Journal of Computational and Applied Mathematics 116 (2000) 23–40

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Condition numbers for singular integral equations in weighted L^2 spaces

C. Laurita

Dipartimento di Matematica, Università degli Studi della Basilicata, Via Nazario Sauro 85, 85100 Potenza, Italy

Received 15 February 1999; received in revised form 15 June 1999

Abstract

The convergence and stability of a discrete collocation method for Cauchy singular integral equations in some weighted Besov spaces are studied. This numerical method results in solving a linear system in order to determine the unknown coefficients of the approximate solution. The author proves that this linear system is well conditioned. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 65R20; 45E05

Keywords: Cauchy singular integral equation; Besov space; Lagrange interpolation; Collocation method; Condition number

1. Introduction

Let us consider the Cauchy singular integral equation with constant coefficients

$$av(x) + \frac{b}{\pi}\int_{-1}^{1} \frac{v(y)}{y-x}\,\mathrm{d}y + \int_{-1}^{1} k(x,y)v(y)\,\mathrm{d}y = f(x), \quad |x| < 1, \tag{1.1}$$

where the first integral in (1.1) has to be understood in the Cauchy principal value sense. Here a and b are given real constants such that $\sqrt{a^2+b^2}=1$, $b \neq 0$, k and f are given complex-valued functions on $[-1,1]^2$ and $[-1,1]$, respectively, v is the unknown solution. It is known that (see [13]), even if f and k are regular functions, the solution v can be unbounded at one or both of the

E-mail address: laurita@matna2.dma.unina.it (C. Laurita)

0377-0427/00/$ - see front matter
PII: S0377-0427(99)00279-4

endpoints 1 and -1. So we can use for it the following representation:

$$v(x) = w^{\alpha,\beta}(x)u(x)$$

with u a smooth function and

$$w^{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta},$$

where, in dependence on the coefficients a and b, the exponents of the Jacobi weight are defined as follows:

$$\begin{aligned} &a - \mathrm{i}b = \mathrm{e}^{\mathrm{i}\pi\alpha_0}, \quad 0 < |\alpha_0| < 1, \\ &\alpha := M + \alpha_0, \quad \beta := N - \alpha_0 \end{aligned} \tag{1.2}$$

and M and N are two integers chosen such that $-1 < \alpha, \beta < 1$. Then the index $\kappa = -(\alpha+\beta) = -(M+N)$ can only take the values $-1, 0, 1$.

By substituting $v = w^{\alpha,\beta}u$ in (1.1) the equation can be written in operator form as

$$(A + K)u = f, \tag{1.3}$$

where we have defined the operators

$$Au(x) := aw^{\alpha,\beta}(x)u(x) + \frac{b}{\pi}\int_{-1}^{1}\frac{w^{\alpha,\beta}(y)u(y)}{y-x}\,\mathrm{d}y, \tag{1.4}$$

$$Ku(x) := \int_{-1}^{1} k(x,y)w^{\alpha,\beta}(y)u(y)\,\mathrm{d}y. \tag{1.5}$$

Several authors have studied collocation and quadrature methods to solve singular integral equations and, under suitable assumptions concerning the smoothness of f and k, convergence, stability results and error estimates have been obtained (see, for instance, [2,3,5,7,9,10] and the references given by the authors).

In [2] the author investigates some numerical aspects related to the computation of the matrix of the linear system associated to the classical collocation method using Gaussian quadrature. In [3] weighted Sobolev-like spaces on $[-1,1]$ are introduced and mapping properties of the operators A and K between this type of spaces are proved. Moreover, the convergence of collocation quadrature method for (1.3) is investigated and error estimates are given in weighted Sobolev-type spaces in the case when $|\alpha| = |\beta| = \frac{1}{2}$. Here, we shall revisit some theorems in [3] proving them for any α and β as in (1.2) by using error estimates of the Lagrange interpolation given in [11].

In this paper we describe a discrete collocation method and study its convergence and stability in some weighted Besov-type spaces.

The singular integral equation (1.3) is approximated by a discrete one. We will prove that the condition number of this finite-dimensional equation does not depend on the number of basis elements when it tends to infinity. However, this need not be true of the matrix of the linear system usually associated to it, as the numerical results given in Section 6 show. Previous estimates, as the referee recalled, with the gratitude of the author, show that the condition number of the matrix of this system is bounded by $C(\lambda_{nk,\max}/\lambda_{nk,\min})^{1/2}$, where C is a constant and λ_{nk}, for $k = 1,\ldots,n$, denotes the kth Christoffel number related to the weight function $w^{-\alpha,-\beta}$.

In order to determine the approximating polynomial solution of (1.3) we suggest to solve an equivalent system whose suitably preconditioned matrix has condition number uniformly bounded by the condition number of the original integral equation.

The system can be solved by using a suitable routine based on one of the known methods for the numerical solution of linear systems. For instance, we used the routine F04ARF from the NAG Fortran Workstation Library Routine Document.

The paper is so organized. In Section 2 we shall state some known preliminary results that can be used in different contexts. In Section 3 we shall describe the collocation method giving convergence and stability results in weighted Besov spaces. In Section 4 the main results about the condition numbers will be stated and in Section 5 we shall give all the proofs. Finally, we present some numerical experiments that confirm our theoretical expectations.

2. Preliminary results

Let $w^{\gamma,\delta}$ be a Jacobi weight function

$$w^{\gamma,\delta}(x) = (1-x)^{\gamma}(1+x)^{\delta}, \quad x \in (-1,1),\ \gamma,\delta > -1$$

and for $X \subset [-1,1]$, $L^2_{\gamma,\delta}(X)$ be the Banach space of all functions u such that

$$\|u\|_{L^2_{\gamma,\delta}(X)} := \left(\int_X |u(x)|^2 w^{\gamma,\delta}(x)\,\mathrm{d}x \right)^{1/2} < \infty.$$

If $X = [-1,1]$ then we use the following notations:

$$L^2_{\gamma,\delta} \equiv L^2_{\gamma,\delta}([-1,1]), \qquad \|u\|_{\gamma,\delta} \equiv \|u\|_{L^2_{\gamma,\delta}([-1,1])}.$$

Now let $\{p_n^{\gamma,\delta}\}_{n=0}^{\infty}$ be the sequence of all orthonormal polynomials with positive leading coefficients associated to the weight $w^{\gamma,\delta}$ such that $\deg p_n^{\gamma,\delta} = n$.

