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On joint Weibull probability density functions

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Abstract

Two important results for the joint probability density function of the Weibull distribution are derived.

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1. Preliminaries

Estimation of the Weibull distribution parameters and its applications were well demonstrated earlier [1–4]. The product of the Weibull density functions and a random variable constructed based on such products are studied. We prove here two results based on such random variables. When multiple data sets from skewed distributions are formed then these results could be of help for conceptualization.

Let X_i be the Weibull random variable with parameters α_i, β_i for all $i = 1, 2, 3, \ldots, n$ and $f(x_i) = (\beta_i/\alpha_i)(x_i/\alpha_i)^{\beta_i-1}\exp\{-(x_i/\alpha_i)^{\beta_i}\}$, where $\alpha_i, \beta_i > 0$. The Laplace transformation for this function i.e. $\int_0^\infty \exp(\lambda_i x_i) f(x_i)\mathrm{d}x_i =$

$$\alpha_i^{\beta_i/(\beta_i-1)}\beta_i/(\beta_i-1)\Gamma\{\beta_i/(\beta_i-1)\}. \tag{1.1}$$

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We can easily verify that

$$\prod_{i=1}^{n}(\beta_i/\alpha_i)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(x_i/\alpha_i)^{\beta_i-1}\exp\left\{-\sum_{i=1}^{n}(x_i/\alpha_i)^{\beta_i}\right\}\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n = 1.$$

The product of the Weibull density functions is given by

$$\prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n}(\beta_i/\alpha_i)(x_i/\alpha_i)^{\beta_i-1}\exp\left\{-\sum_{i=1}^{n}(x_i/\alpha_i)^{\beta_i}\right\}. \qquad (1.2)$$

2. Product Weibull density functions

Theorem 2.1. $\lim_{\beta\to 1}\int\sum_{i=1}^{n}\log\{f(x_i)\}\mathrm{d}x_i = \left(\frac{3x_i^2}{2\alpha_i}\right) - x_i\sum_{i=1}^{n}\left(\frac{x_i}{\alpha_i}\right) - x_i\sum_{i=1}^{n}\log(\alpha_i)$.

Proof. From (1.2), we have

$$\sum_{i=1}^{n}\log f(x_i) = \log\left\{\prod_{i=1}^{n}(\beta_i/\alpha_i)(x_i/\alpha_i)^{\beta_i-1}\right\} - \left\{-\sum_{i=1}^{n}(x_i/\alpha_i)^{\beta_i}\right\}$$
$$= \sum_{i=1}^{n}(\beta_i-1)\log\left(\frac{x_i}{\alpha_i}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha_i}\right)^{\beta_i} + \sum_{i=1}^{n}\log\left(\frac{\beta_i}{\alpha_i}\right)$$

$$\int\sum_{i=1}^{n}\log\left(\frac{x_i}{\alpha_i}\right)^{\beta_i-1}\mathrm{d}x_i = \int\left\{\log\left(\frac{x_1}{\alpha_1}\right)^{\beta_1-1} + \log\left(\frac{x_2}{\alpha_2}\right)^{\beta_2-1} + \cdots + \log\left(\frac{x_n}{\alpha_n}\right)^{\beta_n-1}\right\}\mathrm{d}x_i$$
$$= x_i\log\left(\frac{x_1}{\alpha_1}\right)^{\beta_1-1} + x_i\log\left(\frac{x_2}{\alpha_2}\right)^{\beta_2-1} + \cdots + x_i\log\left(\frac{x_{i-1}}{\alpha_{i-1}}\right)^{\beta_{i-1}-1}$$
$$+ x_i\log\left(\frac{x_i}{\alpha_i}\right)^{\beta_i-1} + x_i\log\left(\frac{x_{i+1}}{\alpha_{i+1}}\right)^{\beta_{i+1}-1} - (\beta_i-1)x_i + \cdots + x_i\log\left(\frac{x_n}{\alpha_n}\right)^{\beta_n-1}$$
$$= x_i\sum_{i=1}^{n}\log\left(\frac{x_i}{\alpha_i}\right)^{\beta_i-1} - (\beta_i-1)x_i$$

