Young’s Symmetrizers for Projective Representations of the Symmetric Group

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Received January 1, 1994; accepted December 30, 1996

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0. INTRODUCTION

It is a basic fact of representation theory that any irreducible representation of a finite group $G$ over the complex field $\mathbb{C}$ can be realized as the left ideal in the group algebra $\mathbb{C}[G]$ generated by a certain element $v$, such that $v^2 = v$. This realization is especially preferable since here the group $G$ acts via left multiplication. But no direct construction of the element $v$ is known for a general $G$. However, if $G$ is the symmetric group $S_n$, there exists an explicit formula for the element $v$ due to Alfred Young [Y].

As early as in 1911, Issai Schur [S] discovered a nontrivial central $\mathbb{Z}$-extension $T_n$ of group $S_n$. That is, the group $S_n$ possesses projective representations which cannot be reduced to linear ones. The group $T_n$ is generated by the elements $t_1, \ldots, t_{n-1}$ and the central element $\xi$ subject to the relations $\xi^2 = 1$ and

$$t_k^2 = \zeta, \quad t_k t_{k+1} t_k = t_{k+1} t_k t_{k+1}; \quad t_k t_{k'} = \xi t_k t_{k'}, k - k' > 0.$$
The homomorphism $T_n : S_n \rightarrow S_n$ is defined by $\zeta \mapsto 1$ and $t_k \mapsto s_k$, where $s_k$ denotes the transposition of $k$ and $k + 1$.

Recently there has been a considerable increase of interest in the projective representations of the group $S_n$. It has turned out that many results concerning linear representations of $S_n$ have remarkable projective counterparts; see for instance [Mo2] and [St1]. An explicit realization of the irreducible projective representations of $S_n$ was produced in [Ni1]. Nevertheless, the question what is the analogue of the element $v \in \mathbb{C}[S_n]$ discovered by Young has been unanswered so far. The aim of the present paper is to answer this question.

The irreducible representations of the group $S_n$ over the field $\mathbb{C}$ are parametrized by the partitions $\lambda$ of the number $n$. Let $\lambda_1, \ldots, \lambda_i$ be the parts of $\lambda$; assume that $\lambda_1 \geq \cdots \geq \lambda_i > 0$. To construct the element $v$ corresponding to the partition $\lambda$, Young introduced the combinatorial objects known now as the standard tableaux [M]. They are the arrays

$$A = \{ A(i, j) | 1 \leq i \leq L, 1 \leq j \leq \lambda_i \}$$

such that each of the numbers $1, \ldots, n$ occurs in $A$ exactly once, and for all possible $i, j$

$$A(i, j) < A(i, j + 1), \quad A(i, j) < A(i + 1, j).$$

Given a partition $\lambda$, there are two distinguished standard tableaux of shape $\lambda$: the row tableau $A'$ and the column tableau $A'$. In the array $A'$ the numbers $1, \ldots, n$ occur consecutively in rows, while in $A'$ they occur consecutively in the columns. Let $Y$ and $Y'$ be the subgroups in $S_n$ consisting of all the elements which preserve each set of the numbers occuring respectively in the same row and in the same column of the tableau $A'$. Let $f$ be the number of the standard tableaux of shape $\lambda$. Then the element $v \in \mathbb{C}[S_n]$ from $[Y]$ is equal to

$$\frac{f}{n!} \cdot \sum_{s \in Y} \sum_{s' \in Y'} s s' (-1)^{s'}, \quad (0.1)$$

where $(-1)^{s'}$ denotes $1$ or $-1$ depending on whether the permutation $s$ is even or odd. The above defined element $v$ is called Young's symmetrizer.

Recently Ivan Cherednik proposed another description of Young's symmetrizer [Ch1]. In that paper an object larger than the group $S_n$ was considered. That is the associative algebra generated by the group algebra $\mathbb{C}[S_n]$ and the pairwise commuting elements $x_1, \ldots, x_n$ subject to the relations

$$x_k s_k = s_k x_k, \quad \text{if} \quad k' \neq k, k + 1; \quad x_k s_k = s_k x_{k+1} = -1.$$
This algebra was also considered in [D] and [L]. It is denoted by $He_n$ and called the degenerate affine Hecke algebra.