For a real number $s \geqslant 0$ we define the following weighted norm:

$$\|u\|_{\gamma,\delta,s} := \left(\sum_{n=0}^{\infty} (1+n)^{2s} |c_n^{\gamma,\delta}(u)|^2 \right)^{1/2}, \tag{2.1}$$

where

$$c_n^{\gamma,\delta}(u) = \int_1^1 u(x)\overline{p_n^{\gamma,\delta}(x)}\,w^{\gamma,\delta}(x)\,\mathrm{d}x$$

is the nth Fourier coefficient of u with respect to the system $\{p_n^{\gamma,\delta}\}_{n=0}^{\infty}$. For $s=0$ we have $\|u\|_{\gamma,\delta,s} = \|u\|_{\gamma,\delta}$. For $s > 0$ the norm defined by (2.1) is equivalent to other two norms which we are going to define below.

At first, let us introduce the kth weighted modulus of smoothness of a function $u \in L^2_{\gamma,\delta}$ [6]

$$\Omega^k_{\varphi}(u,t)_{\gamma,\delta} := \sup_{0<h\leqslant t} \|\Delta^k_{h\varphi} u\|_{L^2_{\gamma,\delta}(I_{hk})},$$

where $\varphi(x)=\sqrt{1-x^2}$, $\Delta_{h\varphi}^k u(x)=\sum_{i=0}^k (-1)^i \binom{k}{j} u(x+(\frac{k}{2}-i)h\varphi(x))$, $k\in\mathcal{N}$, $I_{hk}=[-1+4h^2k^2, 1-4h^2k^2]$ and the error of best approximation by algebraic polynomials of u in $L^2_{\gamma,\delta}$

$$E_n(u)_{\gamma,\delta} := \inf_{P\in\mathcal{P}_n} \|u-P\|_{\gamma,\delta},$$

where $\mathcal{P}_n$ is the set of all algebraic polynomials of degree at most n.

If $s\in\mathcal{N}$ the norm in (2.1) is equivalent to the following Sobolev norm:

$$\|u\|_{\gamma,\delta} + \|u^{(s)}\varphi^s\|_{\gamma,\delta}.$$

For $s\in\mathcal{R}$ and $s>0$, the two norms

$$\|u\|_{\gamma,\delta} + \left(\int_0^1 \left[\frac{\Omega_\varphi^k(u,t)_{\gamma,\delta}}{t^{s+1/2}}\right]^2 \mathrm{d}t\right)^{1/2}, \quad k>s, \tag{2.2}$$

$$\left(\sum_{n=0}^{\infty} [(1+n)^{s-1/2} E_n(u)_{\gamma,\delta}]^2\right)^{1/2} \tag{2.3}$$

are equivalent to the norm in (2.1) [11]. For a real number $s\geqslant 0$ we define the weighted Besov space

$$L^{2,s}_{\gamma,\delta} := \{u\in L^2_{\gamma,\delta}\colon \|u\|_{\gamma,\delta,s}<\infty\} \tag{2.4}$$

endowed with norm (2.1).

It is known (see [3]) that, if $u\in L^{2,s}_{\gamma,\delta}$ with $s>\frac{1}{2}$, in the equivalent class of $L^{2,s}_{\gamma,\delta}$ containing the function u there exists a representative which is locally continuous on $[-1,1]$.

Define, now, the Lagrange fundamental polynomials $l_{nk}^{\gamma,\delta}$, $k=1,\dots,n$ and the Lagrange interpolation operator $L_n^{\gamma,\delta}$ with respect to the zeros $x_{nk}^{\gamma,\delta}$, $k=1,\dots,n$ of $p_n^{\gamma,\delta}$ by

$$l_{nk}^{\gamma,\delta}(x) = \prod_{j=1, j\neq k}^{n} \frac{x-x_{nj}^{\gamma,\delta}}{x_{nk}^{\gamma,\delta}-x_{nj}^{\gamma,\delta}}$$

and

$$L_n^{\gamma,\delta}u = \sum_{k=1}^{n} u(x_{nk}^{\gamma,\delta}) l_{nk}^{\gamma,\delta},$$

respectively. Observe that the Lagrange operator can be defined on $L^{2,s}_{\gamma,\delta}$ if $s>\frac{1}{2}$, according to the previous remark.

In the sequel, the symbol C will stand for some positive constant which can assume different values at different places. From Theorems 3.1 and 3.4 given in [11] the next lemma follows. It gives us estimates of the Lagrange interpolation error in the Besov spaces defined by (2.4).

Lemma 2.1. *For $s>\frac{1}{2}$ and $u\in \boldsymbol{L}^{2,s}_{\gamma,\delta}$ we have*

$$\|u-L_n^{\gamma,\delta}u\|_{\gamma,\delta} \leqslant \frac{C}{\sqrt{n}} \int_0^{1/n} \frac{\Omega_\varphi^k(u,y)_{\gamma,\delta}}{y^{1+1/2}}\,\mathrm{d}y \leqslant Cn^{-s}\|u\|_{\gamma,\delta,s} \tag{2.5}$$

and

$$\|u - L_n^{\gamma,\delta}u\|_{\gamma,\delta,t} \leqslant Cn^{t-s}\left(\int_0^{1/n}\left[\frac{\Omega_\varphi^k(u,y)_{\gamma,\delta}}{y^s+1/2}\right]^2 \mathrm{d}y\right)^{1/2} \leqslant Cn^{t-s}\|u\|_{\gamma,\delta,s} \tag{2.6}$$

if $0 \leqslant t \leqslant s$, *with* C *constant independent of* n *and* u.

At this point it is useful to summarize some known properties of the operators A and K associated with the integral equation.

Here, we shall only consider the case in which $\kappa = 0$, i.e. $\beta = -\alpha$, but analogous results for $\kappa \in \{-1, 1\}$ can be found in [3].

For two Banach spaces E and F, we will denote by $\mathscr{L}(E,F)$ the Banach space of all bounded linear operators between E and F and by $\Phi(E,F)$ the closed subspace of $\mathscr{L}(E,F)$ of all Fredholm operators. For $T \in \Phi(E,F)$ let $\kappa(T)$ denote the index of the operator T.

The norm in $\mathscr{L}(E,F)$ is denoted by $\|\cdot\|_{\star\to\circ}$ if $\|\cdot\|_\star$ and $\|\cdot\|_\circ$ denote the norms in E and F, respectively.

The singular integral operator A defined by (1.4) and its adjoint $\hat{A}$ satisfy the following properties (see [3,9,12,14]).

