On constructive characterizations of \((k, l)\)-sparse graphs

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Abstract

In this paper we study constructive characterizations of graphs satisfying tree-connectivity requirements. The main result is the following: if \(k\) and \(l\) are positive integers and \(l \leq \frac{k}{2}\), then a necessary and sufficient condition is proved for a node being the last node of a construction in a graph having at most \(k|X| - (k + l)\) induced edges in every subset \(X\) of nodes. The arguments and proofs extend those of Frank and Szegő for the case \(l = 1\) [A. Frank, L. Szegő, Constructive characterizations on packing and covering by trees, Discrete Appl. Math. 131 (2) (2003) 347–371].

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1. Constructive characterizations

In this paper we study constructive characterizations of graphs satisfying tree-connectivity requirements. A constructive characterization of a graph property is meant to be a building procedure consisting of some simple operations so that the graphs obtained in this way from some specified initial graph are precisely those having the property. A modest example is the following: a graph is connected if and only if it can be obtained from a node by the following operation: add a new edge connecting an existing node with either an existing node or a new one. Another well-known result is the so called ear-decomposition of 2-connected graphs.

In 1976 Lovász gave a constructive characterization of 2\(k\)-edge-connected graphs. A graph is said to be \(k\)-edge-connected if the deletion of at most \(k - 1\) edges results in a connected graph. From now on, adding an edge means adding a new edge connecting two existing nodes. This new edge can be parallel to existing ones, but it cannot be a loop unless otherwise stated.

**Theorem 1.1** (Lovász [10]). An undirected graph \(G = (V, E)\) is 2\(k\)-edge-connected if and only if \(G\) can be obtained from a single node by the following two operations:

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(i) add a new edge (possibly a loop),
(ii) add a new node \( z \), subdivide \( k \) existing edges by new nodes, and identify the \( k \) subdividing nodes with \( z \).

Operation (ii) is called \textit{pinching} \( k \) edges with \( z \).

Similar constructive characterizations for \((2k + 1)\)-edge-connectivity were given by Mader. A directed counterpart of the previous results is also due to Mader [11].

\( k \)-edge-connectivity is a well-studied way of formulating the notion of high ‘edge-connection’ of an undirected graph but there may be other possibilities, as well. An undirected graph is called \textit{k-tree-connected} if it contains \( k \) edge-disjoint spanning trees. The following constructive characterization of \( k \)-tree-connected graphs was given by Frank in [3] by observing that a combination of a theorem of Mader and a theorem of Tutte gives rise to the following. (For a direct proof, see Tay [13].)

**Theorem 1.2.** An undirected graph \( G = (V, E) \) is \( k \)-tree-connected if and only if \( G \) can be built from a single node by the following three operations:

(i) add a new edge,
(ii) add a new node \( z \) and \( k \) new edges ending at \( z \),
(iii) pinch \( i \) \((1 \leq i \leq k - 1)\) existing edges with a new node \( z \), and add \( k - i \) new edges connecting \( z \) with existing nodes.

What makes a constructive characterization good? Jüttner [8] gave the following building procedure for graphs having a Hamiltonian cycle. Beginning from \( K_3 \) use the following two operations: add a new edge between two existing nodes and subdivide an edge incident to a node of degree 2 by a new node. It is clear that this procedure builds up a graph \( G \) if and only if \( G \) has a Hamiltonian cycle. Since we cannot check in polynomial time whether a graph can be obtained this way or not (unless \( P = NP \)), we do not think that this is a good constructive characterization,

Nash-Williams [12] proved the following theorem concerning coverings by trees. For a graph \( G = (V, E) \), \( \gamma_G(X) \) denotes the number of edges of \( G \) with both end-nodes in \( X \subseteq V \).

**Theorem 1.3** (Nash-Williams [12]). A graph \( G = (V, E) \) is the union of \( k \) edge-disjoint forests if and only if \( \gamma_G(X) \leq k|X| - k \) for all nonempty \( X \subseteq V \).

Frank and Szegő considered two variants of the notion of \( k \)-tree-connectivity in [5]. One of them is the following: a graph \( G \) (with at least 2 nodes) is called \textit{nearly} \( k \)-tree-connected if \( G \) is not \( k \)-tree-connected but adding any new edge to \( G \) results in a \( k \)-tree-connected graph. Let \( K_2^{k-1} \) denote the graph on two nodes with \( k - 1 \) parallel edges. (On the basis of the work of Henneberg [6] and Laman [9], Tay and Whiteley gave the proof of the following theorem in the special case of \( k = 2 \) in [15].)

**Theorem 1.4** (Frank and Szegő [5]). An undirected graph \( G = (V, E) \) is nearly \( k \)-tree-connected if and only if \( G \) can be built from \( K_2^{k-1} \) by applying the following operations:

(O1') add a new node \( z \) and \( k \) new edges ending at \( z \) so that no \( k \) parallel edges can arise,
(O2') choose a subset \( F \) of \( i \) existing edges \((1 \leq i \leq k - 1)\), pinch the elements of \( F \) with a new node \( z \), and add \( k - i \) new edges connecting \( z \) with other nodes so that there are no \( k \) parallel edges in the resulting graph.
Actually, the authors of [5] proved this result in a slightly more general form. They proved the following conjecture for the case where \( l = 1 \). Let \( k, l \) be two integers such that \( k \geq 2 \) and \( 0 \leq l \leq k \). A graph \( G = (V, E) \) is said to be \((k, l)\)-sparse if \( \gamma_G(X) \leq k|X| - (k + l) \) for all \( X \subseteq V, |X| \geq 2 \). (By convention the graph with one single node is \((k, l)\)-sparse.)

