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T-colorings of graphs

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Abstract

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Given a finite set T of positive integers containing $\{0\}$, a T-coloring of a simple graph G is a nonnegative integer function f defined on the vertex set of G, such that if $\{u, v\} \in E(G)$ then $|f(u) - f(v)| \notin T$. The T-span of a T-coloring is defined as the difference of the largest and smallest colors used; the T-span of G, $\mathrm{sp}_T(G)$, is the minimum span over all T-colorings of G. It is known that the T-span of G satisfies $\mathrm{sp}_T(K_{\omega(G)}) \leq \mathrm{sp}_T(G) \leq \mathrm{sp}_T(K_{\chi(G)})$. When T is an r-initial set (Cozzens and Roberts, 1982), or a k multiple of s set (A. Raychaudhuri, 1985), then $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)})$ for all graphs G. Using graph homomorphisms and a special family of graphs, we characterize those T's with equality $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)})$ for all graphs G. We discover new T's with the same result. Furthermore, we get a necessary and sufficient condition of equality $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_m)$ for all graphs G with $\chi(G) = m$.

1. Introduction

Given a finite set T of positive integers containing $\{0\}$, a T-coloring of a simple graph G is a nonnegative integer function f defined on the vertex set of G, such that if $\{u, v\} \in E(G)$ then $|f(u) - f(v)| \notin T$. T-colorings of graphs arose from the channel assignment problem. Hale [1] formulated it in graph theory language. T is the interference set. That is, if we want to assign channels to a pair of adjacent cities or radio stations, then the difference of those two channels used has to avoid the set T. For example, $T = \{0, 7, 14, 15\}$ is the interference set for UHF-television stations. If the set T is $\{0\}$ then T-coloring is the same as proper coloring. The T-span of a T-coloring f, denoted by $\mathrm{sp}_T(f)$, is defined as the difference of the maximum and minimum channels used; and the T-span of a graph G, denoted by $\mathrm{sp}_T(G)$, is the minimum span over all T-colorings of G. It is known that $\mathrm{sp}_T(K_{\omega(G)}) \leq \mathrm{sp}_T(G) \leq \mathrm{sp}_T(K_{\chi(G)})$ for any T-set T and any graph G, where $\omega(G)$ is the maximum clique size of G (Cozzens and Roberts [2]). When T is an r-initial set, i.e., $T = \{0, 1, 2, \ldots, r\} \cup S$, where S contains no multiple of $r + 1$ (Cozzens and Roberts [2]); and when T is a k multiple of s set, i.e., $T = \{0, s, 2s, \ldots, ks\} \cup S$, where $S \subseteq \{s + 1, s + 2, \ldots, ks - 1\}$ (Raychaudhuri

[4]), then

$$\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)}) \quad \text{for all graphs } G. \qquad (*)$$

In this paper, we first introduce a special family of graphs, called T-graphs. A graph G is weakly perfect if $\chi(G) = \omega(G)$. Using graph homomorphisms and the weak perfectness of the T-graphs, we get a necessary and sufficient condition in Section 3 to characterize the sets T. This leads to short proof that (*) holds for the two families above, and new families with this property are presented. In Section 4, we characterize the equality

$$\mathrm{sp}_T(G) = \mathrm{sp}_T(K_m) \quad \text{for all graphs } G \text{ with } \chi(G) = m.$$

2. T-graphs and graph homomorphisms

For a given set T, the T-graph, denoted by G_T, is defined by the following:

$$V(G_T) = \mathbb{Z}^+ \cup \{0\} \quad \text{and} \quad \{x, y\} \in E(G_T) \text{ iff } |x - y| \notin T.$$

The T-graph of order n is the subgraph of G_T induced by the vertices $\{0, 1, 2, \ldots, n-1\}$ of G_T, and is denoted by G_T^n. Since the ordering of the vertices of G_T^n is a T-coloring itself, one has $\mathrm{sp}_T(G_T^n) \leq n-1$ for any n. Given two graphs G and H, a graph homomorphism from G to H is a function $f: V(G) \to V(H)$ such that if $\{u, v\} \in E(G)$ then $\{f(u), f(v)\} \in E(H)$. We say that G is homomorphic to H, if there is a graph homomorphism from G to H, denoted by $G \to H$. If $G \to H$ then $\chi(G) \leq \chi(H)$; if $\chi(G) \leq m$ then $G \to K_m$; and if $K_m \to G$ then $\omega(G) \geq m$. For related work on graph homomorphisms, see [7–8]. From the definitions, we can get the following properties.