The subalgebra $\mathbb{C}[x_1, \ldots, x_n]$ in $He_n$ is maximal commutative. Let $\chi$ be a character of the algebra $\mathbb{C}[x_1, \ldots, x_n]$. Consider the representation of the algebra $He_n$ induced from the character $\chi$. It is called principal series representation, cf. [K] and [R]. The space $M_\chi$ of this representation can be identified with $\mathbb{C}[S_n]$. The element $s_k$ acts in $M_\chi$ via left multiplication, while the action of $x_k$ is determined through the above relations in $He_n$ by

$$x_k \cdot m = (x_k m) \cdot 1, \quad m \in \mathbb{C}[S_n]; \quad x_k \cdot 1 = \chi(x_k).$$

Consider the standard action of the group $S_n$ on the characters $\chi$:

$$s \cdot \chi(x_k) = \chi(x_{s(k)}), \quad s \in S_n.$$ Introduce the element $w \in S_n$ such that for all possible $i, j$

$$w(A'(i, j)) = n - A'(i, j) + 1.$$ Determine now the character $\chi$ by the equalities

$$w \cdot \chi(x_k) = j - i, \quad A'(i, j) = k.$$ It was pointed out in [Ch1] that the right multiplication in $\mathbb{C}[S_n]$ by the element $w$ is an intertwining operator $M_\chi \rightarrow M_w$; for more details see [JKMO].

This approach yields an expression for the element $v$ different from (0.1). For example, if $\lambda$ consists of only one part or each part of $\lambda$ is 1, then $v$ is equal respectively to

$$\frac{1}{n!} \prod_{k=2}^{n} \prod_{k'=1}^{k-1} \left(1 \pm \frac{(k, k')}{k-k'}\right);$$ here $(k, k')$ is the transposition of $k$ and $k'$. These two examples appeared in [KRS].

In the present paper the approach of [Ch1] is employed to construct the projective analogue of the element $v$. It turns out that instead of the group $T_n$ one should consider a central $\mathbb{Z}_2$-extension $D_n$ of the hyperoctahedral group $C_n = S_n \rtimes \mathbb{Z}_2$. The group $D_n$ is the semidirect product of the group $S_n$ and the Clifford group $Cl_n$. The latter is generated by the elements $c_1, \ldots, c_n$ and the central element $\zeta$ subject to the relations $\zeta^2 = 1$ and

$$c_k^2 = \zeta; \quad c_k c_{k'} = \zeta c_{k'} c_k, \quad k > k'.$$
The group $S_n$ acts on the elements $c_1, \ldots, c_n$ via permutations of their indices:

$$sc_s^{-1} = c_{\alpha(k)}, \quad s \in S_n.$$

The factorization map $D_n \to C_n$ is defined by $\zeta \mapsto 1$. We will assume that in the representations $\zeta \mapsto \text{id}$ for both $D_n$ and $T_n$. Denote by $M_n$ the factor-algebra $C[D_n]/(\zeta = -1)$.

By a general theorem on semidirect products of finite groups [Cl], the representations of the group $D_n$ can be constructed from those of $T_n$. This construction is given in Section 1 of the present paper, see also [St2].

On the other hand, there exists an analogue of the algebra $He_n$ corresponding to the group $D_n$; it is introduced in the Section 3. We will denote this algebra by $Se_n$ and call it the degenerate affine Sergeev algebra in honour of Alexander Sergeev, who has used the group $D_n$ to describe the irreducible polynomial representations of the queer classical Lie super-algebra [Se]. The algebra $Se_n$ contains $M_n$ as a subalgebra and there is a homomorphism $\iota$: $Se_n \to M_n$ identical on $M_n$ (Proposition 3.5); cf. [D] and [L]. As well as $He_n$ the algebra $Se_n$ comes equipped with the distinguished maximal commutative subalgebra $C[z_1, \ldots, z_n]$. By considering the intertwining operators between principal series representations of $Se_n$, the analogue of Young's symmetrizer is obtained.