**Conjecture 1.5.** Let \( 1 \leq l < \frac{k+2}{3} \). An undirected graph \( G = (V, E) \) is \((k, l)\)-sparse if and only if \( G \) can be built from a single node by applying the following operations:

(P1) add a new node \( z \) and at most \( k \) new edges ending at \( z \) so that no \( k - l + 1 \) parallel edges can arise.

(P2) Choose a subset \( F \) of \( i \) existing edges \((1 \leq i \leq k - 1)\), pinch the elements of \( F \) with a new node \( z \), and add \( k - i \) new edges connecting \( z \) with other nodes so that there are no \( k - l + 1 \) parallel edges in the resulting graph.

(If \( l = 0 \) is allowed, then Theorem 1.2 is also a special case which has been already verified.) By the fundamental Theorem 1.3 of Nash-Williams, a graph is \((k, l)\)-sparse if and only if the edge-set can be covered by \( k \) spanning forests after adding \( l \) new edges arbitrarily.

We call a graph **highly \( k \)-tree-connected** if the deletion of any existing edge leaves a \( k \)-tree-connected graph. Frank and Király [4] (among others) gave a constructive characterization for highly \( 2 \)-tree-connected graphs. In [5] this was extended for arbitrary \( k \geq 2 \).

We mention a recent result of Berg and Jordán [1] who proved a conjecture of Connelly. A \( 2 \)-connected undirected graph \( G = (V, E) \) is a **generic circuit** if \(|E| = 2|V| - 2 \) and \( \gamma_G(X) \leq 2|X| - 3 \) for all \( 2 \leq |X| \leq |V| - 1 \).

**Theorem 1.6 (Berg and Jordán [1]).** An undirected graph \( G = (V, E) \) is a generic circuit if and only if \( G \) can be built up from \( K_4 \) by the following operation:

- **subdivide an edge \( uv \) by a new node \( z \) and add an edge \( zw \) so that \( w \neq u, v \).**

These graphs have a role in rigidity theory. We also remark that Whiteley in [16] provided some rigidity property of nearly \( k \)-tree-connected graphs; furthermore T. Eren et al. gave the constructive characterization of generically minimally rigid direction based point formations in 3-space in [2] by using the constructive characterization of \((3, 1)\)-sparse graphs given in [5].

Jackson and Jordán consider sparse graphs in connection with rigidity properties in [7]. In [14] Tay proved that for inductive reasons a node of degree at most \( 2k - 1 \) either can be “split off”, or “reduced” to obtain a smaller nearly \( k \)-tree-connected graph. Theorem 1.4 says that there always exists a node which can be “split off”.

The following theorem follows easily from the definition of \((k, l)\)-sparse graphs.

**Theorem 1.7.** Let \( 1 \leq l \leq \frac{k}{2} \). If an undirected graph \( G = (V, E) \) can be built up from a single node by applying the operations (P1) and (P2), then \( G \) is \((k, l)\)-sparse.

Inspired by the previous constructive characterizations we would conjecture that the reverse of the above theorem is also true for all \( k \) and \( l \) satisfying \( \frac{k}{2} \geq l \). But as we show in Section 4, this is not true if \( l \geq \frac{k+2}{3} \).

\( d_G(z) \) (sometimes \( d(z) \)) denotes the degree of a node \( z \) in graph \( G \).
2. Splittings for \((k, l)\)-sparse graphs

For several reasons Conjecture 1.5 fails for \((k, l)\)-sparse graphs if \(\frac{k}{2} < l\). Here we point out one important obstacle: there is no graph on three nodes for which \(|E| = k|V| - (k + l)\). (Indeed, if there was a graph \(G = (V, E)\), then we would have \(|E| \leq 3(k - l)\) since an edge may have multiplicity at most \(k - l\). As \(2k - l > 3k - 3l\), we get a contradiction.)

With the same reasoning the following can be proved.

**Lemma 2.1.** Let \(m \geq 3\) be an integer. There is no graph on \(m\) nodes with \(|E| = km - (k + l)\) satisfying \(\gamma_G(X) \leq k|X| - (k + l)\) for all \(X \subseteq V, |X| \geq 2\) if \(\frac{m - 1}{m + 1}k < l\).

**Proof.** Since \(|E| \leq \frac{m(m - 1)}{2}l\) by the maximal multiplicity of an edge, we have \(km - (k + l) = |E| \leq \frac{m(m - 1)}{2}(k - l)\). But

\[
km - (k + l) - \frac{m(m - 1)}{2}(k - l) = \frac{(m^2 - m - 2)l - (m^2 - 3m + 2)k}{2} = \frac{(m - 2)((m + 1)l - (m - 1)k)}{2} > \frac{1}{2} \left( \frac{m - 1}{m + 1}k - (m - 1)k \right) = 0,
\]
a contradiction. \(\square\)

That is why we study here only the case of \(l \leq \frac{k}{2}\).

In graph \(G\) splitting off a pair \(zu\) and \(zv\) of edges \((u \neq v)\) means that we delete these two edges and add a new edge \(uv\) (maybe parallel to the other existing edges) to \(G\). After applying this operation, \(uv\) is called a split edge. A splitting off in a \((k, l)\)-sparse graph \(G\) is admissible if the resulting graph on node-set \(V - z\) is \((k, l)\)-sparse.

