$\mathcal{F}_\mu(v) \geqslant \frac{1}{4}\|\Delta v\|_2^2 + C_1\bigg(\|\nabla v\|_2^2 + \int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2}\bigg) - C_2$$

for some constants C_1 and C_2 which are dependent on κ and q. This together with (2.16) implies that $\mathcal{F}_\mu$ is coercive, which completes the proof. □

3. Maxwell limit

In this section we prove Theorem 1.3. First, we recall the following lemma from [10].

Lemma 3.1. *If* (w, N) *is a solution of* (2.1)–(2.3), *then we have*

$$w \leqslant 0 \quad in\ \mathbb{R}^2 \backslash \mathcal{P} \quad and \quad N \leqslant 0, \quad -N + f\left(e^w\right) \leqslant 0 \quad in\ \mathbb{R}^2. \tag{3.1}$$

As an immediate consequence of (3.1), we get

$$e^w \leqslant 1, \quad 0 \geqslant N \geqslant f\left(e^w\right) = \frac{2a(e^w - 1)}{(a+1)(e^w + a)} \geqslant \frac{-2a}{a(a+1)} = -\frac{2}{a+1}. \tag{3.2}$$

Lemma 3.1 can be proved by the maximum principle and this gives a great deal of *pointwise* estimates when we consider some asymptotic problems as the Maxwell and the Chern–Simons limits.

For a fixed $q > 0$, we take $\mu > \mu_0$ and write $v_0 = v_{0,\mu}$, $g = g_\mu$, $\mathcal{F} = \mathcal{F}_\mu$. We may assume by (2.18) that μ is independently chosen for all small κ. Letting $v = u - v_0 + \ln a$, we can rewrite (1.5) as

$$\Delta v = 2qf\left(e^{v_0 + v}\right) + g, \tag{3.3}$$

which has a unique solution v^* satisfying $v_0 + v^* \leqslant 0$. For a given $\kappa > 0$, let (v_κ, N_κ) be a variational solution pair of (2.4)–(2.6) obtained by Theorem 1.2. Then Theorem 1.3 is a consequence of the following proposition.

Proposition 3.2. *As* $\kappa \to 0$, *we have*

$$\|N_\kappa\|_{H^k(\mathbb{R}^2)}, \ \|v_\kappa - v^*\|_{H^k(\mathbb{R}^2)} \to 0 \tag{3.4}$$

for any nonnegative integer k.

Proof. The proof is divided into four steps. In this proof, the letter C denotes a generic constant which is independent of κ, and it is occasionally numbered for clarity.

Step 1: We have

$$\|v_\kappa\|_{H^2(\mathbb{R}^2)} \leqslant C, \tag{3.5}$$

where C is independent of κ.

Let φ be a smooth function with compact support. Then, $\mathcal{F}_{\kappa,q}(\varphi) \leqslant C$ as $\kappa \to 0$. It comes from (2.17) that

$$\mathcal{F}(\varphi) \geqslant \mathcal{F}(v_\kappa) \geqslant C\left(\int_{\mathbb{R}^2} |\Delta v_\kappa|^2 + \int_{\mathbb{R}^2} \frac{|v_\kappa|^2}{(1 + |v_\kappa|)^2}\right) - C. \tag{3.6}$$

We also have

$$\int_{\mathbb{R}^2} |\nabla v_\kappa|^2 = -\int_{\mathbb{R}^2} v_\kappa \Delta v_\kappa \leqslant \varepsilon \int_{\mathbb{R}^2} v_\kappa^2 + \frac{1}{4\varepsilon} \int_{\mathbb{R}^2} |\Delta v_\kappa|^2. \tag{3.7}$$

Using the following inequality (see [21]),

$$\int_{\mathbb{R}^2} v^2 \leqslant 2\int_{\mathbb{R}^2} \frac{v^2}{(1+|v|)^2}\cdot\left(1+\int_{\mathbb{R}^2}|\nabla v|^2\right),$$

we infer from (3.6) and (3.7) that $\|v_\kappa\|_{L^2(\mathbb{R}^2)}$ and $\|\Delta v_\kappa\|_{L^2(\mathbb{R}^2)}$ are uniformly bounded, which implies that $\|v_\kappa\|_{H^2(\mathbb{R}^2)} \leqslant C$. By the Sobolev Imbedding Theorem, $\|v_\kappa\|_{L^\infty(\mathbb{R}^2)} \leqslant C_0$ for some positive constant C_0 which is independent of κ.