Theorem 2.2. *The singular integral operator* A *defined by* (1.4) *belongs to* $\Phi(L^2_{\alpha,-\alpha}, L^2_{-\alpha,\alpha})$ *with* $\kappa(A) = 0$. *Moreover, its adjoint operator* $\hat{A}$ *belongs to* $\mathscr{L}(L^2_{-\alpha,\alpha}, \boldsymbol{L}^2_{\alpha,-\alpha})$ *and is given by*

$$(\hat{A}f)(x) = aw^{-\alpha,\alpha}(x)f(x) - \frac{b}{\pi}\int_{-1}^{1}\frac{w^{-\alpha,\alpha}(y)f(y)}{y-x}\,\mathrm{d}y. \tag{2.7}$$

A is invertible and $\hat{A}$ is its inverse operator. Moreover, the relations

$$\begin{aligned} A p_n^{\alpha,-\alpha}(x) &= (-1)^M p_n^{-\alpha,\alpha}(x), \\ \hat{A} p_n^{-\alpha,\alpha}(x) &= (-1)^M p_n^{\alpha,-\alpha}(x) \end{aligned} \tag{2.8}$$

hold for all $x \in (-1, 1)$.

The following result gives us a generalization of the mapping properties of the operator A in some pairs of weighted Besov spaces.

Theorem 2.3. *For all* $s \geqslant 0$, *the operator* A *belongs to* $\mathscr{L}(L^{2,s}_{\alpha,-\alpha}, L^{2,s}_{-\alpha,\alpha})$ *and the adjoint operator* $\hat{A}$ *belongs to* $\mathscr{L}(L^{2,s}_{-\alpha,\alpha}, L^{2,s}_{\alpha,-\alpha})$. *Moreover, A is a bijection and the inverse operator is* $\hat{A}$.

Now let us study the properties of the integral operator K defined by (1.5). It is known that if the kernel $k(x,y)$ satisfies the following condition:

$$\int_{-1}^{1}\int_{-1}^{1}|k(x,y)|^2 w^{\alpha,-\alpha}(y)\,\mathrm{d}y\,w^{-\alpha,\alpha}(x)\,\mathrm{d}x < \infty,$$

the operator K is a linear compact operator between the spaces $\boldsymbol{L}^2_{\alpha,-\alpha}$ and $\boldsymbol{L}^2_{-\alpha,\alpha}$. To investigate the mapping properties of K in spaces of type (2.4), we suppose that the following smoothness conditions on the kernel $k(x,y)$ are fulfilled.

Let

$$\begin{aligned} &k(\cdot,y)\in L^{2,s}_{-\alpha,\alpha}, \text{ uniformly with respect to } y\in[-1,1],\\ &k(x,\cdot)\in L^{2,r}_{\alpha,-\alpha}, \text{ uniformly with respect to } x\in[-1,1] \end{aligned} \tag{2.9}$$

with some positive real numbers s and r which will be specified later. The term "uniformly" means that

$$\|k(\cdot,y)\|_{-\alpha,\alpha,s}\leqslant C_1 \quad \text{and} \quad \|k(x,\cdot)\|_{\alpha,-\alpha,r}\leqslant C_2$$

for all $x,y\in[-1,1]$, where C_1 and C_2 are suitable positive constants independent of both x and y. We have [3]:

Theorem 2.4. *Under hypotheses* (2.9) *the integral operator K belongs to $\mathscr{L}(L^2_{\alpha,-\alpha},L^{2,t}_{-\alpha,\alpha})$ for all $t\leqslant s$. Moreover, it is compact for all $t<s$.*

As a consequence of the previous results, we can deduce the following one concerning the solvability of the equation $(A+K)u=f$.

Corollary 2.5. *Under hypotheses* (2.9), *assume that* $\ker(A+K)=\{0\}$ *in $L^2_{\alpha,-\alpha}$. Then $A+K : L^{2,t}_{\alpha,-\alpha}\to L^{2,t}_{-\alpha,\alpha}$ is an invertible linear bounded operator for all t, $0\leqslant t<s$.*

3. A discrete collocation method

In the present section we shall describe a discrete collocation method to solve the singular integral equation (1.3) in the case $\kappa=0$, i.e. $\beta=-\alpha$ [5].

Assuming that the kernel $k(x,y)$ satisfies conditions (2.9), we shall give some results concerning the convergence and stability of the method in weighted Besov-type spaces. This method involves in approximating the unknown solution u by a polynomial of degree $n-1$ of the kind

$$u_n(x)=\sum_{j=0}^{n-1} a_j\, p_j^{\alpha,-\alpha}(x), \tag{3.1}$$

where a_j, $j=0,1,\ldots,n-1$ are unknown constants.

In order to evaluate the coefficients a_j, $j=0,1,\ldots,n-1$, we require that u_n is the solution of (1.3) at the points of a given matrix

$$T=\{x_{nk}\colon k=1,\ldots,n, \quad n=1,2,\ldots\}$$

which we call *collocation knots*, i.e.

$$Au_n(x_{nk})+Ku_n(x_{nk})=f(x_{nk}), \quad k=1,\ldots,n. \tag{3.2}$$

Eq. (3.2) can be rewritten, according to (3.1) and (2.8), as follows:

$$\sum_{j=0}^{n-1} a_j[(-1)^M p_j^{-\alpha,\alpha}(x_{nk})+Kp_j^{\alpha,-\alpha}(x_{nk})]=f(x_{nk}), \quad k=1,\ldots,n. \tag{3.3}$$

Eq. (3.3) represents a linear system of n equations in the n unknowns $a_j, j=0,1,\ldots,n-1$.

Let us observe that in order to compute the coefficients of the unknowns a_j, we have to evaluate the integrals $Kp_i^{\alpha,-\alpha}(x_{nk})$, $j=0,1,\dots,n-1$, $k=1,\dots,n$, but "a priori" we do not know if they can be exactly evaluated. Therefore, to avoid this problem, we can apply a suitable quadrature formula on N knots to compute them, replacing, in such way, the integral operator K by a discrete one K_N. Then, system (3.2) becomes

$$Au_n(x_{nk}) + K_N u_n(x_{nk}) = f(x_{nk}), \quad k = 1,\dots,n \tag{3.4}$$

and the method is called *discrete*. In the sequel, we shall consider a particular discrete collocation method and study its convergence and stability properties.

At first, recall that the Gaussian rule of order n with respect to a general Jacobi weight function $w^{\gamma,\delta}$ is given by

$$\int_{-1}^{1} u(y)w^{\gamma,\delta}(y)\,\mathrm{d}y \approx \sum_{j=1}^{n} u(t_{nj}^{\gamma,\delta})\lambda_{nj}^{\gamma,\delta}, \tag{3.5}$$

where $t_{nj}^{\gamma,\delta}$, $j=1,\dots,n$ are the zeros of $p_n^{\gamma,\delta}$, and $\lambda_{nj}^{\gamma,\delta}$, $j=1,\dots,n$ are the Christoffel numbers defined by

$$\lambda_{nj}^{\gamma,\delta} = \int_{-1}^{1} l_{nj}^{\gamma,\delta}(y)w^{\gamma,\delta}(y)\,\mathrm{d}y. \tag{3.6}$$

Remark that this quadrature formula is exact for all polynomials u of degree less than $2n$.