**Definition 2.2.** Let \(b_G\) denote the following function for any \(X \subseteq V, |X| \geq 2\):

\[
b_G(X) := k|X| - (k + l) - \gamma_G(X).
\]

By this definition a graph \(G = (V, E)\) is \((k, l)\)-sparse if and only if \(b_G(X) \geq 0\) for all subsets \(X \subseteq V, |X| \geq 2\). If \(b_G(X) = 0\) and \(X \neq V\), then \(X\) is said to be a \(G\)-tight (or for short tight) set. Furthermore, \(G\) is a union of \(k\) edge-disjoint spanning trees after adding arbitrary \(l\) edges if and only if \(G\) is \((k, l)\)-sparse and \(b_G(V) = 0\). We will abbreviate \(b_G\) as \(b\).

**Observation 2.3.** Splitting off \(zu\) and \(zv\) at node \(z\) is not admissible if and only if there exists a tight subset in \(V - z\) containing \(u\) and \(v\).

We say that splitting off \(j\) disjoint pairs of edges \((1 \leq j \leq k - 1)\) at node \(z\) is admissible if it consists of admissible splittings. Obviously the order of the pairs in a splitting sequence is irrelevant. The length of a splitting sequence \(S\) is the number of its pairs and it is denoted by \(|S|\). \(G_S\) denotes the graph obtained after applying the splitting sequence \(S\).

An admissible splitting sequence at node \(z\) of length \(d_G(z) - k\) (which number is denoted by \(i\)) is called a full splitting for \(d_G(z) \geq k + 1\). That is, a full splitting at \(z\) is the inverse of operation (P2). For the sake of convenience, at a node \(z\) with degree at most \(k\) the inverse of operation (P1) (that is, the deletion of \(z\) and all of its adjacent edges) is also called a full splitting. The main result of this paper is a necessary and sufficient condition of a node admitting a full splitting. We hope that it will lead to a proof of Conjecture 1.5 just like in the special case of \(l = 1\) in [5].
Observation 2.4. $b_G(X)$ is an upper bound for the number of split edges induced by $X \subseteq V - z$ provided by an admissible sequence of splittings at some node $z$.

The next four claims are about $(k, l)$-sparse graphs. $d_G(X, Y)$ is defined to be the number of edges between the node-sets $X - Y$ and $Y - X$. If $X = \{u\}$, $Y = \{v\}$, then we use $d_G(u, v)$.

Claim 2.5. If $X, Y \subseteq V$ and $|X \cap Y| \geq 2$, then

$$b(X) + b(Y) = b(X \cap Y) + b(X \cup Y) + d_G(X, Y).$$

Proof. $b(X) + b(Y) = k|X| - (k + l) - \gamma_G(X) + k|Y| - (k + l) - \gamma_G(Y) = k(|X| + |Y|) - 2(k + l) - (\gamma_G(X) + \gamma_G(Y)) = k|X \cap Y| - (k + l) - \gamma_G(X \cap Y) + k|X \cup Y| - (k + l) - \gamma_G(X \cup Y) + d_G(X, Y) = b(X \cap Y) + b(X \cup Y) + d_G(X, Y). \qed$

Claim 2.6. If $X, Y \subseteq V$ and $|X \cap Y| = 1$, then

$$b(X) + b(Y) = b(X \cup Y) - l + d_G(X, Y).$$

Proof. $b(X) + b(Y) = k|X| - (k + l) - \gamma_G(X) + k|Y| - (k + l) - \gamma_G(Y) = k(|X| + |Y| - 1) - (k + l) - l - (\gamma_G(X) + \gamma_G(Y)) = b(X \cup Y) - (k + l) - l - (\gamma_G(X \cup Y) - d_G(X, Y)) = b(X \cup Y) - l + d_G(X, Y). \qed$

Claim 2.7. If $X_1, X_2, X_3 \subseteq V$ and $|X_j \cap X_m| = 1$ for $1 \leq j < m \leq 3$ and $|X_1 \cap X_2 \cap X_3| = 0$, then

$$b\left(\bigcup_{j=1}^{3} X_j\right) \leq \sum_{j=1}^{3} b(X_j) - k + 2l.$$

Proof. $b(\bigcup_{j=1}^{3} X_j) = k|\bigcup_{j=1}^{3} X_j| - (k + l) - \gamma_G(\bigcup_{j=1}^{3} X_j) \leq k(\sum_{j=1}^{3} |X_j| - 3) - (k + l) - \sum_{j=1}^{3} \gamma_G(X_j) = \sum_{j=1}^{3} (k|X_j| - (k + l) - \gamma_G(X_j)) - k + 2l = \sum_{j=1}^{3} b(X_j) - k + 2l. \qed$

Remark. In particular, all of $X_1, X_2, X_3$ cannot be tight at the same time for $k \geq 2l + 1$. If $k = 2l$ and $X_1, X_2, X_3$ are tight sets, then $\bigcup_{j=1}^{3} X_j$ is also tight. These special cases will be used frequently in the paper and also indicate why we will consider only the case of $l \leq \frac{k}{2}$.

Claim 2.8. Let $l \leq \frac{k}{2}$ and $G = (V, E)$ be a $(k, l)$-sparse graph. Let $z \in V$ and $X \subseteq V - z$ be a maximal tight set containing the distinct nodes $c_1, c_2$. Let $d$ be a node in $V - X - z$. If there is a tight set in $V - z$ containing $c_1$ and $d$, then there is no tight set in $V - z$ containing $c_2$ and $d$.