Step 2: As $\kappa \to 0$,

$$\|N_\kappa\|_{H^2(\mathbb{R}^2)} \to 0. \tag{3.8}$$

Let $\kappa_j \searrow 0$ be any sequence and write $(v_{\kappa_j}, N_{\kappa_j}) = (v_j, N_j)$. Since N_j and ΔN_j are uniformly bounded in $L^\infty(\mathbb{R}^2)$, we can extract a subsequence, still denoted by (v_j, N_j), and a function N_0 such that $N_j \to N_0$ in $C^{1,\gamma}(K)$ for any $\gamma \in [0,1)$ and for any compact sets K. Let us integrate (2.4) and (2.5) to obtain

$$2q\int_{\mathbb{R}^2}\left(-N_j + f\left(e^{v_0+v_j}\right)\right) = -4\pi d, \tag{3.9}$$

$$2q\int_{\mathbb{R}^2} f'\left(e^{v_0+v_j}\right)e^{v_0+v_j}N_j = \kappa_j^2 q^2\int_{\mathbb{R}^2}\left(-N_j + f\left(e^{v_0+v_j}\right)\right) = -2\pi d\kappa_j^2 q. \tag{3.10}$$

Multiplying (2.5) by N_j, we deduce from (3.2) and (3.9) that

$$\int_{\mathbb{R}^2}\left(|\nabla N_j|^2 + 2qf'\left(e^{v_0+v_j}\right)e^{v_0+v_j}N_j^2\right) = \kappa_j^2 q^2\int_{\mathbb{R}^2}\left(-N_j + f\left(e^{v_0+v_j}\right)\right)N_j \leqslant \frac{4\pi d\kappa_j^2 q}{a+1}.$$

Hence, $N_0 \equiv 0$. Since $f'(t)$ is a decreasing function for $t \geqslant 0$, we derive from (3.10) that

$$\begin{aligned}-\pi d\kappa^2 &< \int_{\mathbb{R}^2\setminus\Omega_\delta} f'\left(e^{v_0+v_j}\right)e^{v_0+v_j}N_j \leqslant \int_{\mathbb{R}^2\setminus\Omega_\delta} f'\left(e^{v_0-C_0}\right)e^{v_0-C_0}N_j \\ &\leqslant C\int_{\mathbb{R}^2\setminus\Omega_\delta} N_j < 0,\end{aligned}$$

where Ω_δ is given by (2.13). Thus, it is seen that

$$\int_{\mathbb{R}^2}|N_j| = \int_{\Omega_\delta}|N_j| + \int_{\mathbb{R}^2\setminus\Omega_\delta}|N_j| \leqslant \|N_j\|_{L^\infty(\Omega_\delta)}\cdot|\Omega_\delta| + \int_{\mathbb{R}^2\setminus\Omega_\delta}|N_j| \to 0 \tag{3.11}$$

as $\kappa_j \to 0$. Consequently, $\|N_j\|_2^2 \leqslant \|N_j\|_{L^\infty(\mathbb{R}^2)}\|N_j\|_{L^1(\mathbb{R}^2)} = o(1)$. On the other hand, it follows from (3.9) that

$$\int_{\mathbb{R}^2} \left(-N_j + f\left(e^{v_0+v_j}\right)\right)^2 \leqslant \left(\|N_j\|_{L^\infty(\mathbb{R}^2)} + \|f\|_{L^\infty(0,\infty)}\right) \int_{\mathbb{R}^2} \left|-N_j + f\left(e^{v_0+v_j}\right)\right| \leqslant C.$$

Thus, by (2.5)

$$\|\Delta N_j\|_2 \leqslant C\kappa_j^2 \left\|-N_j + f\left(e^{v_0+v_j}\right)\right\|_2 + C\|N_j\|_2 = o(1).$$

Now the Calderon–Zygmund inequality implies that $\|N_j\|_{H^2(\mathbb{R}^2)} = o(1)$ as $\kappa_j \to 0$. Since κ_j was arbitrary, we conclude that $\|N_\kappa\|_{H^2(\mathbb{R}^2)} \to 0$ as $\kappa \to 0$.

Step 3: As $\kappa \to 0$, we have

$$\left\|v_\kappa - v^*\right\|_{H^2(\mathbb{R}^2)} = o(1). \tag{3.12}$$

Subtracting (2.4) from (3.3), we obtain

$$\Delta\left(v^* - v_\kappa\right) = 2qN_\kappa + 2q\left(f\left(e^{v_0+v^*}\right) - f\left(e^{v_0+v_\kappa}\right)\right). \tag{3.13}$$

Multiplying this equation by $v^* - v_\kappa$ and integrating by parts, we are led to

$$-\int_{\mathbb{R}^2} N_\kappa\left(v^* - v_\kappa\right) = \frac{1}{2q}\int_{\mathbb{R}^2} \left|\nabla\left(v^* - v_\kappa\right)\right|^2 + \int_{\mathbb{R}^2} \left(f\left(e^{v_0+v^*}\right) - f\left(e^{v_0+v_\kappa}\right)\right)\left(v^* - v_\kappa\right). \tag{3.14}$$