Properties. (i) *if* $G \to H$ *then* $\mathrm{sp}_T(G) \leq \mathrm{sp}_T(H)$.

(ii) $\mathrm{sp}_T(G) \leq n-1$ *iff* $G \to G_T^n$.

(iii) *If n is the minimum number such that* $\chi(G_T^n) \geq \chi(G)$, *then* $\mathrm{sp}_T(G) \geq n-1$.

(iv) *If* $\mathrm{sp}_T(G) < n$ *then* $\chi(G) \leq \chi(G_T^n)$.

(v) *If* $\chi(G) \leq \omega(G_T^n)$ *then* $\mathrm{sp}_T(G) \leq n-1$.

Proof. (i) If f is a homomorphism from G to H, and g is a T-coloring of H, then the composition function $g \circ f$ is a T-coloring of G. Hence $\mathrm{sp}_T(G) \leq \mathrm{sp}_T(H)$.

(ii) ($\Leftarrow$) From Property (i), and $\mathrm{sp}_T(G_T^n) \leq n-1$.

($\Rightarrow$) Suppose f is a T-coloring of G attaining $\mathrm{sp}_T(f) = \mathrm{sp}_T(G) \leq n-1$. Without loss of generality, we can assume the colors that f uses are in the set $\{0, 1, \ldots, n-1\}$. If $\{u, v\} \in E(G)$, then $|f(u) - f(v)| \notin T$, i.e. $\{f(u), f(v)\} \in E(G_T^n)$. Hence f is also a homomorphism from G to G_T^n.

(iii) If n is the minimum number with $\chi(G_T^n) \geq \chi(G)$ but $\mathrm{sp}_T(G) < n-1$, by (ii) $G \to G_T^{n-1}$, which implies $\chi(G) \leq \chi(G_T^{n-1})$. This contradicts the minimality of n.

(iv) If $\mathrm{sp}_T(G) < n$, by (ii) $G \to G_T^n$, so $\chi(G) \leqslant G_T^n$.
(v) If $\omega(G_T^n) \geqslant \chi(G)$ then

$$G \to K_{\omega(G_T^n)} \to G_T^n,$$

so $G \to G_T^n$, which implies $\mathrm{sp}_T(G) \leqslant n-1$. □

3. Main theorem

The following Lemma 3.1 describes the relation between the minimum span of a complete graph K_m and the clique size of the T-graph of size n. Then we present the main theorem.

Lemma 3.1. *The number n is the minimum such that $\omega(G_T^n) = m$ iff $\mathrm{sp}_T(K_m) = n-1$.*

Proof. ($\Rightarrow$) Suppose n is the minimum such that $\omega(G_T^n) = m$. Therefore $K_m \to G_T^n$, so $\mathrm{sp}_T(K_m) \leqslant n-1$. Now if $\mathrm{sp}_T(K_m) < n-1$, by Property (ii), we get $K_m \to G_T^{n-1}$, so $\omega(G_T^{n-1}) \geqslant m$. This contradicts the minimality of n.

($\Leftarrow$) If $\mathrm{sp}_T(K_m) = n-1$, then $K_m \to G_T^n$. Therefore $\omega(G_T^n) \geqslant m$. If $\omega(G_T^{n-1}) = m$, then by Property (v), we get $\mathrm{sp}_T(K_m) \leqslant n-2$, which contradicts $\mathrm{sp}_T(K_m) = n-1$. Hence n is the minimum with $\omega(G_T^n) = m$. □

Theorem 3.2. *Given T, the following are equivalent:*
(i) $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)})$ *for all graphs G,*
(ii) $\mathrm{sp}_T(G_T^n) = \mathrm{sp}_T(K_{\chi(G_T^n)})$ *for all n,*
(iii) *G_T^n is weakly perfect for all n,*
(iv) *G_T^n is weakly perfect for all n with some graph H such that* $\mathrm{sp}_T(H) = n-1$.

Proof. (i)$\Rightarrow$(ii), (iii)$\Rightarrow$(iv) are trivial.