Proof. According to Claim 2.5, $P \cap X = \{c_1\}$ since $X$ is maximal. By Claims 2.5 and 2.7 we obtain that there is no tight set containing $c_2$ and $d$. \qed

Let $G$ be a $(k, l)$-sparse graph. Since $\sum_{v \in V} d_G(v) = 2|E| \leq 2k|V| - 2(k + l) < 2k|V|$, there is a node $z \in G$ with $d_G(z) \leq 2k - 1$.

Claim 2.9. Let $G = (V, E)$ be a $(k, l)$-sparse graph. $d_G(u, v) \leq k - l$ for any two nodes $u, v$.

Proof. By the definition of $(k, l)$-sparse graphs, $\gamma_G(\{u, v\}) \leq k|\{u, v\}| - (k + l) = k - l$ for set $\{u, v\}. \qed$
3. **Full splittings in \((k, l)\)-sparse graphs**

In this section we derive a necessary and sufficient condition for an arbitrary specified node to admit a full splitting. The proof follows the line of the proof given by Frank and Szegő for the case \(l = 1\) in [5]. The techniques are the same; the biggest difference is that Proposition 3.4 was straightforward in that special case, since only one splitting off was needed.

Let \(k \geq 2\) and \(0 \leq l \leq \frac{k}{2}\). Let \(G\) be a \((k, l)\)-sparse graph. Consider a node \(z\) with degree at most \(2k - 1\) for which there is no full splitting. If \(d_G(z) \leq k\), then the deletion of \(z\) and its adjacent edges results in a \((k, l)\)-sparse graph, and hence \(d_G(z) \geq k + 1\).

Assume that a longest admissible splitting sequence \(S\) at \(z\) is not a full splitting. Since \(z\) does not admit a full splitting, \(|S| < i := d_G(z) - k\).

Let \(N_D(w)\) denote the set of the neighbours of a node \(w\) in graph \(D\).

**Claim 3.1.** If \(|N_{G_S}(z)| \geq 2\), then there exists a maximal \(G_S\)-tight subset \(P_{\max}\) of \(V - z\) including \(N_{G_S}(z)\).

**Proof.** Let \(z a\) and \(z b\) denote two non-parallel edges in \(G_S\). Since \((z a, z b)\) is not an admissible splitting off, there is a tight set \(X \subseteq V - z\) containing \(a\) and \(b\). According to Claim 2.5, there is a maximal tight set \(P_{\max} \subseteq V - z\) containing \(a\) and \(b\).

If there is another neighbour \(c\) of \(z\) which is not in \(P_{\max}\), then there is a tight set \(Y \subseteq V - z\) containing \(a\) and \(c\), since \((z a, z c)\) is not an admissible splitting off. Since \(P\) is maximal, \(Y \cap P_{\max} = \{a\}\). By Claim 2.8 \((z b, z c)\) is an admissible splitting off, a contradiction, that is, \(P_{\max}\) contains all the neighbours of \(z\). \(\Box\)

**Claim 3.2.** If \(|N_{G_S}(z)| \geq 2\), then there exists a split edge whose end-nodes are disjoint from the nodes of \(P_{\max}\).

**Proof.** Since there is no admissible splitting off at \(z\) in \(G_S\), according to Claim 3.1 there exists \(P_{\max} \subseteq V - z\). Let \(j, h, m\) denote the number of split edges with exactly, respectively, 2, 1, 0 end-nodes in \(P_{\max}\). \(j + h + m = |S| < i\) since \(S\) is not full.

\[
\begin{align*}
 k|P_{\max} + z| - (k + l) & \geq \gamma_G(P_{\max} + z) = \gamma_{G_S}(P_{\max}) + j + h + d_{G_S}(z, P_{\max}) \\
 &= \gamma_{G_S}(P_{\max}) + j + h + (k + i - 2(j + h + m)) \\
 &= \gamma_{G_S}(P_{\max}) + k + (i - (j + h + m)) - m > k|P_{\max}| - (k + l) + k - m \\
 &= k|P_{\max} + z| - (k + l) - m,
\end{align*}
\]

which implies \(m > 0\). \(\Box\)

**Claim 3.3.** If \(|N_{G_S}(z)| \geq 2\), then \(|N_{G_S}(z)| = 2\). There is a neighbour \(s\) of \(z\) for which \(d_{G_S}(z, s) = 1\).

**Proof.** First assume that \(|N_{G_S}(z)| \geq 3\). Let \(a_1, a_2, a_3\) denote three of these nodes. By Claim 3.2 there is a split edge \(uv\) disjoint from \(P_{\max}\). Let \(J = \{1, 2, 3\}\).

By Claim 2.8, \(S - (zu, vz) \cup (zu, za_j)\) is an admissible splitting sequence for at least two elements \(j\) of \(J\). The same is true for \(S - (zu, vz) \cup (vz, za_j)\). Hence we may assume that \(S - (zu, vz) \cup (zu, za_1)\) and \(S - (zu, vz) \cup (vz, za_2)\) are both admissible splitting sequences. We claim that \(S' := S - (zu, vz) \cup (zu, za_1) \cup (vz, za_2)\) is an admissible splitting sequence. If not, then there is a tight set \(Y\) in \(G_S - z\) containing \(u, v, a_1, a_2\). Then, according to Claim 2.5,
$P_{\text{max}} \cup Y$ is a tight set in $G_{S} - z$ contradicting the maximality of $P_{\text{max}}$. The length of $S'$ is greater than the length of $S$, a contradiction.