We note that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \left(f\left(e^{v_0+v^*}\right) - f\left(e^{v_0+v_\kappa}\right)\right)\left(v^* - v_\kappa\right) \\
&\quad = 2a\int_{\mathbb{R}^2} \frac{e^{v_0+v_\kappa}(e^{v^*-v_\kappa} - 1)(v^* - v_\kappa)}{(e^{v_0+v^*} + a)(e^{v_0+v_\kappa} + a)} \geqslant \frac{2a}{(1+a)^2}\int_{\mathbb{R}^2} e^{v_0 - C_0}\left(v^* - v_\kappa\right)^2 \\
&\quad = \frac{2ae^{-C_0}}{(1+a)^2}\left(\int_{\Omega_\delta} e^{v_0}\left(v^* - v_\kappa\right)^2 + \int_{\mathbb{R}^2\setminus\Omega_\delta} e^{v_0}\left(v^* - v_\kappa\right)^2\right) := \frac{2ae^{-C_0}}{(1+a)^2}(I + II).
\end{aligned}$$

Here, the inequality follows from the fact that $v^* - v_\kappa \geqslant 0$. Obviously we have

$$II \geqslant \inf_{\mathbb{R}^2\setminus\Omega_\delta} e^{v_0} \cdot \int_{\mathbb{R}^2\setminus\Omega_\delta} \left(v^* - v_\kappa\right)^2.$$

As in the proof of Proposition 2.1, we obtain that $e^{-v_0} \leqslant C|x - p_j|^{-2d}$ for $x \in B_\delta(p_j)$. Hence, if we choose $\alpha \in (0, 1/(d+1))$, then

$$\int_{\Omega_\delta} e^{-v_0\frac{\alpha}{1-\alpha}} \leqslant C,$$

which implies that

$$\begin{aligned}\int\limits_{\Omega_\delta}(v^*-v_\kappa)^2 &\leqslant \sup_{\Omega_\delta}(v^*-v_\kappa)^{2-2\alpha}\int\limits_{\Omega_\delta}(v^*-v_\kappa)^{2\alpha}\\ &\leqslant C\Big(\int\limits_{\Omega_\delta}e^{-v_0\frac{\alpha}{1-\alpha}}\Big)^{1-\alpha}\Big(\int\limits_{\Omega_\delta}e^{v_0}(v^*-v_\kappa)^2\Big)^{\alpha}\\ &\leqslant C\Big(\int\limits_{\Omega_\delta}e^{v_0}(v^*-v_\kappa)^2\Big)^{\alpha}=CI^{\alpha}.\end{aligned}$$

As a consequence,

$$\int\limits_{\mathbb{R}^2}\big(f(e^{v_0+v^*})-f(e^{v_0+v_\kappa})\big)(v^*-v_\kappa)\geqslant C_1\Big[\Big(\int\limits_{\Omega_\delta}(v^*-v_\kappa)^2\Big)^{\frac{1}{\alpha}}+\int\limits_{\mathbb{R}^2\backslash\Omega_\delta}(v^*-v_\kappa)^2\Big] \tag{3.15}$$

for some constant $C_1>0$. On the other hand, since $\|N_\kappa\|_2=o(1)$, we see that

$$\begin{aligned}&-\int\limits_{\mathbb{R}^2}N_\kappa(v^*-v_\kappa)\\ &\quad\leqslant\Big(\int\limits_{\Omega_\delta}|N_\kappa|^2\Big)^{\frac12}\Big(\int\limits_{\Omega_\delta}(v^*-v_\kappa)^2\Big)^{\frac12}+\Big(\int\limits_{\mathbb{R}^2\backslash\Omega_\delta}|N_\kappa|^2\Big)^{\frac12}\Big(\int\limits_{\mathbb{R}^2\backslash\Omega_\delta}(v^*-v_\kappa)^2\Big)^{\frac12}\\ &\quad\leqslant\frac{C_1}{2}\Big\{\Big(\int\limits_{\Omega_\delta}(v^*-v_\kappa)^2\Big)^{\frac1\alpha}+\int\limits_{\mathbb{R}^2\backslash\Omega_\delta}(v^*-v_\kappa)^2\Big\}+C\Big\{\Big(\int\limits_{\Omega_\delta}|N_\kappa|^2\Big)^{\frac{1}{2-\alpha}}+\int\limits_{\mathbb{R}^2\backslash\Omega_\delta}|N_\kappa|^2\Big\}\\ &\quad\leqslant\frac{C_1}{2}\Big\{\Big(\int\limits_{\Omega_\delta}(v^*-v_\kappa)^2\Big)^{\frac1\alpha}+\int\limits_{\mathbb{R}^2\backslash\Omega_\delta}(v^*-v_\kappa)^2\Big\}+o(1).\end{aligned}$$