(ii)$\Rightarrow$(iii): It's enough to show that, for any n, $\mathrm{sp}_T(G_T^n) = \mathrm{sp}_T(K_{\chi(G_T^n)})$ implies $\omega(G_T^n) = \chi(G_T^n)$. Suppose $\mathrm{sp}_T(G_T^n) = \mathrm{sp}_T(K_{\chi(G_T^n)}) = n_0 = 1$, since $\mathrm{sp}_T(G_T^n) \leqslant n-1$, so $n_0 \leqslant n$. Also by Lemma 3.1, n_0 is the minimum such that $\omega(G_T^{n_0}) = \chi(G_T^n)$. Therefore $\omega(G_T^n) \geqslant \omega(G_T^{n_0}) = \chi(G_T^n)$, so we get $\omega(G_T^n) = \chi(G_T^n)$.

(iv)$\Rightarrow$(i): Let $\mathrm{sp}_T(G) = n_0 - 1$ and $\mathrm{sp}_T(K_{\chi(G)}) = n-1$, and suppose $n_0 < n$. Then $G \to G_T^{n_0}$, $\chi(G_T^n) = \omega(G_T^n)$, and $\chi(G_T^{n_0}) = \omega(G_T^{n_0})$. By Lemma 3.1, n is the minimum with $\omega(G_T^n) = \chi(G)$. Therefore, $\chi(G_T^{n_0}) = \omega(G_T^{n_0}) < \omega(G_T^n) = \chi(G)$. This contradicts $G \to G_T^{n_0}$. Hence $n_0 = n$. □

If G is weakly perfect then $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)}) = \mathrm{sp}_T(K_{\omega(G)})$ for any T, but the reverse is not always true. From the claim of (ii)$\Rightarrow$(iii) above, we obtain the following corollary for T-graphs.

Corollary 3.3. *For any T and any n,* $\mathrm{sp}_T(G_T^n) = \mathrm{sp}_T(K_{\chi(G_T^n)})$ *iff G_T^n is weakly perfect.*

Next we discuss the two known families of T those satisfy (*).

Corollary 3.4 [2, 9]. *If T is an r-initial set, then (*) holds and* $\mathrm{sp}_T(K_m) = (r+1)(m-1)$.

Proof. By Theorem 3.1, to get (*), it is enough to show that G_T^n is weakly perfect for all n. In G_T, for any $k \geq 0$ the set of vertices $\{0, r+1, 2(r+1), \ldots, k(r+1)\}$ forms a clique, so $\omega(G_T^{k(r+1)+1}) \geq k+1$ for any $k \geq 0$ (See Fig. 1 as an example). Therefore it's sufficient to show that $\chi(G_T^{k(r+1)}) \leq k$. Let f be the coloring defined by

$$f(i) = \left\lfloor \frac{i}{r+1} \right\rfloor + 1.$$

If $\{a, b\} \in G_T^{k(r+1)}$ then $|a-b| \notin T$, i.e. $|a-b| \geq r+1$, so

$$\left\lfloor \frac{a}{r+1} \right\rfloor \neq \left\lfloor \frac{b}{r+1} \right\rfloor.$$

Hence $f(a) \neq f(b)$, f is a proper coloring and

$$|\mathrm{Range}(f)| = \left\lfloor \frac{k(r+1)-1}{r+1} \right\rfloor + 1 = k,$$

so

$$\chi(G_T^{k(r+1)}) = \omega(G_T^{k(r+1)}) = k.$$

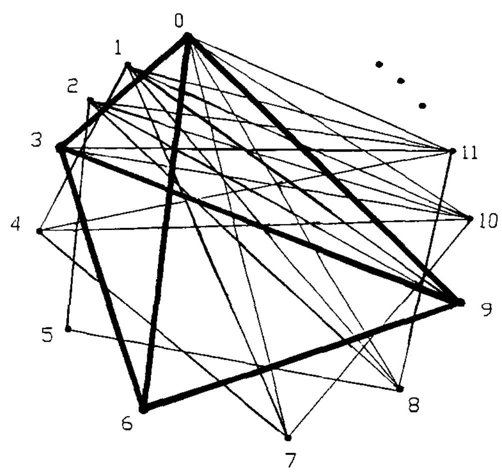

Fig. 1. $T = \{0, 1, 2, 4, 5\}$ (T is an 3-initial set).