Let $\lambda_1 > \cdots > \lambda_{l}$. Then the partition $\lambda$ is called strict. The irreducible representations $\rho$ of the group $D_n$ such that $\rho(\zeta) = -\text{id}$, are parametrized by the pairs $(\lambda, \delta)$ where $\delta = (\pm 1)^l$. Denote by $\rho_{\lambda, \delta}$ the representation corresponding to the pair $(\lambda, \delta)$. In Section 7 for each strict partition $\lambda$ an element $\psi_\lambda \in M_n$ is introduced such that the left ideal $V_\lambda = M_n \psi_\lambda$ is a sub-representation of a certain principal series representation of $Se_n$. The definition (7.6) of $\psi_\lambda$ employs a limiting process called fusion procedure [Ch2]; see Section 5. In particular, the proof of Theorem 5.6 provides an explicit expression for the element $\psi_\lambda$; see Remark 9.6 and Example 9.7. The subalgebra $M_n$ of $Se_n$ acts in $V_\lambda$ via left multiplication and the action of $Se_n$ in $V_\lambda$ factors through the homomorphism $\iota$ (Theorem 7.2). But contrary to the case of $S_n$ the representation of the group $D_n$ in the space $V_\lambda$ so obtained is reducible. If $l$ is even, it splits into the direct sum of $2^l$ copies of $\rho_{\lambda, 1}$. If $l$ is odd, it splits into the direct sum of $2^{(l-1)/2}$ copies of $\rho_{\lambda, 1} \oplus \rho_{\lambda, -1}$ (Theorem 8.3).

The vector space $V_\lambda$ comes equipped with a distinguished basis, which is an analogue of Young's basis in an irreducible representation of the group $S_n$; cf. [Ch1], [GM] and [Mu]. By considering the action of $D_n$ on the vectors of this basis, one obtains the same explicit realization of an irreducible representation of the group $T_n$ as was produced in [N1].
Denote by $Z_n$ the factor-algebra $\mathbb{C}\left[C_n^n\right]/\langle \zeta = -1 \rangle$, then for a certain element $w_\lambda \in S_n$ we have the expansion

$$
\psi_\lambda = w_\lambda + \sum_{w \in Z_n} z_w w, \quad z_w \in Z_n
$$

where $z_w \neq 0$ only if $\text{length}(w) < \text{length}(w_\lambda)$. The analogue $v_\lambda \in M_n$ of Young's symmetrizer is then equal to $\psi_\lambda w_\lambda^{-1}$ up to a scalar factor.

If $\lambda$ consists of only one part then the element $v_\lambda \in M_n$ is equal to

$$
\frac{1}{n!} \prod_{k=2}^{n} \left( \prod_{k' = 1}^{k-1} \left( 1 - \frac{(k, k')}{u - u'} \right) \right),
$$

where

$$
u = \sqrt{k(k-1)} \quad \text{and} \quad u' = \sqrt{k'(k'-1)}.
$$

In this case the equality $v_\lambda^2 = v_\lambda$ is established (Corollary 9.3). In the general case the definition of $v_\lambda$ is justified by Theorem 9.5. The proof of that theorem is based on the results of Section 6 which form the central part of the present article. We always have the equality $v_\lambda^2 = z_\lambda v_\lambda$ for some non-zero element $z_\lambda \in Z_n$ (Theorem 9.2). It may be conjectured that $z_\lambda = 1$ for any strict partition $\lambda$. This conjecture will be considered elsewhere.

1. IRREDUCIBLE REPRESENTATIONS OF THE GROUP $D_n$

In this section we shall establish the relationship between the irreducible representations of the group $D_n$ and those of the group $T_n$. Let $\lambda$ be a strict partition of $n$ into $l$ parts. The irreducible representations $\tau$ of the group $T_n$ such that $\tau(\zeta) = \text{id}$, were labelled in [S] by all the pairs $(\lambda, e)$ where $e = (\pm 1)^{n-1}$. Denote by $\tau_{\lambda, e}$ the representation corresponding to the pair $(\lambda, e)$; let $U_{\lambda, e}$ be the representation space. An explicit realization of the representation $\tau_{\lambda, e}$ was produced in [N1].

Denote by $B_n$ the inverse image in $T_n$ of the alternating subgroup $A_n \subset S_n$. The following facts are also contained in [S].

**Proposition 1.1.** (a) If $n - l$ is odd then the restrictions of $\tau_{\lambda, 1}$ and $\tau_{\lambda, -1}$ onto the subgroup $B_n$ are the same irreducible representation. We may assume

$$
\tau_{\lambda, -1}(t_k) = -\tau_{\lambda, 1}(t_k). \quad (1.1)
$$
(b) If \( n - 1 \) is even than the restriction of \( \tau_{\zeta, 1} \) onto \( B_n \) splits into two irreducible components. There exists an element \( \alpha_j \in \text{End}(U_{\zeta}) \) such that
\[
\alpha_j \tau_{\zeta, 1}(t_k) = - \tau_{\zeta, 1}(t_k) \alpha_j, \quad k = 1, \ldots, n - 1; \quad \alpha_j^2 = \text{id}. \quad (1.2)
\]

(c) All irreducible representations of the group \( B_n \) occured in (a), (b) are pairwise non-equivalent and constitute a complete family of irreducible representations.