Now introduce some notations. By $y_{nj} \equiv y_{nj}^{\alpha,-\alpha}$, $j=1,\dots,n$, denote the zeros of the polynomial $p_n^{\alpha,-\alpha}$ and by $x_{nk} \equiv x_{nk}^{-\alpha,\alpha}$, $k=1,\dots,n$ denote the zeros of $p_n^{-\alpha,\alpha}$ with α defined in (1.2).

Let us consider the approximating equation

$$Au_n + Ku_n = f \tag{3.7}$$

and apply the quadrature formula (3.5) to evaluate $Ku_n(x)$. In this way, we obtain the discrete approximating equation

$$Au_n + K_n u_n = f, \tag{3.8}$$

where we have defined the operator K_n as follows:

$$K_n u(x) := \int_{-1}^{1} [L_{n,y}^{\alpha,-\alpha} k(x,y)]u(y)w^{\alpha,-\alpha}(y)\,\mathrm{d}y.$$

The subscript "y" of $L_{n,y}^{\alpha,-\alpha}$ means that the interpolation is done with respect to the variable y.

Now choosing the knots x_{nk}, $k=1,\dots,n$ as collocation points for the approximating equation (3.8) we get the following system:

$$\sum_{j=0}^{n-1} a_j \left[-\frac{b}{\sin \pi\alpha}\, p_j^{-\alpha,\alpha}(x_{nk}) + \sum_{i=1}^{n} k(x_{nk}, y_{ni})p_j^{\alpha,-\alpha}(y_{ni})\lambda_{ni}^{\alpha,-\alpha} \right] = f(x_{nk}) \tag{3.9}$$

with $k=1,\dots,n$.

In place of the previous system, to compute the unknown constants a_j, $j=0,1,\dots,n-1$ we solve the following equivalent one:

$$\sqrt{\lambda_{nk}^{-\alpha,\alpha}}(A+K_n)u_n(x_{nk}) = \sqrt{\lambda_{nk}^{-\alpha,\alpha}}f(x_{nk}), \quad k=1,\dots,n. \tag{3.10}$$

Let us observe that (3.10) can be obtained by multiplying from the left both the matrix of the coefficients and the vector of the right-hand side of Eq. (3.9) by the nonsingular diagonal matrix

$$\operatorname{diag}(\sqrt{\lambda_{n1}^{-\alpha,\alpha}},\dots,\sqrt{\lambda_{nn}^{-\alpha,\alpha}}).$$

At this moment we only say that the reason of this choice is related to the condition number of the matrix of the system which we have to solve (see Theorem 4.2).

By considering (3.9) let us proceed to study the consistence of our discrete method. If we denote by $L_n^{-\alpha,\alpha}$ the Lagrange interpolation operator with respect to the nodes x_{nk}, $k=1,\dots,n$, system (3.9) results equivalent to the following operator equation:

$$L_n^{-\alpha,\alpha}(A+K_n)u_n = L_n^{-\alpha,\alpha}f. \tag{3.11}$$

Since $L_n^{-\alpha,\alpha}$ is a projection on $\mathscr{P}_{n-1}$, (3.11) can be rewritten in the form

$$Au_n + L_n^{-\alpha,\alpha}K_nu_n = L_n^{-\alpha,\alpha}f \tag{3.12}$$

also according to (2.8). Remark that (3.12) only makes sense if $s>\frac{1}{2}$ and $r>\frac{1}{2}$. Furthermore, let us observe that the described method coincides with a discrete projection method. The discrete approximating equation (3.12) can be obtained by projecting (3.7) on $\mathscr{P}_{n-1}$ by means of the interpolating operator $L_n^{-\alpha,\alpha}$.

In order to prove the convergence of (3.12) we must estimate the following operator norm: $\|K-L_m^{-\alpha,\alpha}K_n\|_{\alpha,-\alpha\to-\alpha,\alpha,t}$.

Remark. The following lemma and theorem are generalizations of Lemma 4.4 and Theorem 4.6 in [3], respectively, that hold with $|\alpha|=\frac{1}{2}$ and $r\geqslant s+\frac{1}{2}$. We remark that analogous results holds for $\kappa\in\{-1,1\}$.

Lemma 3.1. *Under hypotheses* (2.9), *let* $u\in L^2_{\alpha,-\alpha}$, $s>\frac{1}{2}$, $r\geqslant s$. *Then for all* t, $0\leqslant t\leqslant s$ the estimate

$$\|(K-L_n^{-\alpha,\alpha}K_n)u\|_{-\alpha,\alpha,t}\leqslant Cn^{t-s}\|u\|_{\alpha,-\alpha} \tag{3.13}$$

holds with C constant independent of n and u.

As a consequence of the previous lemma we deduce:

Theorem 3.2. *Assume the homogeneous equation* $(A+K)u=0$ *has only the trivial solution in* $L^2_{\alpha,-\alpha}$. *Further, suppose that* $k(x,y)$ *satisfies* (2.9) *with* $s>\frac{1}{2}$, $r\geqslant s$ *and* $f\in L^{2,s}_{-\alpha,\alpha}$. *Then for all sufficiently large n Eq.* (3.12) *has a unique polynomial solution* $u_n\in\mathscr{P}_{n-1}$. *If u and* u_n *denote the uniquely determined solution of* (1.3) *and* (3.13), *respectively, the following error estimate,*

$$\|u-u_n\|_{\alpha,-\alpha,t}\leqslant Cn^{t-s}\|u\|_{\alpha,-\alpha,s}, \tag{3.14}$$

holds for all t, $0\leqslant t<s$, *with C constant independent of n and u.*

4. Condition numbers

In Section 3 we described a discrete method for the numerical solution of Cauchy singular integral equations. The method, we saw, consists in solving the finite-dimensional equations (3.12) or the equivalent linear system of equations (3.10). In this section we look at the relation between the condition numbers of these approximating problems and the condition number of the original integral equation (1.3).

Recall that if T is an invertible operator between normed spaces the *condition number* of T is defined as

$$\mathrm{cond}(T) := \|T\| \cdot \|T^{-1}\|.$$

To understand the meaning of this definition let us consider a general operator equation

$$Au = f$$

and a second one obtained from it by introducing a perturbation δ in the data f:

$$Au_\delta = f + \delta.$$

It is a standard argument to show

$$\frac{\|u - u_\delta\|}{\|u\|} \leqslant \mathrm{cond}(A) \cdot \frac{\|\delta\|}{\|f\|}. \tag{4.1}$$

From (4.1) it results that if the condition number takes small values, then small perturbations δ in the data lead to small perturbations in the solution. In this case we say that the problem is *well conditioned*. If cond(A) increases the solution u of the equation is increasingly sensitive to small changes δ in the data f (see, for instance, [4]).