On the other hand, this also implies that $n = k(r+1)+1$ is the minimum such that $\omega(G_T^n) = k+1$. By Lemma 3.1, $\mathrm{sp}_T(K_m) = (m-1)(r+1)$. This finishes the proof. □

Corollary 3.5 [5]. *If T is a k multiple of s set, then* (∗) *holds and*

$$\mathrm{sp}_T(K_m) = sd(k+1)+q-1, \quad m = ds+q, \quad d \geq 0, \quad 1 \leq q \leq s.$$

Proof. In G_T, $[0, s-1] \cup \{ks+s\}$ is a clique. Furthermore, for any $d \geq 0$ and $0 \leq q \leq s$, $\omega(G_T^{ds(k+1)+q}) \geq sd+q$ (see Fig. 2 as an example). Hence, for (∗), it will be enough to show that $\chi(G_T^{ds(k+1)}) = sd$ for all $d \geq 1$. We can color G_T by f:

$$f(i) = j+1+s\left\lfloor \frac{i}{s(k+1)} \right\rfloor, \quad \text{where } j \equiv i \pmod{s} \text{ and } 0 \leq j < s.$$

We now show that f is a proper coloring. Because the colors f uses are increasing periodically, we only have to check the first period, i.e., for $i \in [0, s(k+1)-1]$. If $\{a, b\} \in E(G_T)$ and $a, b \in [0, s(k+1)-1]$, then $|a-b| \notin T$. Since $T \supseteq \{0, s, 2s, \ldots, ks\}$, this implies $f(a) \neq f(b)$. Also, we can get that $n = ds(k+1)+q$ is the minimum such that $\omega(G_T^n) = sd+q$ for all $1 \leq q \leq s$ and $d \geq 0$. So by Lemma 3.1, we can get the minimum spans for complete graphs. This finishes the proof. □

For a k multiple of s set T, the maximum cliques in the family of T-graphs come out periodically. For example, when $T = \{0, 3, 4, 5, 6\}$ (Fig. 2),

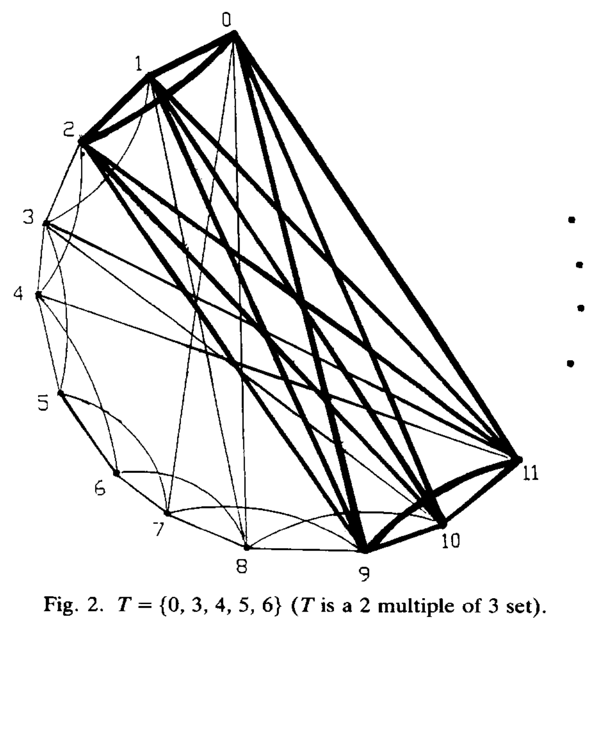

Fig. 2. $T = \{0, 3, 4, 5, 6\}$ (T is a 2 multiple of 3 set).

$\{0, 1, 2, 9, 10, 11, 18, 19, 20, \ldots\}$ is a maximum clique in G_T. In the above proof, we properly color the vertices in each period by the modularity, and use different colors for each period. Now we may extend T by union with another set S', where all numbers in S' are greater than or equal to $s(k+2)$, and S' does not contain any number of

$$[s(2k+1)+1, s(2k+3)-1] \cup [s(3k+2)+1, s(3k+4)-1]$$
$$\cup\, [s(4k+3)+1, s(4k+5)-1] \cup \cdots.$$

Since $\{0, s, 2s, \ldots, ks\} \subseteq T \cup S'$, this implies the modular coloring used in the above proof is still proper for $G_{T\cup S'}$. Without breaking the maximum clique, we let S' avoid those values to maintain that the maximum clique in G_T is also a maximum clique in $G_{T\cup S'}$. We call $T \cup S'$ an extended k multiple of s set. This implies the following.

Corollary 3.6. *If T is an extended k multiple of s set, then T has the same result as a k multiple of s set as in Corollary* 3.4.