$$\int\sum_{i=1}^{n}\left(\frac{x_i}{\alpha_i}\right)^{\beta_i}\mathrm{d}x_i = x_i\left(\frac{x_1}{\alpha_1}\right)^{\beta_1} + x_i\left(\frac{x_2}{\alpha_2}\right)^{\beta_2} + \cdots + \frac{x_i}{\beta_i+1}\left(\frac{x_i}{\alpha_i}\right)^{\beta_i} + \cdots + x_i\left(\frac{x_n}{\alpha_n}\right)^{\beta_n}$$
$$= x_i\sum_{i=1}^{n}\left(\frac{x_i}{\alpha_i}\right)^{\beta_i} - x_i\left(\frac{x_i}{\alpha_i}\right)^{\beta_i}\left(1+\frac{1}{\beta_i+1}\right)$$

$$\int\sum_{i=1}^{n}\log\left(\frac{\beta_i}{\alpha_i}\right)\mathrm{d}x_i = x_i\sum_{i=1}^{n}\log\left(\frac{\beta_i}{\alpha_i}\right).$$

Therefore,

$$\int\sum_{i=1}^{n}\log f(x_i)\mathrm{d}x_i = x_i\sum_{i=1}^{n}\log\left(\frac{x_i}{\alpha_i}\right)^{\beta_i-1} - (\beta_i-1)x_i - x_i\sum_{i=1}^{n}\left(\frac{x_i}{\alpha_i}\right)^{\beta_i}$$

$$+x_i\left(\frac{x_i}{\alpha_i}\right)^{\beta_i}\left(1+\frac{1}{\beta_i+1}\right)+x_i\sum_{i=1}^{n}\log\left(\frac{\beta_i}{\alpha_i}\right).$$

Hence as $\beta_i \to 1$ the RHS of above equation will become the required expression. □

Remark 2.2. When n variables $X_1, X_2, X_3 \ldots X_n$ follow $f(x_1), f(x_2), f(x_3) \ldots f(x_n)$ then the n-variable Laplace transformation (in this case it can also be called the moment generating function) can be written using (1.1) as

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty \exp(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n)f(x_1)f(x_2)\cdots f(x_n)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n$$
$$\times\prod_{i=1}^{n}(\alpha_i)^{\beta_i/(\beta_i-1)}\beta_i/(\beta_i-1)\Gamma\{\beta_i/(\beta_i-1)\}.$$

Let $g(Y_{ij}) = \prod_{i=1}^{n} f(y_{ij})$ for all $j = 1, 2, 3 \ldots m$. Suppose $Y_{i1}, Y_{i2}, Y_{i3} \ldots Y_{im}$ are the identically independent random variables with $Y_{ij} \sim g(Y_{ij})$. Let $\mathfrak{R} = \mathrm{Min}(Y_{i1}, Y_{i2}, \ldots, Y_{im})$ then $P(\mathfrak{R} > z) = P\{\mathrm{Min}(Y_{i1}, Y_{i2}, \ldots, Y_{im}) > z\} = P\left(\bigcap_{j=1}^{m} Y_{ij} > z\right) = \prod_{j=1}^{m} P(Y_{ij} > z) = P(Y_{ij} > z)^m$. Now $P(Y_{ij} > z) = \int_z^\infty g(Y_{ij})\,\mathrm{d}y_i = \int_z^\infty \prod_{i=1}^{n} f(y_{ij})\,\mathrm{d}y_i =$

$$\prod_{k=1}^{i-1}(\beta_{ik}/\alpha_{ik})(y_{ik}/\alpha_{ik})^{\beta_{ik}-1}\exp\left\{-\sum_{k=1}^{i-1}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\}\prod_{k=1}^{i-1}(\beta_{ik}/\alpha_{ik})(y_{ik}/\alpha_{ik})^{\beta_{ik}-1}$$
$$\times\exp\left\{-\sum_{k=i+1}^{n}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\}(\beta_{ii}/\alpha_{ii})(1/\alpha_{ii})^{\beta_{ii}-1}\int_z^\infty y_{ii}^{\beta_{ii}-1}\exp\{-(y_{ii}/\alpha_{ii})\}\,\mathrm{d}y_{ii}.$$