For the discrete numerical method we have studied before, we can prove the convergence of the condition number of the approximating equation (3.12) to that of the original one (1.3), as $n \to \infty$. We shall suppose the operators mapping between the spaces $L^2_{\alpha,-\alpha}$ and $L^2_{-\alpha,\alpha}$ (in Theorem 3.2 we take $t = 0$). Remark that, for simplicity of notations, in the sequel sometimes, we shall omit to write the subscripts $\star \to \circ$ in the symbol $\|\cdot\|_{\star\to\circ}$ introduced to denote the norm of an operator acting from two normed space E and F equipped with the norm $\|\cdot\|_\star$ and $\|\cdot\|_\circ$, respectively.

For the condition number of the approximating equation (3.12) the following result holds.

Theorem 4.1. *Under the hypotheses of Theorem* 3.2, *one has*

$$\mathrm{cond}(A + L_n^{-\alpha,\alpha} K_n) \to \mathrm{cond}(A + K) \quad \text{as } n \to \infty \tag{4.2}$$

and consequently, *for sufficiently large* n,

$$\mathrm{cond}(A + L_n^{-\alpha,\alpha} K_n) \leqslant C\,\mathrm{cond}\,(A + K) \tag{4.3}$$

with C *constant independent of* n.

Observe that the previous relations also hold if we consider the operators acting between more general Besov-type spaces as in (2.4).

Condition (4.3) means that if the integral equation

$$(A+K)u=f$$

is well conditioned, the same happens for its approximating one

$$(A+L_n^{-\alpha,\alpha}K_n)u_n=L_n^{-\alpha,\alpha}f.$$

We shall see that, in this case, also the linear system (3.10) is well conditioned.

At first rewrite (3.10) in matrix form as follows:

$$(A_n+B_n)\boldsymbol{a}=\boldsymbol{f}_n, \tag{4.4}$$

where

$$A_n := \left(-\sqrt{\lambda_{nk}^{-\alpha,\alpha}}\frac{b}{\sin\pi\alpha}p_j^{-\alpha,\alpha}(x_{nk})\right)_{k=1,j=0}^{n,n-1},$$

$$B_n := \left(\sqrt{\lambda_{nk}^{-\alpha,\alpha}}\sum_{i=1}^{n}k(x_{nk},y_{ni})p_j^{\alpha,-\alpha}(y_{ni})\lambda_{ni}^{\alpha,-\alpha}\right)_{k=1,j=0}^{n,n-1},$$

$$\boldsymbol{a} := (a_0,a_1,\ldots,a_{n-1})^{\mathrm{T}},$$

$$\boldsymbol{f}_n := [(\sqrt{\lambda_{nk}^{-\alpha,\alpha}}f(x_{nk}))_{k=1}^{n}]^{\mathrm{T}}.$$

A_n and B_n are linear mappings from $\mathscr{C}^n$ into $\mathscr{C}^n$. We shall consider $\mathscr{C}^n$ endowed with the euclidean norm

$$\|\boldsymbol{v}\|_2=\left(\sum_{i=1}^{n}|v_i|^2\right)^{1/2}$$

for $\boldsymbol{v}=(v_1,\ldots,v_n)^{\mathrm{T}}\in\mathscr{C}^n$, and the space of all $n\times n$ matrices with complex coefficients equipped with the norm induced by $\|\cdot\|_2$ on $\mathscr{C}^n$, i.e. with the spectral norm

$$\|T_n\|_2=\sup_{\|\boldsymbol{v}\|_2=1}\|T_n\boldsymbol{v}\|_2.$$

We have the following main result.

Theorem 4.2. *Under the hypotheses of Theorem* 3.2, *for all sufficiently large n the matrix* A_n+B_n *is regular and*

$$\operatorname{cond}(A_n+B_n)\leqslant C\operatorname{cond}(A+K), \tag{4.5}$$

where C is a constant independent of n.

Relation (4.5) shows that the matrix of the linear system (3.10) will be well conditioned if the integral equation (1.3) is also well conditioned. For this it is important that the matrix of the system (3.9) is multiplied by the diagonal matrix

$$\operatorname{diag}(\sqrt{\lambda_{n1}^{-\alpha,\alpha}},\ldots,\sqrt{\lambda_{nn}^{-\alpha,\alpha}}).$$

5. Proofs

Proof of Lemma 3.1. At first observe that it holds

$$(K - L_n^{-\alpha,\alpha}K_n)u = (K - L_n^{-\alpha,\alpha}K)u + L_n^{-\alpha,\alpha}(K - K_n)u.$$

Thus,

$$\|(K - L_n^{-\alpha,\alpha}K_n)u\|_{-\alpha,\alpha,t} \leqslant \|(K - L_n^{-\alpha,\alpha}K)u\|_{-\alpha,\alpha,t} + \|L_n^{-\alpha,\alpha}(K - K_n)u\|_{-\alpha,\alpha,t}.$$

By (2.6), applied to Ku, one has

$$\begin{aligned}\|(K - L_n^{-\alpha,\alpha}K)u\|_{-\alpha,\alpha,t} &\leqslant \text{const} \cdot n^{t-s}\|Ku\|_{-\alpha,\alpha,s} \\ &\leqslant \text{const} \cdot n^{t-s}\|u\|_{\alpha,-\alpha}.\end{aligned}$$

The last inequality follows from the boundedness of the operator K.

Consider, now, $L_n^{-\alpha,\alpha}(K - K_n)u$, which is a polynomial of degree $n-1$, and estimate $\|L_n^{-\alpha,\alpha}(K - K_n)u\|_{-\alpha,\alpha,t}$.

At first we have, for all $x \in [-1, 1]$,

$$\begin{aligned}|(K - K_n)u(x)| &= \left|\int_{-1}^{1} [k(x,y) - L_{n,y}^{\alpha,-\alpha}k(x,y)]u(y)w^{\alpha,-\alpha}(y)\,\mathrm{d}y\right| \\ &\leqslant \left(\int_{-1}^{1} [k(x,y) - L_{n,y}^{\alpha,-\alpha}k(x,y)]^2 w^{\alpha,-\alpha}(y)\,\mathrm{d}y\right)^{1/2} \left(\int_{-1}^{1} |u(y)|^2 w^{\alpha,-\alpha}(y)\,\mathrm{d}y\right)^{1/2} \\ &= \|u\|_{\alpha,-\alpha}\|k(x,\cdot) - L_{n,y}^{\alpha,-\alpha}k(x,\cdot)\|_{\alpha,-\alpha}\end{aligned}$$

by applying the Schwarz inequality.