Example. If $T = \{0, 3, 6, 12\}$, *or* $T = \{0, 3, 6, 12, 13, 14, 15\}$ *then for all graphs* G, $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)})$, *and* $\mathrm{sp}_T(K_m)$ *has the same value as in Corollary* 3.4.

It would be nice to characterize all sets T for which (*) holds. While we cannot do this we now present other new families of T's with the same property as the above cases. We let N_p denote the infinite set of multiples of p, $\{p, 2p, 3p, \ldots\}$.

Theorem 3.7. *Suppose* $T = ([0, a+b] - \{a+1\}) \cup S$, *where* $a = cp$, $p \geq 2$, $b \geq 2$, $i(a+1) \notin N_p$ *and* $(a+b+1) + i(a+1) \notin N_p$ *for all* $i = 0, 1, 2, \ldots, p-1$, *and* S *has the following properties*:

(1) *All numbers in S are less than $pa + p + b$.*
(2) $N_p \cap [0, pa+p+b-1] \subseteq S$.
(3) $i(a+1) \notin S$, *and* $(a+b+1) + i(a+1) \notin S$ *for all* $i = 0, 1, 2, \ldots, p-1$.

Then $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)})$ *for all graphs* G, *and*

$$\mathrm{sp}_T(K_m) = k(pa+p+b) + (l-1)(a+1), \quad m = kp + l, \quad k \geq 0, \quad 1 \leq l \leq p.$$

Note that in this theorem, the set T is neither a k multiple of s set, nor an r-initial set since $p(a+1)$ is in T. Before we prove this theorem, let us look at the simpler special cases $p = 2$, and $b = 2, 4$ respectively.

Corollary 3.8. *If* $T = [0, a] \cup \{a+2, a+4, a+6, \ldots, 2(a+1)\} \cup R$, *where a is an even integer and* $R \subseteq \{a+5, a+7, \ldots, 2(a+1)+1\}$ *then* (*) *holds, and*

$$\mathrm{sp}_T(K_m) = \begin{cases} k(2a+4), & \text{if } m = 2k+1,\ k \geq 0; \\ (k-1)(2a+4) + a + 1, & \text{if } m = 2k,\ k \geq 1. \end{cases}$$

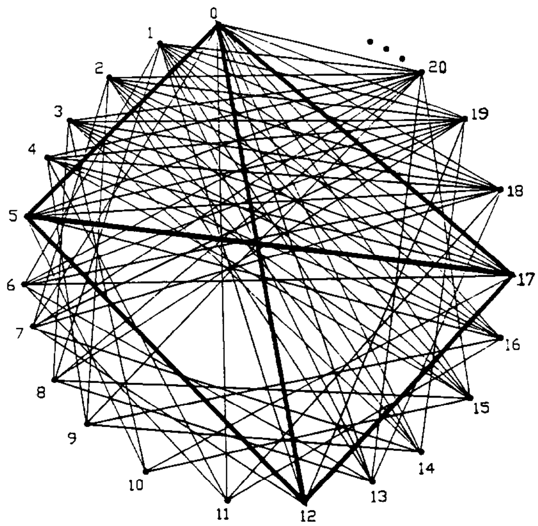

Fig. 3. $T = \{0, 1, 2, 3, 4, 6, 8, 9, 10\}$.

Proof. In G_T, for any $k \geq 0$, $\omega(G_T^{k(2a+4)+1}) \geq 2k+1$ and $\omega(G_T^{k(2a+4)+a+2}) \geq 2(k+1)$ (see Fig. 3 as an example). Hence, it is enough to show the following:

(i) $\chi(G_T^{k(2a+4)}) \leq 2k$, $k \geq 1$ and

(ii) $\chi(G_T^{k(2a+4)+a+1}) \leq 2k+1$, $k \geq 0$.