Thus, we conclude from (3.14) and (3.15) that

$$\frac{1}{2q}\int\limits_{\mathbb{R}^2}\big|\nabla(v^*-v_\kappa)\big|^2+\frac{C_1}{2}\Big\{\Big(\int\limits_{\Omega_\delta}(v^*-v_\kappa)^2\Big)^{\frac1\alpha}+\int\limits_{\mathbb{R}^2\backslash\Omega_\delta}(v^*-v_\kappa)^2\Big\}\leqslant o(1).$$

In the sequel,

$$\|v^*-v_\kappa\|_2=o(1).$$

Moreover, by (3.13) and the mean value theorem,

$$\big\|\Delta(v^*-v_\kappa)\big\|_2\leqslant C\big(\|N_\kappa\|_2+\|v^*-v_\kappa\|_2\big)=o(1).$$

This enables us to arrive at (3.12) by the Calderon–Zygmund inequality.

Step 4: It remains to show that (3.4) holds for $k \geqslant 3$. This can be achieved by induction. For example consider $k = 3$. It comes from (3.13) that

$$\begin{aligned}
&\Delta\partial_j\big(v^* - v_\kappa\big)\\
&\quad = 2q\partial_j N_\kappa + 2qf'\big(Ue^{v^*}\big)\partial_j Ue^{v^*} - 2qf'\big(Ue^{v_\kappa}\big)\partial_j Ue^{v_\kappa}\\
&\qquad + 2qUf'\big(Ue^{v^*}\big)e^{v^*}\partial_j v^* - 2qUf'\big(Ue^{v_\kappa}\big)e^{v_\kappa}\partial_j v_\kappa\\
&\quad \equiv 2q\partial_j N_\kappa + \partial_j U F_1(\varphi_1, \varphi_2) - \partial_j U F_1(\psi_{\kappa 1}, \psi_{\kappa 2}) + U\big(F_2(\varphi_1, \varphi_2, \varphi_3) - F_2(\psi_{\kappa 1}, \psi_{\kappa 2}, \psi_{\kappa 3})\big),
\end{aligned}$$

where $U = e^{v_0}$ is a smooth bounded function and

$$\begin{aligned}
F_1(t_1, t_2) &= 2qt_1t_2, & F_2(t_1, t_2, t_3) &= 2qt_1t_2t_3,\\
\varphi_1 &= f'\big(Ue^{v^*}\big), & \varphi_2 &= e^{v^*}, & \varphi_3 &= \partial_j v^*,\\
\psi_{\kappa 1} &= f'\big(Ue^{v_\kappa}\big), & \psi_{\kappa 2} &= e^{v_\kappa}, & \psi_{\kappa 3} &= \partial_j v_\kappa.
\end{aligned}$$

Since $\Delta\partial_j(v^* - v_\kappa) \in L^2(\mathbb{R}^2)$ uniformly by induction, we have $\partial_j(v^* - v_\kappa) \in H^2(\mathbb{R}^2)$ uniformly. In particular, $\|\partial_j(v^* - v_\kappa)\|_{L^\infty(\mathbb{R}^2)} \leqslant C$. By induction, it is obvious that for each $j = 1, 2, 3$,

$$\|\partial_j N_\kappa\|_2,\ \|\varphi_j - \psi_{\kappa j}\|_2 \to 0.$$

Hence,

$$\begin{aligned}
&\big\|F_2(\varphi_1, \varphi_2, \varphi_3) - F_2(\psi_{\kappa 1}, \psi_{\kappa 2}, \psi_{\kappa 3})\big\|_2\\
&\quad \leqslant \big\|F_2(\varphi_1, \varphi_2, \varphi_3) - F_2(\psi_{\kappa 1}, \varphi_2, \varphi_3)\big\|_2 + \big\|F_2(\psi_{\kappa 1}, \varphi_2, \varphi_3) - F_2(\psi_{\kappa 1}, \psi_{\kappa 2}, \varphi_3)\big\|_2\\
&\qquad + \big\|F_2(\psi_{\kappa 1}, \psi_{\kappa 2}, \varphi_3) - F_2(\psi_{\kappa 1}, \psi_{\kappa 2}, \psi_{\kappa 3})\big\|_2\\
&\quad \leqslant C\big(\|\varphi_1 - \psi_{\kappa 1}\|_2 + \|\varphi_2 - \psi_{\kappa 2}\|_2 + \|\varphi_3 - \psi_{\kappa 3}\|_2\big) \to 0.
\end{aligned}$$