From hypotheses (2.9) on the kernel and the estimates of Lagrange interpolation given in Lemma 2.1, it follows

$$|(K - K_n)u(x)| \leqslant Cn^{-r}\|u\|_{\alpha,-\alpha}. \tag{5.1}$$

Here and in the sequel C stands for a constant independent of n and u. Moreover, using the Gaussian formula, one has

$$\begin{aligned}\|L_n^{-\alpha,\alpha}(K - K_n)u\|_{-\alpha,\alpha} &= \left(\int_{-1}^{1} \left|L_n^{-\alpha,\alpha}(K - K_n)u(x)\right|^2 w^{-\alpha,\alpha}(x)\,\mathrm{d}x\right)^{1/2} \\ &= \left(\sum_{k=1}^{n} \lambda_{nk}^{-\alpha,\alpha}[(K - K_n)u(x_{nk})]^2\right)^{1/2}.\end{aligned}$$

By (5.1) we get

$$\|L_n^{-\alpha,\alpha}(K - K_n)u\|_{-\alpha,\alpha} \leqslant Cn^{-r}\|u\|_{\alpha,-\alpha}. \tag{5.2}$$

Thus, since $r \geqslant s$, the thesis is proved when $t = 0$.

For $t>0$, by the equivalence of the norms defined in (2.1) and (2.3), we shall estimate

$$\left(\sum_{i=0}^{n-1}[(1+i)^{t-1/2}E_i(L_n^{-\alpha,\alpha}(K-K_n)u)_{-\alpha,\alpha}]^2\right)^{1/2}.$$

Since

$$E_i(L_n^{-\alpha,\alpha}(K-K_n)u)_{-\alpha,\alpha}\leqslant\|L_n^{-\alpha,\alpha}(K-K_n)u\|_{-\alpha,\alpha},$$

one has, by (5.2),

$$\left(\sum_{i=0}^{n-1}[(1+i)^{t-1/2}E_i(L_n^{-\alpha,\alpha}(K-K_n)u)_{-\alpha,\alpha}]^2\right)^{1/2}$$

$$\leqslant Cn^{-r}\|u\|_{\alpha,-\alpha}\left(\sum_{i=0}^{n-1}(1+i)^{2t-1}\right)^{1/2}\leqslant Cn^{t-r}\|u\|_{\alpha,-\alpha}.$$

Then we conclude

$$\|L_n^{-\alpha,\alpha}(K-K_n)u\|_{-\alpha,\alpha,t}\leqslant Cn^{t-r}\|u\|_{\alpha,-\alpha}\leqslant Cn^{t-s}\|u\|_{\alpha,-\alpha}$$

from which we finally deduce (3.13). The interested reader can also see the proof of Lemma 3.10 in [8]. □

Proof of Theorem 3.2. From Corollary 2.5 we derive the existence of $(A+K)^{-1}$ as an element of $\mathscr{L}(L^{2,t}_{-\alpha,\alpha},L^{2,t}_{\alpha,-\alpha})$, for all t, $0\leqslant t<s$. We want to show that for sufficiently large n the operators $A+L_n^{-\alpha,\alpha}K_n$, are invertible in $\mathscr{L}(L^{2,t}_{\alpha,-\alpha},L^{2,t}_{-\alpha,\alpha})$, with $0\leqslant t<s$. Applying (3.13) we have

$$\|(A+K)-(A+L_n^{-\alpha,\alpha}K_n)\|_{\alpha,-\alpha,t\to-\alpha,\alpha,t}\leqslant\|K-L_n^{-\alpha,\alpha}K_n\|_{\alpha,-\alpha,t\to-\alpha,\alpha,t}\leqslant Cn^{t-s}\tag{5.3}$$

which tends to zero as $n\to\infty$.

For a well-known result (see, for instance, [1]) we can deduce that for sufficiently large n the inverses $(A+L_n^{-\alpha,\alpha}K_n)^{-1}$ exist and are uniformly bounded with respect to n in the norm of $\mathscr{L}(L^{2,t}_{-\alpha,\alpha},L^{2,t}_{\alpha,-\alpha})$. More precisely, one has

$$\|(A+L_n^{-\alpha,\alpha}K_n)^{-1}\|\leqslant\frac{\|(A+K)^{-1}\|}{1-\|(A+K)^{-1}\|\cdot\|K-L_n^{-\alpha,\alpha}K_n\|},\tag{5.4}$$

which assures the stability of the method. Then (3.12) is uniquely solvable. Moreover its solution is a polynomial $u_n\in\mathscr{P}_{n-1}$. In fact, from (3.12) we derive

$$Au_n=L_n^{-\alpha,\alpha}(f-K_nu_n)$$

which, by multiplying by $\hat{A}$ on the left-hand side, becomes

$$u_n=\hat{A}[L_n^{-\alpha,\alpha}(f-K_nu_n)].\tag{5.5}$$

Since (2.8) holds, u_n belongs to $\mathscr{P}_{n-1}$.

To obtain the error estimates (3.14) observe

$$u_n-u=(A+L_n^{-\alpha,\alpha}K_n)^{-1}[L_n^{-\alpha,\alpha}f-(A+L_n^{-\alpha,\alpha}K_n)u].$$

Then, adding and subtracting f in the square brackets, from (2.6) and (3.13) we deduce

$$\begin{aligned}\|u-u_n\|_{\alpha,-\alpha,t} &\leqslant C[\|f-L_n^{-\alpha,\alpha}f\|_{-\alpha,\alpha,t}+\|(K-L_n^{-\alpha,\alpha}K_n)u\|_{-\alpha,\alpha,t}]\\ &\leqslant C(n^{t-s}\|f\|_{\alpha,-\alpha,s}+n^{t-s}\|u\|_{\alpha,-\alpha})\\ &\leqslant Cn^{t-s}\|u\|_{\alpha,-\alpha,s}\end{aligned}$$

for all t, $0\leqslant t<s$. To write the last inequality we have considered that, since $f=(A+K)^{-1}u$ it holds

$$\|f\|_{-\alpha,\alpha,s}\leqslant C\|u\|_{\alpha,-\alpha,s}. \qquad \square$$

Proof of Theorem 4.1. From (5.3) it follows

$$A+L_n^{-\alpha,\alpha}K_n \to A+K \quad \text{as } n\to\infty,$$

in the operator norm, from which

$$\|A+L_n^{-\alpha,\alpha}K_n\| \to \|A+K\| \quad \text{as } n\to\infty$$

by the continuity of the norm. Furthermore, this implies

$$\|(A+L_n^{-\alpha,\alpha}K_n)^{-1}\| \to \|(A+K)^{-1}\|$$

and (4.2) follows.