To show (i), since G_T^{2a+4} is isomorphic to the subgraph of G_T induced by the vertices $[2a+4, 2(2a+4)-1]$, it will be enough to claim that $\chi(G_T^{2a+4}) \leq 2$. Color G_T^{2a+4} by:

$$f(i) = \begin{cases} 1, & i \text{ is even;} \\ 2, & i \text{ is odd.} \end{cases}$$

If $\{x, y\} \in E(G_T^{2a+4})$, then $x, y \leq 2a+3$ and $|x-y| \notin T$. Since T contains all the even integers less than or equal to $2(k+1)$, $|x-y|$ must be odd which implies $f(x) \neq f(y)$, i.e., f is a proper coloring. On the other hand, the vertex set $[k(2a+4), k(2a+4)+a]$ is independent, so (ii) follows from (i). This also implies that $n = k(2a+4)+1$ is the minimum such that $\omega(G_T^n) = 2k+1$, and $n = k(2a+4)+a+2$ is the minimum such that $\omega(G_T^n) = 2(k+1)$. So by Lemma 3.1, we complete the proof. □

Corollary 3.9. *If*

$$T = ([0, a+4] - \{a+1\}) \cup \{a+6, a+8, \ldots, 2(a+1), 2(a+1)+2\} \cup R$$

where a is even and $R \subseteq \{a+7, a+9, \ldots, 2(a+1)+3\}$ *then* (*) *holds, and*

$$\mathrm{sp}_T(K_m) = \begin{cases} k(2a+6), & m = 2k+1,\ k \geq 0; \\ (k-1)(2a+6)+a+1, & m = 2k,\ k \geq 1. \end{cases}$$

Proof. Similar argument to the above corollary. □

Proof of Theorem 3.7. First notice that $(p-1)(a+1)+a+b+1=pa+p+b$. Since S has the properties (1) and (3), the set of vertices $\{0, a+1, 2(a+1), \ldots, (p-1)(a+1), pa+p+b\}$ in G_T forms a clique. Hence, $\omega(G_T^{pa+p+b+1}) \geqslant p+1$. Moreover, we get

$$\omega(G_T^{k(pa+p+b)+q(a+1)+1}) \geqslant kp+q+1, \quad \text{for all } k \geqslant 0 \text{ and } 0 \leqslant q \leqslant p-1.$$

By Theorem 3.2, it is enough to show that $\chi(G_T^{k(pa+p+b)+q(a+1)+1}) \leqslant kp+q+1$ for $0 \leqslant q \leqslant p-1$ and $k \geqslant 0$. Any $a+1$ consecutive vertices in G_T form an independent set, and the subgraph of G_T induced by the vertices $[k(pa+p+b), (k+1)(pa+p+b)-1]$ is isomorphic to G_T^{pa+p+b} for all $k \geqslant 1$. Therefore, it is sufficient to show that $\chi(G_T^{pa+p+b}) \leqslant p$. We define the coloring f on G_T^{pa+p+b} by the following:

$$f(i) = i - \left\lfloor \frac{i}{p} \right\rfloor p + 1.$$

If $\{i, j\} \in E(G_T^{pa+p+b})$, then $i, j \leqslant pa+p+b-1$. Since S contains all multiples of p, one has that $f(i) \neq f(j)$. The coloring f uses p colors. This shows that T has the property (*). It is easy to see that $n = k(pa+p+b)+q(a+1)$ is the minimum number such that $\omega(G_T^{n+1}) = kp+q+1$, $k \geqslant 0$, $0 \leqslant q \leqslant p-1$. By Lemma 3.1, we get the minimum span of K_m. □

Using the same method of extending a k multiple of s set, we can also extend the T's in the above three cases to get more families of T with the property (*). It's tedious, so we will not state them here.

A greedy algorithm to T-color a complete graph K_m is defined as: Order the vertices of K_m by $\{1, 2, 3, \ldots, m\}$. Suppose the vertices $[1, i-1]$ have been colored, then color the vertex i by the smallest integer that will not contradict the definition of a T-coloring, and keep going to the last vertex. If the T-coloring done by the greedy algorithm attains the minimum span of K_m, then we say that greedy works for K_m ([6]). Notice that for all T's in this section, the greedy algorithm works for all complete graphs K_m. To end this section, we state the following conjecture.

Conjecture 3.10. If T has the property (*), then the greedy algorithm gets $\mathrm{sp}_T(K_m)$ for all $m \geqslant 1$.

4. Graphs G with $\chi(G) = m$

The property (*) is very strong for T. That is, only a few of T's have that property but the most of T's do not. In this section, we consider the weaker property where equality $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)})$ holds only for the graphs G with fixed chromatic number instead for all graphs G.

Theorem 4.1. *The T-span* $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_m) = n - 1$ *for all graphs G with* $\chi(G) = m$ *iff* $\omega(G_T^n) = \chi(G_T^n) = m$ *and* $\chi(G_T^{n-1}) < \chi(G_T^n)$.