Now assume that $|N_{G_{S}}(z)| = 2$. Let $s$ and $t$ be the two neighbours of $z$ and assume that $d_{G_{S}}(z, s) \geq 2$ and $d_{G_{S}}(z, t) \geq 2$. By Claim 3.2 there is a split edge $uv$ disjoint from $P_{\text{max}}$. According to Claim 2.8, $S - (zu, zv) \cup (zu, zt)$ or $S - (zu, zv) \cup (zu, zs)$ is an admissible splitting sequence. This also holds for $zv$ instead of $zu$.

First we may assume that $S - (zu, zv) \cup (zu, zt)$ and $S - (zu, zv) \cup (zv, zt)$ are admissible. If $S - (zu, zv) \cup (zu, zs) \cup (zv, zs)$ is not admissible, then there is a $G_{S}$-tight set $Y \subseteq V - z$ containing $s, u, v$. Then $S - (zu, zv) \cup (zu, zs) \cup (zv, zt)$ is admissible by Claim 2.7.

Second we may assume that $S - (zu, zv) \cup (zu, zt)$ and $S - (zu, zv) \cup (zs, zv)$ are admissible. Then $S - (zu, zv) \cup (zu, zt) \cup (zv, zs)$ is admissible by Claim 2.7, a contradiction.

Since there is no other case, we are done. □

Now we prove that if $d_{G}(z)$ is at most $k + l$, then a full splitting always exists at $z$.

**Proposition 3.4.** Let $G$ be a $(k, l)$-sparse graph. If $z \in V$ has degree at most $k + l$, then there exists a full splitting at $z$.

**Proof.** If $d_{G}(z)$ is at most $k$, then if we delete $z$ with its adjacent edges, we obviously get a $(k, l)$-sparse graph.

We claim that there always exists a full splitting at a node $z$ with degree $k + i$ where $1 \leq i \leq l$. There is no $G$-tight set $X \subseteq V - z$ which contains all the neighbours of $z$ because, if there was one, then we would have $b_{G}(X + z) = b_{G}(X) + k - d_{G}(z) \leq 0 + k - (k + 1) < 0$ which contradicts that $G$ is $(k, l)$-sparse. Since there are no edges with multiplicity greater than $k - l$, the neighbour-set of $z$ in $G$ has at least two elements, so by Observation 2.3 and Claim 2.7 there is an admissible splitting off at $z$. Hence the longest admissible splitting sequence at $z$ has length at least $l$.

Let $S$ be a longest admissible splitting sequence at $z$. If $|S| \geq i$, then we are done. If $h := |S| < i$, then $d_{G_{S}}(z) \geq d_{G}(z) - 2(i - 1) = k + i - 2i + 2 = k - i + 2 \geq k - l + 2$. Hence by Claim 2.9, $|N_{G_{S}}(z)| \geq 3$ or $|N_{G_{S}}(z)| = 2$ and both neighbours are joined to $z$ by at least two edges. By Claim 3.3 $S$ is not longest, a contradiction. □

Let $i := d_{G}(z) - k$ (here $2 \leq i \leq k - 1$). Call a node $z$ small if $k - 1 \leq d_{G}(z) \leq 2k - 1$.

**Theorem 3.5.** A small node $z$ of $G$ does not admit a full splitting if and only if $z$ has a neighbour $t$ and there is a family $\mathcal{P}_{z}$ of subsets of $V - z$ with at least two elements such that:

\[
X \cap Y = \{t\} \quad \text{for } X, Y \in \mathcal{P}_{z},
\]

\[
\sum_{X \in \mathcal{P}_{z}} b(X) < d_{G}(z, t) - (k - i) - d_{G}(z, V - z - \cup \mathcal{P}_{z}),
\]

where $\cup \mathcal{P}_{z}$ denotes $\bigcup_{X \in \mathcal{P}_{z}} X$.

**Proof.** Suppose first that $t$ and $\mathcal{P}_{z}$ satisfy (\text{*}), (\text{**}) and let $S$ be an admissible splitting sequence (see Fig. 1). The number of split edges incident to $t$ with other end-nodes outside of $\cup \mathcal{P}_{z}$ is at most $d_{G}(z, V - z - \cup \mathcal{P}_{z})$. The number of split edges incident to $t$ with their other end-nodes in $\cup \mathcal{P}_{z}$ is at most $\sum_{X \in \mathcal{P}_{z}} b(X)$. Since a full splitting at $z$ means deleting $k - i$ edges incident to $z$ and splitting off all the other edges of $z$ (so that the graph obtained this way is $(k, l)$-sparse), we would have at least $d_{G}(z, t) - (k - i)$ split edges incident to $t$ which implies by (\text{**}) that $S$ is not full.
To see the other direction, let $S$ be a longest admissible splitting sequence at $z$ for which the following pair is lexicographically maximal: $(|N_{G_S}(z)|, |P_{\text{max}}|)$ where $P_{\text{max}}$ denotes a maximal tight set in $G_S$ which includes $N_{G_S}(z)$ but does not contain $z$. If there is no such tight set, then let $P_{\text{max}} := \emptyset$. Since $z$ does not admit a full splitting, $|S| < i$.

By Claim 3.3 there are only the following two cases. An edge not incident to $t$ is called $t$-disjoint.