Relation (4.3) follows immediately from (4.2), since $\text{cond}(A+K)\geqslant 1$ and

$$\gamma_n := \frac{\text{cond}(A+L_n^{-\alpha,\alpha}K_n)}{\text{cond}(A+K)} \to 1,$$

which implies $\gamma_n\leqslant C$. $\square$

Before proving the main result of this paper, let us make some useful remarks.

Remark. Let $w^{\gamma,\delta}$ be a Jacobi weight function.

1. If $u_n\in\mathscr{P}_{n-1}$ and

$$u_n(x)=\sum_{j=0}^{n-1}a_j p_j^{\gamma,\delta}(x), \quad x\in[-1,1],$$

one has

$$\|u_n\|_{\gamma,\delta}^2=\sum_{j=0}^{n-1}|a_j|^2.$$

Hence it holds

$$\|u_n\|_{\gamma,\delta}=\|\boldsymbol{a}\|_2, \tag{5.6}$$

where $\boldsymbol{a}=(a_0,a_1,\ldots,a_{n-1})^{\mathrm{T}}$.

2. For a function $f \in \boldsymbol{L}^{2,s}_{\gamma,\delta}$, for which $L_n^{\gamma,\delta} f$ is defined, we have

$$\|L_n^{\gamma,\delta} f\|^2_{\gamma,\delta} = \int_{-1}^{1} |L_n^{\gamma,\delta} f(x)|^2 w^{\gamma,\delta}(x)\,\mathrm{d}x$$

$$= \sum_{k=1}^{n} |f(x_{nk}^{\gamma,\delta})|^2 \lambda_{nk}^{\gamma,\delta},$$

i.e.,

$$\|L_n^{\gamma,\delta} f\|_{\gamma,\delta} = \|\boldsymbol{f}_n\|_2, \tag{5.7}$$

where $\boldsymbol{f}_n = [(\sqrt{\lambda_{nk}^{-\alpha,\alpha}} f(x_{nk}))_{k=1}^n]^{\mathrm{T}}$.

Proof of Theorem 4.2. The nonsingularity of the matrix $A_n + B_n$ follows from the Theorem 3.2 and the equivalence between (3.12) and (3.10).

To estimate $\mathrm{cond}(A_n + B_n)$ we have to estimate both the norm $\|A_n + B_n\|_2$ and $\|(A_n + B_n)^{-1}\|_2$.

Let $\boldsymbol{b} \in \mathscr{C}^n$ be arbitrary with $\boldsymbol{b} = (b_0, b_1, \ldots, b_{n-1})^{\mathrm{T}}$ and $\boldsymbol{g} = (A_n + B_n)\boldsymbol{b}$.

Define

$$u_n(x) = \sum_{j=0}^{n-1} b_j p_j^{\alpha,-\alpha}(x).$$

Now, let f be such that

$$f = (A + L_n^{-\alpha,\alpha} K_n) u_n.$$

Thus f is a polynomial belonging to $\mathscr{P}_{n-1}$ and

$$\boldsymbol{g} = [(\sqrt{\lambda_{nk}^{-\alpha,\alpha}} f(x_{nk}))_{k=1}^n]^{\mathrm{T}}.$$

Taking into account (5.6) and (5.7) one has

$$\|\boldsymbol{g}\|_2 = \|f\|_{-\alpha,\alpha}$$

$$\leqslant \|A + L_n^{-\alpha,\alpha} K_n\|_{\alpha,-\alpha \to -\alpha,\alpha} \|u_n\|_{\alpha,-\alpha}$$

$$= \|A + L_n^{-\alpha,\alpha} K_n\|_{\alpha,-\alpha \to -\alpha,\alpha} \|\boldsymbol{b}\|_2,$$

hence according to the definition of the operator norm,

$$\|A_n + B_n\|_2 \leqslant \|A + L_n^{-\alpha,\alpha} K_n\|_{\alpha,-\alpha \to -\alpha,\alpha}. \tag{5.8}$$

Now, consider the solution of the system

$$(A_n + B_n)\boldsymbol{b} = \boldsymbol{g} \tag{5.9}$$

for arbitrary $\boldsymbol{g} \in \mathscr{C}^n$, with $\boldsymbol{g} = [(\sqrt{\lambda_{nk}^{-\alpha,\alpha}} g_k)_{k=1}^n]^{\mathrm{T}}$.

Choose $f \in \boldsymbol{L}^2_{-\alpha,\alpha}$ such that

$$\|L_n^{-\alpha,\alpha} f\|_{-\alpha,\alpha} = \|\boldsymbol{g}\|_2.$$

To this end, according to (5.7) it is sufficient to take a function f which satisfies $f(x_{nk}) = g_k$, $k = 1,\dots,n$. For this function f, let

$$u_n = (A + L_n^{-\alpha,\alpha} K_n)^{-1}(L_n^{-\alpha,\alpha} f),$$

that is a polynomial, as it was shown in the proof of Theorem 3.2. If we expand u_n in Fourier series in the space $\boldsymbol{L}^2_{\alpha,-\alpha}$ we obtain

$$u_n(x) = \sum_{j=0}^{n-1} b_j p_j^{\alpha,-\alpha}(x),$$

where b_j, $j = 0, 1, \dots, n-1$ are the components of the vector $\boldsymbol{b}$ defined by (5.9). Thus, one has

$$\|u_n\|_{\alpha,-\alpha} \leqslant \|(A + L_n^{-\alpha,\alpha} K_n)^{-1}\|_{-\alpha,\alpha\to\alpha,-\alpha} \|L_n^{-\alpha,\alpha} f\|_{-\alpha,\alpha}$$
$$= \|(A + L_n^{-\alpha,\alpha} K_n)^{-1}\|_{-\alpha,\alpha\to\alpha,-\alpha} \|\boldsymbol{g}\|_2.$$

From (5.6) it follows

$$\|\boldsymbol{b}\|_2 \leqslant \|(A + L_n^{-\alpha,\alpha} K_n)^{-1}\|_{-\alpha,\alpha\to\alpha,-\alpha} \|\boldsymbol{g}\|_2$$

and then

$$\|(A_n + B_n)^{-1}\|_2 \leqslant \|(A + L_n^{-\alpha,\alpha} K_n)^{-1}\|_{-\alpha,\alpha\to\alpha,-\alpha}. \tag{5.10}$$

Comparing (5.8) and (5.10) with (4.3) we obtain the thesis. □

6. Numerical results

In this section we shall give some results obtained by applying the numerical method described in Section 3 at some test equations of type (1.1). In all the following examples the analytical solution of (1.1) are unknown. Then, Tables 1–3 show the values of the approximate solution $u_n(x)$ in some equispaced points, for increasing values of $n \in \mathcal{N}$. Moreover Table 4 contains the evaluations of the condition number both of the matrix of the linear system (3.9), which we denote by M_n, and of the matrix of the preconditioned one (3.10), which we denote by PM_n, for increasing values of $n \in \mathcal{N}$. The reported numerical results agree with our theoretical expectations. In order to solve the linear system (4.4) we used the routine F04ARF from the NAG Fortran Workstation Library Routine Document.