Proof. ($\Rightarrow$) Since $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_m) = n - 1$, by Lemma 3.1, n is the minimum such that $\omega(G_T^n) = m$. Now, suppose $m = \omega(G_T^n) < \chi(G_T^n)$, then there exists some $n_0 < n$ such that $\chi(G_T^{n_0}) = m$. Hence, $\mathrm{sp}_T(G_T^{n_0}) = \mathrm{sp}_T(K_m) = n - 1$. But $\mathrm{sp}_T(G_T^{n_0}) \leq n_0 - 1 < n - 1$. This is a contradiction. Therefore, $\omega(G_T^n) = \chi(G_T^n) = m$. Next, if we suppose $\chi(G_T^{n-1}) = \chi(G_T^n) = m$, then $\mathrm{sp}_T(G_T^{n-1}) = \mathrm{sp}_T(K_m) = n - 1$. But this contradicts $\mathrm{sp}_T(G_T^{n-1}) \leq n - 2$.

($\Leftarrow$) Since $\omega(G_T^{n-1}) \leq \chi(G_T^{n-1}) < \chi(G_T^n) = \omega(G_T^n) = m$, so n is the minimum number such that $\omega(G_T^n) = m$. By Lemma 3.1, $\mathrm{sp}_T(K_m) = n - 1$. Suppose there is a graph G with $\chi(G) = m$ but $\mathrm{sp}_T(G) \leq \mathrm{sp}_T(K_m) = n - 1$, i.e., $\mathrm{sp}_T(G) \leq n - 2$. By Property (i), $G \to G_T^{n-1}$, then $m = \chi(G) \leq (G_T^{n-1}) < m$. This is a contradiction. □

From Theorem 4.1, if we want to check the truth of

$$\mathrm{sp}_T(G) = \mathrm{sp}_T(K_m) \quad \text{for all graphs } G \text{ with } \chi(G) = m, \qquad (**)$$

we first find out the smallest integer n such that $\omega(G_T^n) = m$. By Lemma 3.1, $\mathrm{sp}_T(K_m) = n - 1$. Secondly, we check the graph G_T^n. If both $\omega(G_T^n) = \chi(G_T^n)$ and $\chi(G_T^{n-1}) < \chi(G_T^n)$ are true, then we can get (**). Otherwise, we are able to find a counterexample by looking at G_T^n.

Example. *If* $T = \{0, 2, 3, 5\}$, *then for any graphs G with* $\chi(G) = 4$, $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_4) = 8$.

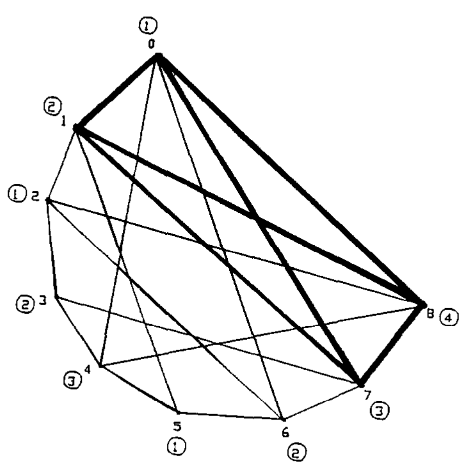

Fig. 4. $T = \{0, 2, 3, 5\}$.

Proof. From the graph G_T^9 (Fig. 4) and by Lemma 3.1, we can get $\mathrm{sp}_T(K_3) = 7$ and $\mathrm{sp}_T(K_4) = 8$. The circled number at each vertex is a proper coloring. Since $\omega(G_T^9) = 4$ and $\omega(G_T^8) = 3$, hence $\chi(G_T^9) = 4$ and $\chi(G_T^8) = 3$. Therefore by Theorem 4.1, we get $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_4) = 8$ for all graphs G with $\chi(G) = 4$. □

For the above case, $\chi(G_T^8) = \omega(G_T^8) = 3$ but $\chi(G_T^7) = 3$. By Theorem 4.1, there exists some graph G with $\chi(G) = 3$ but $\mathrm{sp}_T(G) < \mathrm{sp}_T(K_{\chi(G)})$. For example, the 5-cycle is such a graph (Actually, G_T^5 is a 5-cycle!). Since $\{0, 1, 2, 3, 4\}$ is a T-coloring of C_5, $\mathrm{sp}_T(C_5) \leq 4$, but $\mathrm{sp}_T(K_3) = 7$.

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