Similarly, $\|F_1(\varphi_1, \varphi_2) - F_1(\psi_{\kappa 1}, \psi_{\kappa 2})\|_2 \to 0$. As a consequence, we obtain that $\|\partial_j(v^* - v_\kappa)\|_{H^2(\mathbb{R}^2)} \to 0$, which proves the estimate for $v^* - v_\kappa$ when $k = 3$. A similar argument gives the estimate for N_κ when $k = 3$. The convergence in higher norms is computed in a similar way and we omit the detail. □

4. Chern–Simons limit

In this section, we prove Theorem 1.4. Although (2.1) and (2.2) form a system of equations, they have various types of maximum principle structures which will be the main tool for the proof of the Chern–Simons limits in this section. Throughout this section, let $\kappa > 0$ and $\mu > \mu_0$ be fixed. We write $v_0 = v_{0,\mu}$ and $\mathcal{F} = \mathcal{F}_\mu$. Given $q > 0$, let (v_q, N_q) be a pair of solutions of (2.4)–(2.6) given by Theorem 1.2. Setting $v = u - v_0 + \ln a$, we can rewrite (1.6) as

$$\Delta v = \frac{4}{\kappa^2} e^{v_0+v} f\big(e^{v_0+v}\big) f'\big(e^{v_0+v}\big) + g. \tag{4.1}$$

Theorem 1.4 is a consequence of the following proposition.

Proposition 4.1. *There is a subsequence, still denoted by (v_q, N_q), and a pair of functions (v_*, N_*) such that*

$$(v_q, N_q) \to (v_*, N_*) \quad in\ C^k_{loc}(\mathbb{R}^2) \times C^k_{loc}(\mathbb{R}^2)$$

for any nonnegative integer k. Furthermore, v_* is a solution of (4.1) *and $N_* = f(e^{v_0+v_*})$.*

Hereafter, B_r denotes the ball of radius r centered at the origin. In order to prove the above proposition, we often use the following interior gradient estimate (see [1]).

Lemma 4.2. *Suppose that $-\Delta u = f$ in $\Omega \subset \mathbb{R}^n$. Then,*

$$\left|\nabla u(x)\right|^2 \leqslant C\|u\|_{L^\infty(\Omega)}\left(\|f\|_{L^\infty(\Omega)} + \frac{1}{\mathrm{dist}(x, \partial\Omega)}\|u\|_{L^\infty(\Omega)}\right)$$

for $x \in \Omega$. Here, C depends only on Ω.

Proof of Proposition 4.1. We split the proof into four steps. In this proof, the letter C will denote a positive generic constant which is independent of q and may vary from line to line.

Step 1: Let us define

$$\begin{cases} \varphi_q = q\left(-N_q + f\left(Ue^{v_q}\right)\right), \\ \psi_q = q\left(-\kappa^2\varphi_q + 2f'\left(Ue^{v_q}\right)Ue^{v_q}N_q\right), \end{cases} \tag{4.2}$$

where $U = e^{v_0}$ is a smooth bounded function on $\mathbb{R}^2$ such that $\|\nabla^k U\|_{L^\infty(\mathbb{R}^2)} \leqslant C(k)$. Then, it holds from (2.4) and (2.5) that

$$\Delta v_q = 2\varphi_q + g, \qquad \Delta N_q = \psi_q. \tag{4.3}$$

Moreover, a straightforward calculation yields that

$$\begin{aligned} \frac{1}{q}\Delta\varphi_q &= \left(\kappa^2 q + 2f'\left(Ue^{v_q}\right)Ue^{v_q}\right)\varphi_q + f''\left(Ue^{v_q}\right)e^{2v_q}|\nabla U + U\nabla v_q|^2 \\ &\quad + f'\left(Ue^{v_q}\right)\left\{-2qUe^{v_q}N_q + e^{v_q}\Delta U + 2e^{v_q}\nabla U\cdot\nabla v_q + Ue^{v_q}|\nabla v_q|^2 + gUe^{v_q}\right\} \\ &\equiv \left(\kappa^2 q + 2f'\left(Ue^{v_q}\right)Ue^{v_q}\right)\varphi_q + \sigma_q, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \frac{1}{q}\Delta\psi_q &= \left(\kappa^2 q + 2f'\left(Ue^{v_q}\right)Ue^{v_q}\right)\psi_q + 2f'''\left(Ue^{v_q}\right)Ue^{3v_q}N_q|\nabla U + U\nabla v_q|^2 \\ &\quad + f''\left(Ue^{v_q}\right)e^{2v_q}\left\{\left(4N_q - \kappa^2 q\right)|\nabla U + U\nabla v_q|^2 + 4U\nabla N_q\cdot(\nabla U + U\nabla v_q) \right. \\ &\quad \left. + 2UN_q\left(\Delta U + 2\nabla U\cdot\nabla v_q + 2U\varphi_q + gU + U|\nabla v_q|^2\right)\right\} \\ &\quad + f'\left(Ue^{v_q}\right)e^{v_q}\left\{\left(4N - 2\kappa^2 q\right)U\varphi_q + 4\nabla N_q\cdot(\nabla U + U\nabla v_q) \right. \\ &\quad \left. + \left(2N_q - \kappa^2 q\right)\left(\Delta U + 2\nabla U\cdot\nabla v_q + gU + U|\nabla v_q|^2\right)\right\} \\ &\equiv \left(\kappa^2 q + 2f'\left(Ue^{v_q}\right)Ue^{v_q}\right)\psi_q + \eta_q. \end{aligned} \tag{4.5}$$