Example 1. Let $k(x, y) = \frac{1}{2}|x - y|^{9/2}$, $f(x) = e^x$, $\alpha = \frac{1}{4}$.

Table 4 shows that the matrix of system (3.10) is better conditioned than the other one of (3.9). Moreover, the sequence $\{\mathrm{cond}(M_n)\}$ increases according to the fact that $\mathrm{cond}(M_n)$ is bounded by $C(\lambda_{nk,\max}^{-\alpha,\alpha}/\lambda_{nk,\min}^{-\alpha,\alpha})^{1/2} \leqslant Cm^{3/4}$. The other one $\{\mathrm{cond}(PM_n)\}$ is convergent as n goes to infinity.

Example 2. Let $k(x, y) = \sin xy$, $f(x) = \cos x$, $\alpha = \frac{1}{4}$.

Tables 5–7 show the values of u_n in the indicated points for n not greater than 16, since this number of collocation knots is sufficient to obtain values affected by an error of the order of the

Table 1

n	$x=-1$	$x=-0.75$	$x=-0.5$
4	1.0	0.3	−0.3
8	1.581	0.3224	−0.3694
16	1.5814033	0.3224690	−0.3694414
32	1.581403368	0.322469088	−0.369441421
64	1.5814033681	0.32246908888	−0.3694414210
128	1.5814033681383	0.3224690888856	−0.369441421035
256	1.58140336813832	0.322469088885668	−0.369441421035047

Table 2

n	$x=-0.25$	$x=0$	$x=0.25$
4	−0.7	−0.6	−0.2
8	−0.6579	−0.6258	−0.2685
16	−0.6579681	−0.625833	−0.268565
32	−0.657968182	−0.62583353	−0.26856552
64	−0.6579681828	−0.625833536	−0.2685655278
128	−0.6579681828605	−0.62583353610	−0.268565527880
256	−0.657968182860581	−0.625833536106916	−0.268565527880797

Table 3

n	$x=0.5$	$x=0.75$	$x=1$
4	0.5	1.8	3.0
8	0.49991	1.8328	3.933
16	0.499919636	1.8328139	3.9333238
32	0.4999196362	1.832814007	3.933323831
64	0.49991963622	1.83281400731	3.9333238313
128	0.4999196362236	1.8328140073171	3.93332383133502
256	0.499919636223606	1.83281400731719	3.93332383133502

Table 4

n	cond(M_n)	cond(PM_n)
4	6.01860795304063	5.8
8	6.47953551154866	5.8679
16	6.85524676319908	5.86795
32	8.54306931543816	5.86795426
64	13.94559701601129	5.8679542697
128	23.33074273594644	5.867954269744
256	39.16610787128132	5.86795426974467

Table 5

n	$x=-1$	$x=-0.75$	$x=-0.5$
4	−1.21	−1.12	−0.95
8	−1.218270	−1.122685	−0.952635
16	−1.21827086124490	−1.12268592319859	−0.952635785271222

Table 6

n	$x=-0.25$	$x=0$	$x=0.25$
4	−0.7	−0.40	−4.1E-002
8	−0.710578	−0.40329	−4.1676E-002
16	−0.710578894130081	−0.403295118915972	−4.167629643073395E−002

Table 7

n	$x=0.5$	$x=0.75$	$x=1$
4	0.35	0.78	1.2
8	0.3597372	0.783452	1.20991
16	0.359737266725713	0.783452894062918	1.20991817145819

Table 8

n	cond(M_n)	cond(PM_n)
4	3.13569573227304	2.9177
8	5.37576172467813	2.91770077260
16	9.02242049433755	2.91770077260696
32	15.06890340132042	2.91770077260697
64	25.21776822540110	2.91770077260696
128	42.29378489222648	2.91770077260696
256	71.02717508568738	2.91770077260696
512	119.3660554766741	2.91770077260697

machine precision. Nevertheless, we report the conditions numbers of the two systems for n taking more large values (Table 8).

The convergence of the sequence $\{\text{cond}(PM_n)\}$, as n goes to infinity, is very fast and faster than in the previous example. Surely this depends on the smoothness properties of the function $k(x,y)$.

Acknowledgements

The author thanks Professor G. Mastroianni, Doctor M.G. Russo and Prof. P. Junghanns for their helpful suggestions. She also thanks the referees for the accurate reading of the paper and the remarks for its improvement.

References

[1] K.E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge University Press, Cambridge, 1997.

[2] D. Berthold, On the stability of the classical collocation method for Cauchy-type singular integral equations, J. Comput. Appl. Math. 43 (1992) 171–184.

[3] D. Berthold, W. Hoppe, B. Silbermann, A fast algorithm for solving the generalized airfoil equation, J. Comput. Appl. Math. 43 (1992) 185–219.

[4] D. Bini, M. Capovani, O. Menchi, Metodi numerici per l'algebra lineare, Zanichelli, Bologna, 1993.

[5] M.R. Capobianco, M.G. Russo, Uniform convergence estimates for a collocation method for the Cauchy singular integral equation, J. Integral Equations Appl. 9(1) (1997) 21–45.

[6] Z. Ditzian, V. Totik, Moduli of Smoothness, SCMG Springer, New York, 1987.

[7] M.A. Golberg, The convergence of a collocation method for a class of Cauchy singular integral equations, J. Math. Anal. Appl. 100 (1984) 500–512.

[8] P. Junghanns, Product integration for the generalized airfoil equation, in: E. Schock (Ed.), Beiträge zur Angewandten Analysis und Informatik, Shaker Verlag, Aachen, 1994, pp. 171–188.

[9] P. Junghanns, B. Silbermann, Theorie der Näherungsverfahren für singuläre Integralgleichungen auf Intervallen, Math. Nachr. 103 (1981) 199–224.

[10] G. Mastroianni, S. Prössdorf, A quadrature method for Cauchy integral equations with weakly singular perturbation kernel, J. Integral Equations Appl. 4(2) (1992) 205–228.

[11] G. Mastroianni, M.G. Russo, Lagrange interpolation in weighted Besov spaces, Constr. Approx. 14 (1999) 1–33.

[12] S.G. Mikhlin, S. Prössdorf, Singular Integral Operators, Akademie Verlag, Berlin, 1986.

[13] N.I. Muskhelishvili, Singular Integral Equations, Noordhoff, Groningen, 1953.

[14] S. Prössdorf, B. Silbermann, Numerical Analysis for Integral and Related Operator Equations, Akademie Verlag, Berlin, 1991.