**Case 1.** $|N_{G_S}(z)| = 2$ and $z$ has a neighbour $s$ for which $d_{G_S}(z, s) = 1$.

Let $u \in V - t - s$ be an arbitrary node for which there is a $t$-disjoint split edge $uv$ (by Claim 3.2 there exists such an edge). There is a tight set $X \subseteq V - z$ containing $u$ and $t$; otherwise $S' := S - (zu, zv) \cup (zu, zt)$ is another longest admissible splitting sequence for which if $v \neq s$, then $|N_{G_{S'}}(z)| = 3$, if $v = s$ and $d_{G_S}(z, t) \geq 3$, then $d_{G_{S'}}(z, t) \geq d_{G_{S'}}(z, s) \geq 2$, which is a contradiction by Claim 3.3. If $v = s$ and $d_{G_S}(z, t) = 2$ and $d_{G_S}(z, s) = 1$, then by Claim 3.2 there is a split edge $ab$ which is disjoint from $P_{\text{max}}$. We may suppose $a \neq u$.

Since $S^* := S - (zu, zs) \cup (za, zs) \cup (zu, zt)$ is not admissible, we have a tight set in $G_S$ containing $a, s, t, u$ showing that there is a tight set containing $t$ and $u$.

Let $P_u \subseteq V - z$ be a tight set including $u$ and $t$ and containing the minimal number of $t$-disjoint split edges which is inclusion-wise maximal. Similarly, there is a tight set $X \subseteq V - z$ containing $s$ and $t$; otherwise $S \cup (zs, zt)$ is a longer admissible splitting sequence than $S$. Let $P_t$ be such a tight set containing the minimal number of $t$-disjoint split edges which is inclusion-wise maximal.

Let $\mathcal{P}_z := \{X \subseteq V - z : \exists u \in V \text{ incident to a } t\text{-disjoint split edge such that } X = P_u \text{ or } X = P_t\}$. For nodes $u \neq v$, $P_u$ can be equal to $P_t$, but there is only one copy of them in $\mathcal{P}_z$. Now we prove some essential properties of $\mathcal{P}_z$.

**Proposition 3.6.** There is no $t$-disjoint split edge in any member $X$ of $\mathcal{P}_z$.

**Proof.** Let us assume $X = P_u$ and $X \neq P_t$. By the definition of $P_u$ we have a $t$-disjoint split edge $uv$. First suppose $v \notin P_u$. Then $P_v \cap P_u = \{t\}$ according to the existence of (split) edge $uv$ and Claim 2.5. Let us suppose indirectly that there is a $t$-disjoint split edge $ab$ in $P_u$. We may suppose that $b \neq u$.

$S - (za, zb) - (zu, zv) \cup (zt, zu) \cup (zv, za)$ would be another longest splitting sequence with one more remaining neighbour of $z$, so it cannot be admissible, that is, there is a set $Y \subseteq V - z$ containing $a, u, v, t$, which is tight in $G_S$. $Y$ does not contain $b$, and hence the tight set $Y \cap P_u$ contains a smaller number of split edges than $P_u$, a contradiction.

Suppose that $v = s$ and $v \in P_u$. Consider a split edge $cd$ which is disjoint from $P_{\text{max}}$ and hence from $P_u$ (such an edge exists according to Claim 3.2). By the previous two paragraphs tight sets $P_c$ and $P_d$ do not contain $t$-disjoint split edges. According to Claim 2.5, $P_c \cap P_{\text{max}} = \{t\}$.
According to Claim 2.8, $S' := S - (zc, zd) \cup (zc, zs)$ is an admissible splitting sequence. For $S'' := S' - (zu, zv) \cup (zt, zu)$, the cardinality of $N_{G_{S''}}(z) = \{t, s, d\}$ is 3; hence $S''$ cannot be admissible, that is, there is a tight set $Y \subseteq V - z$ containing $c, s, u, t$ in $G_{S'}$. $Y \cup P_{\max}$ (in $G_{S'}$) contradicts the choice of $S$ by the maximality of $P_{\max}$.

Now assume that $X = P_3$. Let us suppose indirectly that there is a $t$-disjoint split edge $ab$ in $P_3$. $S' := S - (za, zb) \cup (zt, zs)$ is an admissible splitting sequence with three remaining neighbours of $z$ in $G_{S'}$, which is a contradiction by Claim 3.3. \hfill \Box

Now it follows that (**) holds for $P_z$.

**Case 2.** $|N_{G_{S}}(z)| = 1$. Let $t$ denote the only neighbour of $z$ in $G_{S}$.

**Claim 3.7.** There exists a $t$-disjoint split edge.

**Proof.** Let $p$ and $m$ be the number of split edges incident to and not incident to $t$, respectively. Since $S$ is not full, $p + m = |S| < i$. In the original graph $G$ by Claim 2.9,

$$k - 1 \geq d_G(z, t) = d_G(z) - p - 2m = k + i - p - 2m = k + (i - p - m) - m > k - m,$$

which implies that $m > 1$. \hfill \Box

Since $S$ is not a full splitting, $d_{G_{S}}(z) \geq k + i - 2(i - 1) = k - i + 2 \geq 3$. Now we define $P_z$. Let $u \in V - t$ be an arbitrary node for which there is a $t$-disjoint split edge $uv$. There is a tight set $X \subseteq V - z$ containing $u$ and $t$; otherwise $S'' := S - (zu, zv) \cup (zt, zt)$ is an other longest admissible splitting sequence for which $|N_{G_{S''}}(z)| = 2$, which contradicts the choice of $S$. Let $P_u$ be such a tight set containing the minimal number of $t$-disjoint split edges which is inclusion-wise maximal. Let $P_z := \{X \subseteq V - z : \exists u \in V$ incident to a $t$-disjoint split edge such that $X = P_u\}$.(The only difference from Case 1 is that there is no set $P_3$ here.)