Therefore,

$$P(\mathfrak{R} > z) = \left(\prod_{k=1}^{i-1}(\beta_{ik}/\alpha_{ik})(y_{ik}/\alpha_{ik})^{\beta_{ik}-1}\prod_{k=1}^{i-1}(\beta_{ik}/\alpha_{ik})(y_{ik}/\alpha_{ik})^{\beta_{ik}-1}\right)^m$$
$$\times\exp\left\{-\sum_{k=1}^{i-1}(y_{ik}/\alpha_{ik})^{\beta_{ik}}-\sum_{k=i+1}^{n}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\}^m\exp(-z/\alpha_{ii})^{m\beta_{ii}}$$
$$>\exp\left\{-m^2z\beta_{ii}/\alpha_{ii}\sum_{k=1}^{i-1}(y_{ik}/\alpha_{ik})^{\beta_{ik}}-m^2z\beta_{ii}/\alpha_{ii}\sum_{k=i+1}^{n}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\}$$
$$=\exp\left\{-m^2z\beta_{ii}/\alpha_{ii}\sum_{k=1}^{i-1}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\}\Big/\exp\left\{m^2z\beta_{ii}/\alpha_{ii}\sum_{k=i+1}^{n}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\} \quad (2.1)$$

and

$$P(\mathfrak{R}\le z) < \frac{\exp\left\{m^2z\beta_{ii}/\alpha_{ii}\sum_{k=i+1}^{n}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\}-\exp\left\{-m^2z\beta_{ii}/\alpha_{ii}\sum_{k=1}^{i-1}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\}}{\exp\left\{m^2z\beta_{ii}/\alpha_{ii}\sum_{k=i+1}^{n}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\}} \quad (2.2)$$

from (2.1) we can write

$$\exp\left\{-m^2 z\beta_{ii}/\alpha_{ii}\sum_{k=1}^{i-1}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\} < P(\mathfrak{R} > z)\exp\left\{m^2 z\beta_{ii}/\alpha_{ii}\sum_{k=i+1}^{n}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\}$$
$$< \exp\left\{m^2 z\beta_{ii}/\alpha_{ii}\sum_{k=i+1}^{n}(y_{ik}/\alpha_{ik})^{\beta_{ik}}\right\}. \tag{2.3}$$

Inequality (2.3) can also be obtained from (2.2).

Theorem 2.3. *For a given* $\mathfrak{R}$ *as above,* $\sum_{k=1}^{i-1} -(y_{ik}/\alpha_{ik})^{\beta_{ik}} < \sum_{k=i+1}^{n}(y_{ik}/\alpha_{ik})^{\beta_{ik}}$.

Proof. Taking logarithms for (2.3) and through some algebraic manipulations we arrive at the required inequality. □

The Weibull distribution has been shown to be the best model for life testing problems. Bivariate and multivariate forms were also explored by the researchers in a classical way [5,6]. Lee discussed properties of the bivariate distributions of the form $F(x_1, x_2) = \exp\{-(\alpha_1 x_1^{\beta} + \alpha_2 x_2^{\beta})^{\mu}\}$ for $\alpha_i > 0$, $x_i \geq 0$, $i = 1, 2$, $\beta > 0$ and $0 < \mu \leq 1$. He further showed that two random variables having such a distribution function can be represented in terms of independent random variables and are useful in generating random samples. The Weibull distribution is known for its well known applications in statistical quality control charts [7] and for problems arising in reliability for assessing the importance of the individual components of a system.

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