Step 2: We have

$$\|v_q\|_{L^\infty(\mathbb{R}^2)},\ \|v_q\|_{H^1(\mathbb{R}^2)} < C \tag{4.6}$$

for any $q > 0$.

Let ξ be a smooth function with compact support. Then $\mathcal{F}(\xi)$ is well defined, and satisfies $\mathcal{F}(\xi) \leqslant C(1+q+q^2)$. It comes from (2.17) that

$$\mathcal{F}(v_q) \geqslant C\left(\|\Delta v_q\|_2^2 + q^2\|\nabla v_q\|_2^2 + q^2 \int\limits_{\mathbb{R}^2} \frac{v_q^2}{(1+|v_q|)^2}\right) - Cq^2.$$

Since $\mathcal{F}(v_q) \leqslant \mathcal{F}(\xi)$, we deduce that for all large q,

$$\int\limits_{\mathbb{R}^2} \left(|\nabla v_q|^2 + \frac{v_q^2}{(1+|v_q|)^2}\right) \leqslant C.$$

Consequently, $\|v_q\|_{H^1(\mathbb{R}^2)} \leqslant C$ by the inequality (2.16).

On the other hand, we derive from (2.4) and (2.5) that

$$\Delta\left(v_q + \frac{2}{\kappa^2 q} N_q\right) = g + \frac{4}{\kappa^2} f'\left(e^{v_0+v_q}\right) e^{v_0+v_q} N_q.$$

Given $x \in \mathbb{R}^2$, let us define $W_{x,q}(y) = (v_q + \frac{2}{\kappa^2 q} N_q)(x+y)$ for $|y| \leqslant 1$. By (3.2), $\Delta W_{x,q} \in L^\infty(\mathbb{R}^2)$ uniformly as $q \to \infty$. Then, it comes from the standard elliptic estimates that for all large q,

$$\begin{aligned}\|W_{x,q}\|_{H^2(B_{1/2})} &\leqslant C\left(\|W_{x,q}\|_{L^2(B_1)} + \|\Delta W_{x,q}\|_{L^2(B_1)}\right)\\ &\leqslant C\left(\|v_q\|_{L^2(\mathbb{R}^2)} + 1\right) \leqslant C,\end{aligned}$$

where C is independent of x and q. Now, the Sobolev Imbedding Theorem implies that $|W_{x,q}(0)| \leqslant \|W_{x,q}\|_{L^\infty(B_{1/2})} \leqslant C$. Hence we conclude from (3.2) that $\|v_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C$.

Step 3: For any $q > 0$

$$\|\nabla v_q\|_{L^\infty(\mathbb{R}^2)},\ \|\nabla N_q\|_{L^\infty(\mathbb{R}^2)},\ \|\varphi_q\|_{L^\infty(\mathbb{R}^2)},\ \|\psi_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C, \tag{4.7}$$

where C is independent of q.

Given $x \in \mathbb{R}^2$, let us define $\rho_{x,q}(y) = v_q(x+y)$ for $|y| \leqslant 1$. Then by (2.4),

$$\|\Delta \rho_{x,q}\|_{L^\infty(B_1)} \leqslant C(q+1)$$

for all large q. Since $\rho_{x,q} \in L^\infty(B_1)$ uniformly by (4.6), Lemma 4.2 implies that

$$\left|\nabla \rho_{x,q}(0)\right|^2 \leqslant C\|\rho_{x,q}\|_{L^\infty(B_1)}\left(\|\Delta \rho_{x,q}\|_{L^\infty(B_1)} + \|\rho_{x,q}\|_{L^\infty(B_1)}\right) \leqslant C(q+1)$$

for all large q. Here, C is independent of x and q. As a consequence, $\|\nabla v_q\|^2_{L^\infty(\mathbb{R}^2)} \leqslant C(q+1)$ and hence $\|\sigma_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C(q+1)$ for all large q. Now, let z_q be the minimum point of φ_q. Since $\Delta\varphi_q(z_q) \geqslant 0$, we draw from (4.4) that

$$\varphi_q(z_q) \geqslant -\frac{\sigma_q}{\kappa^2 q + 2f'(Ue^{v_q})Ue^{v_q}}\bigg|_{z_q} \geqslant -C.$$