**Proposition 3.8.** There is no $t$-disjoint split edge in an arbitrary element of $P_z$.

**Proof.** See the first two paragraphs of the proof of Proposition 3.6.

Now it follows that (**) holds for $P_z$.

**Claim 3.9.** Let $X, Y$ be two distinct members of $P_z$, $X \cap Y = \{t\}$.

**Proof.** Let us suppose $X = P_u$ and $Y = P_v$ for some $u, v \in V$. By Propositions 3.6 and 3.8, $P_u \not\subset P_v$. If $|P_u \cap P_v| \geq 2$, then by Claim 2.5 $d_{G_{S}}(P_u, P_v) = 0$ and $P_u \cup P_v$ is tight. Since it does not contain any $t$-disjoint split edge, it contradicts the maximal choice of $P_u$. \hfill \Box

Hence (*) holds for $P_z$.

We have shown that if a small node $z$ does not admit a full splitting, then the neighbour $t$ of $z$ and set-system $P_z$ satisfy both (*) and (**). \hfill \Box

We state the following easy consequence of Theorem 3.5. The neighbour $t$ of $z$ in Theorem 3.5 is called the blocking node of $z$.

**Corollary 3.10.** Let $z$ be a small node in a $(k, l)$-sparse graph $G$. If $z$ does not admit a full splitting, then the blocking node $t$ of $z$ is unique.
4. Counterexamples

In this section we give a \((k, l)\)-sparse graph for any \(k \geq 3\), \(\frac{k+2}{3} \leq l \leq \frac{k}{2}\), which cannot be obtained by the operations of Theorem 1.7. This is surprising because we managed to prove almost all the ingredients of the proof of the constructive characterization of \((k, 1)\)-sparse graphs (given in [5]) also for these graphs. We remark that the graphs we give in the smallest cases \((4, 2)\) and \((6, 3)\) have 60 and 85 nodes, respectively.

Let us consider \(m:=3k-l+2\) copies of the following graph \(G_1 = (V_1, E_1)\) and let the subscripts go from 1 to \(m\). Graph \(G_1\) has \(|V_1| = 5\) nodes \(a_1, b_1, c_1, d_1, z_1\) and \(|E_1| = k|V_1|-(k+l)=4k-l\) edges. Edges \(a_1d_1, b_1d_1, c_1d_1, z_1d_1\) have multiplicity \(k-l\), \(b_1z_1, c_1z_1\) has \(l\), \(a_1b_1\) has multiplicity \(l-1\), \(a_1z_1\) has multiplicity 1, and all the other edges have multiplicity 0. See Fig. 2; the multiplicities of the edges are shown in the figure.

It is easy to see that \(G_1\) is \((k, l)\)-sparse since it can be obtained by the operations (e.g. by the following order: \(z_1, d_1, c_1, b_1, a_1\)).

Let \(G_{(k,l)} = (V_{(k,l)}, E_{(k,l)})\) where \(V_{(k,l)} := \bigcup_{j=1}^{m} V_{j}\), \(E_{(k,l)} := \bigcup_{j=1}^{m} E_{j} \cup E^{*}\) and \(E^{*} := K_{1} \cup K_{2} \cup K_{3} \cup K_{1,2} \cup K_{3,2} \cup K_{1,3}\), where

\[
K_1 = \{a_i a_j : 1 \leq i < j \leq k + 1\}
\]

\[
K_2 = \{c_1 c_j : 2k - l + 3 \leq j \leq 3k - l + 2\} \cup \{c_i c_j : 2k - l + 3 \leq i < j \leq 3k - l + 2\}
\]

\[
K_3 = \{b_1 b_j : k + 2 \leq j \leq 2k - l + 2\} \cup \{b_i b_j : k + 2 \leq i < j \leq 2k - l + 2\}
\]

\[
K_{1,2} = \{b_1 a_j : 2 \leq i \leq k + 1, k + 2 \leq j \leq 2k - l + 2\}
\]

\[
K_{3,2} = \{b_i c_j : 2k - l + 3 \leq i \leq 3k - l + 2, k + 2 \leq j \leq 2k - l + 2\}
\]

\[
K_{1,3} = \{c_i a_j : 2 \leq i \leq k + 1, 2k - l + 3 \leq j \leq 3k - l + 2\}
\]

See Fig. 3. We will use the following two facts about \(E^{*}\):

- \(d_{E^{*}}(v) \leq k\) for all \(v \in V\),
- \(d_{G_{(k,l)}}(V_i, V_j) = 1\) for all \(1 \leq i < j \leq 3k - l + 2\).

It is clear that \(|V_{(k,l)}| = 5m = 5(3k - l + 2) = 15k - 5l + 10\) and \(|E_{(k,l)}| = m|E_1| + |E^{*}| = m(4k - l) + \frac{1}{2}m(3k - l + 1)\). In \(G_{(k,l)}\) we have the following degrees for any \(1 \leq j \leq m\):

\[
d(a_j) = d(b_j) = d(b_j) = 2k.
\]
Hence the only small nodes are $z_j$-s. Since \{a_j, d_j\}, \{b_j, d_j\}, \{c_j, d_j\} are tight sets, there is no full splitting at $z_j$, and hence graph $G_{(k,l)}$ cannot be obtained by the operations.

It remains to see that $G_{(k,l)}$ is $(k, l)$-sparse for the given $k$ and $l$. We are going to prove that $b(X) \geq 0$ for all $X \subseteq V_{(k,l)}$. It can be shown easily that if $X \subseteq V_{(k,l)}$ includes at least two nodes of $V_j$ for some $j$, then $b(X) \geq b(X \cup V_j)$. Hence it is enough to prove that the condition holds for subsets $X$ for which either $V_j \subseteq X$ or $|X \cap V_j| \leq 1$ for all $j$.

Let $n$ denote the number of $V_j$’s that are included entirely in $X$ and $r$ denote the number of $V_j$’s having a one-element intersection with $X$. $|X| = 5n + r$; hence we must prove

$$|E[X]| \leq k|X| - (k + l) = k(5n + r) - (k + l) = 5kn + kr - k - l. \quad (1)$$

We have

$$|E[X] - E^*| = n|E_1| = n(4k - l),$$

$$|E[X] \cap E^*| \leq \frac{n(n + r - 1) + rk}{2},$$

since $d(V_i, V_j) = 1$ and $d(a_i, V - V_i) = d(c_i, V - V_i) = k$, $d(b_i, V - V_i) = k - l + 1 < k$ for all $i, j$. Hence

$$|E[X]| = |E[X] - E^*| + |E[X] \cap E^*| \leq n(4k - l) + \frac{n(n + r - 1) + kr}{2}. \quad (2)$$

We will prove that the difference of the right hand sides of (1) and (2) is at least 0, which implies that $G$ is $(k, l)$-sparse. Let us make a computation, but first multiply by 2,

$$2(5kn + kr - k - l) - 2\left(n(4k - l) + \frac{n(n + r - 1) + kr}{2}\right),$$

$$= (10kn + 2kr - 2k - 2l) - (8kn - 2ln + n^2 + nr - n + kr),$$

$$= 10kn + 2kr - 2k - 2l - 8kn + 2ln - n^2 - nr + n - kr,$$

$$= 2kn + kr - 2k - 2l + 2ln - n^2 - nr + n,$$

$$= (n + r)(k - n) + n(k + 2l + 1) - 2(k + l). \quad (3)$$
If \(2 \leq n \leq k\), then (3) is obviously at least 0. \(n + r \leq m = 3k - l + 2\). If \(n > k\), then we continue the computation:

\[
\begin{align*}
&\geq m(k - n) + n(k + 2l + 1) - 2(k + l) \\
&= (3k - l + 2)(k - n) + n(k + 2l + 1) - 2(k + l) \\
&= (3k - l + 2)k + n(3l - 2k - 1) - 2(k + l) \\
&\geq 3l - 2k - 1 < 0,
\end{align*}
\]

since \(3l - 2k - 1 < 0\),

\[
\begin{align*}
&\geq (3k - l + 2)k + (3k - l + 2)(3l - 2k - 1) - 2(k + l) \\
&= (3k - l + 2)(3l - k - 1) - 2(k + l) \\
&= (3k - l + 2)(3l - k - 2) + (3k - l + 2) - 2k - 2l \\
&= (3k - l + 2)(3l - k - 2) + (k - 3l + 2) \\
&= (3k - l + 1)(3l - k - 2).
\end{align*}
\]

(4)

Since \(l \geq \frac{k + 2}{3}\), that is, \(3l \geq k + 2\), (4) is at least 0. If \(n = 1\) or 0, \(E[X] \leq k|X| - (k + l)\) can be shown with a much shorter computation. Hence we have proved that \(G\) is really \((k, l)\)-sparse.

5. Conclusion and open problems

This paper investigates possible constructive characterizations of \((k, l)\)-sparse graphs. One type of construction is considered: the node-sets of the graphs are grown one by one. The simplest case (if \(l = 0\), that is, tree-connectivity) was solved before. After the construction of \((2, 1)\)-sparse graphs, Frank and Szegő constructed \((k, 1)\)-sparse graphs recently.

The present author tried to extend the proof of Frank and Szegő for \(l \leq \frac{k}{2}\). A necessary and sufficient condition was given for a node being the last node of a construction. This was one of the main ingredients of the proof for the case \(l = 1\). The remaining part of the proof of the construction is the following: if there is a \((k, l)\)-sparse graph with smallest degree at least \(k + l + 1\), then it is impossible that neither of the nodes of degree at most 2 can be the last node of a construction, that is, by Theorem 3.5 there exists a set-system \(P_z\) for every such a node \(z\). However a graph was given in which the set-systems exist at the same time for every node of small degree, that is, which cannot be obtained by the operations in question, if \(\frac{k + 2}{3} \leq l \leq \frac{k}{2}\).

Conjecture 1.5 says that the construction may work in the cases remaining.

Another important question is that of finding an appropriate constructive characterization theorem for \((k, l)\)-sparse graphs if \(\frac{k + 2}{3} \leq l \leq \frac{k}{2}\). One possibility is the following.

Question 5.1. If \(i = k\) is allowed in (P2), is the reverse of Theorem 1.7 true?

Certainly, this can be allowed in the cases which are already proved but it is not necessary.

Are the examples of Section 4 the graphs with the smallest number of nodes? We think they are.

We may have to allow operations which glue together bigger graphs and the nodes are not considered one by one for \((k, l)\)-sparse graphs if \(\frac{k}{3} \leq l \leq k\).

A graph is defined as \([k, m]\)-sparse if \(0 \leq m \leq k\) and \(\gamma_G(X) \leq k|X| - m\) for all \(X \subseteq V, |X| \geq 2\). These graphs have a direct connection not to covering by trees but to covering by so-called 1-trees, and they may have a similar construction.
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