Since $\varphi_q \leqslant 0$ by (3.1), this implies that $\|\varphi_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C$ for all large q. With this estimate, we can get a better estimate for ∇v_q. Indeed, since $\|\varphi_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C$ and $\|v_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C$, we obtain $\|\nabla v_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C$ applying Lemma 4.2 to (4.3) as above. In the sequel, the first and the third inequalities of (4.7) are verified.

On the other hand, since $\|N_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C$ and $\|\Delta N_q\|_{L^\infty(\mathbb{R}^2)} \leqslant Cq$ by (2.5) and (3.2), we deduce from Lemma 4.2 that $\|\nabla N_q\|^2_{L^\infty(\mathbb{R}^2)} \leqslant Cq$. Hence, $\|\eta_q\|_{L^\infty(\mathbb{R}^2)} \leqslant Cq$. Applying the maximum principle to (4.5), we have

$$\|\psi_q\|_{L^\infty(\mathbb{R}^2)} \leqslant \left\|\frac{\eta_q}{\kappa^2 q + 2f'(Ue^{v_q})Ue^{v_q}}\right\|_{L^\infty(\mathbb{R}^2)} \leqslant C.$$

With this estimate and Lemma 4.2, the second equation of (4.3) yields a better estimate $\|\nabla N\|^2_{L^\infty(\mathbb{R}^2)} \leqslant C$.

Step 4: It has been verified that v_q, N_q, and $\Delta v_q, \Delta N_q$ are uniformly bounded in $L^\infty(\mathbb{R}^2)$. Hence v_q and N_q are uniformly bounded in $W^{2,p}(K)$ for any $p > 1$ and a compact subset $K \subset \mathbb{R}^2$. Then there exist a subsequence, still denoted by (v_q, N_q), and a pair of functions (v_*, N_*) such that

$$\|v_q - v_*\|_{C^{1,\gamma}(K)} \to 0, \qquad \|N_q - N_*\|_{C^{1,\gamma}(K)} \to 0 \tag{4.8}$$

for $0 \leqslant \gamma < 1$. Since $\|\varphi_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C$ and $\|\psi_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C$, we infer that $N_* = f(e^{v_0+v_*})$ and

$$\varphi_q \to \frac{2}{\kappa^2} e^{v_0+v_*} f\big(e^{v_0+v_*}\big) f'\big(e^{v_0+v_*}\big).$$

Multiplying the first equation of (4.3) by a test function $\xi \in C_c^\infty(\mathbb{R}^2)$ and letting $q \to \infty$, we derive that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2} \nabla v_q \cdot \nabla\xi + (2\varphi_q + g)\xi \\ &\to \int_{\mathbb{R}^2} \nabla v_* \cdot \nabla\xi + \frac{4}{\kappa^2} e^{v_0+v_*} f\big(e^{v_0+v_*}\big) f'\big(e^{v_0+v_*}\big)\xi + g\xi, \end{aligned}$$

which implies that v_* is a weak solution of (4.1).

In order to complete the proof of Proposition 4.1, it is enough to show that for every nonnegative integer k,

$$\big\|\partial^k\varphi_q\big\|_{L^\infty(\mathbb{R}^2)},\ \big\|\partial^k\psi_q\big\|_{L^\infty(\mathbb{R}^2)},\ \big\|\nabla\partial^k v_q\big\|_{L^\infty(\mathbb{R}^2)},\ \big\|\nabla\partial^k N_q\big\|_{L^\infty(\mathbb{R}^2)} \leqslant C. \tag{4.9}$$

We proceed by induction on k. The case for $k=0$ is a consequence of (4.7). Now we suppose that (4.9) is valid for a nonnegative integer $k-1$. It follows from the first equation of (4.3) that

$$\Delta\partial^k v_q = 2\partial^k\varphi_q + \partial^k g. \tag{4.10}$$

Since $\partial^k\varphi_q$ contains the derivatives of v_q and N_q with orders up to k, it comes from the induction assumption that $\|\Delta\partial^k v_q\|_{L^\infty(\mathbb{R}^2)} \leqslant Cq$ and $\|\partial^k v_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C$. Here and in the sequel, C is a generic constant depending only on k. Given $x \in \mathbb{R}^2$, if we set $\rho^{(k)}_{x,q}(y) = \partial^k v_q(x+y)$ for $|y| \leqslant 1$, then by Lemma 4.2,

$$\begin{aligned}\left|\nabla\partial^k v_q(x)\right|^2 = \left|\nabla\rho^{(k)}_{x,q}(0)\right|^2 &\leqslant C\left\|\rho^{(k)}_{x,q}\right\|_{L^\infty(B_1)}\left(\left\|\Delta\rho^{(k)}_{x,q}\right\|_{L^\infty(B_1)} + \left\|\rho^{(k)}_{x,q}\right\|_{L^\infty(B_1)}\right)\\ &\leqslant C\left\|\partial^k v_q\right\|_{L^\infty(\mathbb{R}^2)}\left(\left\|\Delta\partial^k v_q\right\|_{L^\infty(\mathbb{R}^2)} + \left\|\partial^k v_q\right\|_{L^\infty(\mathbb{R}^2)}\right) \leqslant Cq\end{aligned}$$

for all large q. As a consequence, $\|\nabla\partial^k v_q\|^2_{L^\infty(\mathbb{R}^2)} \leqslant Cq$.

We note from (4.4) that

$$\frac{1}{q}\Delta\partial^k\varphi_q = \left(\kappa^2 q + 2f'\left(Ue^{v_q}\right)Ue^{v_q}\right)\partial^k\varphi_q + H_{k,q} + \partial^k\sigma_q,$$

where $H_{k,q}$ contains derivatives of φ_q with orders less than k and derivatives of v_q with orders up to k. Since $\|\partial^k e^{v_0}\|_{L^\infty(\mathbb{R}^2)} \leqslant C$, it follows from the induction assumption that $\|H_{k,q}\|_{L^\infty(\mathbb{R}^2)} \leqslant C$. Since $\|\nabla\partial^k v_q\|^2_{L^\infty(\mathbb{R}^2)} \leqslant Cq$, we see from the induction assumption that $\|\partial^k\sigma_q\|_{L^\infty(\mathbb{R}^2)} \leqslant Cq$ for all large q. Therefore, we infer from the maximum principle that

$$\left\|\partial^k\varphi_q\right\|_{L^\infty(\mathbb{R}^2)} \leqslant \left\|\frac{H_{k,q} + \partial^k\sigma_q}{\kappa^2 q + 2f'(Ue^{v_q})Ue^{v_q}}\right\|_{L^\infty(\mathbb{R}^2)} \leqslant C. \tag{4.11}$$

Returning to (4.10), we derive from Lemma 4.2 that $\|\nabla\partial^k v_q\|^2_{L^\infty(\mathbb{R}^2)} \leqslant C$. Hence, the first and the third estimates of (4.9) hold true for k.

It follows from the second equation of (4.3) that

$$\Delta\partial^k N_q = \partial^k\psi_q. \tag{4.12}$$

Since $\partial^k\psi_q$ contains the derivatives of v_q, N_q, and φ_q with orders up to k, we deduce that $\|\Delta\partial^k N_q\|_{L^\infty(\mathbb{R}^2)} \leqslant \|\partial^k\psi_q\|_{L^\infty(\mathbb{R}^2)} \leqslant Cq$. Since $\|\partial^k N_q\|_{L^\infty(\mathbb{R}^2)} \leqslant C$ by induction assumption, we derive from Lemma 4.2 that $\|\nabla\partial^k N_q\|^2_{L^\infty(\mathbb{R}^2)} \leqslant Cq$. It comes from (4.5) that

$$\frac{1}{q}\Delta\partial^k\psi_q = \left(\kappa^2 q + 2f'\left(Ue^{v_q}\right)Ue^{v_q}\right)\partial^k\psi_q + G_{k,q} + \partial^k\eta_q,$$

where $G_{k,q}$ contains derivatives of ψ_q with orders less than k and derivatives of v_q with orders up to k. Furthermore, $\partial^k\eta_q$ contains derivatives of v_q and N_q with orders up to $k+1$. Therefore, $\|G_{k,q}\|_{L^\infty(\mathbb{R}^2)} \leqslant C$ and $\|\partial^k\eta_q\|_{L^\infty(\mathbb{R}^2)} \leqslant Cq$. Then, the maximum principle yields that for all large q,

$$\left\|\partial^k\psi_q\right\|_{L^\infty(\mathbb{R}^2)} \leqslant \left\|\frac{G_{k,q} + \partial^k\eta_q}{\kappa^2 q + 2f'(Ue^{v_q})Ue^{v_q}}\right\|_{L^\infty(\mathbb{R}^2)} \leqslant C.$$

Going back to (4.12), we infer from this estimate and Lemma 4.2 that $\|\nabla\partial^k N\|^2_{L^\infty(\mathbb{R}^2)} \leqslant C$. Thus, we have proved the second and the fourth estimates of (4.9), which finishes the proof of proposition. $\square$

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