We will denote the image of an element \( c \in Cl_n \) in the factor-algebra \( Z_n \) by the same letter. Introduce the \( Z_2 \)-gradation in \( Z_n \) by \( \deg(c) = 1 \).

**Proposition 1.2.** (a) The map \( t_k \mapsto (c_k - c_{k+1})/\sqrt{2} \) defines algebra homomorphism \( \eta: C[T_n] \to Z_n \).

(b) For any \( c \in Cl_n \) and \( s \in S_n \) the following equality holds in \( Z_n \):
\[
scs^{-1} = \eta(t) \eta(t^{-1}) \cdot (-1)^{\deg(s) \deg(c)},
\]
where \( t \) is an inverse image of \( s \) in \( T_n \).

**Proof.** Part (a) is obtained by verifying directly that the elements \( (c_k - c_{k+1})/\sqrt{2} \in Z_n \) satisfy the defining relations for the generators \( t_k \) in the group \( T_n \) when \( \zeta = -1 \).

Due to (a) it suffices to prove (b) only for \( s = s_k \) and \( c = c_{c_\xi} \), where \( k = 1, \ldots, n - 1 \) and \( k' = 1, \ldots, n \). In this case we shall assume that \( t = t_k \).

Then for \( k' \neq k, k + 1 \) the equality in (b) is evident; for \( k' = k, k + 1 \) it is again verified directly.

In this section we provide a construction of the irreducible representations of the group \( D_n \) from those of the group \( T_n \). This construction is a version of the general one given in [Cl] for the irreducible representations of semi-direct products of finite groups, see also [St2]. It will depend on the parity of \( n \). Let \( \delta = (\pm 1)^\gamma \).

Let \( n \) be even, then there is only one irreducible representation \( \nu \) of the algebra \( Z_n \). Let \( U = (C^2)^{\otimes n/2} \) be the representation space. The element \( c_0 = c_1 \cdots c_{n/2} \cdot t^{\delta n - 1}/2 \in Z_n \) has the properties similar to (1.2):
\[
c_0 c_k = - c_k c_0, \quad k = 1, \ldots, n; \quad c_0^2 = 1. \quad (1.3)
\]
Due to (1.3) and to Proposition 1.2, the following maps define a representation of the group \( D_n \) in the space \( W_{\zeta, e} = U \otimes U_{\zeta, e} \):
\[
s_k \mapsto i \cdot \nu(c_0 \eta(t_k)) \otimes \tau_{\zeta, 1}(t_k), \quad k = 1, \ldots, n - 1;
\]
\[
c_k \mapsto \nu(c_k) \otimes \text{id}, \quad k = 1, \ldots, n; \quad \zeta \mapsto -\text{id}.
\]
Denote the representation so obtained by \( \rho_\lambda, \epsilon \). Put \( \rho_\lambda = \rho_{\lambda,1} \) if \( l \) is even, and \( \rho_\lambda = \rho_{\lambda,1} \oplus \rho_{\lambda,-1} \) if \( l \) is odd. Let us observe that in this case
\[
\epsilon = (\pm 1)^{n-l} = (\pm 1)^l = \delta.
\]

Now let \( n \) be odd, then there are two irreducible non-equivalent representations \( v \) and \( v' \) of the algebra \( Z_n \) with the common representation space \( U = (\mathbb{C}^2)^{\otimes (n-1)/2} \); we may assume that \( v(c_k) = -v(c_k) \).

Endow the space \( \text{Mat}_{2\times 2} \mathbb{C} \) of the complex \( 2 \times 2 \)-matrices with the Pauli basis
\[
E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Due to Proposition 1.2, the following maps define a representation of the group \( D_n \) in the space \( W_\lambda = \mathbb{C}^2 \otimes U \otimes U_{\lambda,\epsilon} \):
\[
s_k \mapsto I \otimes v(\eta(t_k)) \otimes \tau_{\lambda,\epsilon}(t_k), \quad k = 1, \ldots, n-1; \\
c_k \mapsto J \otimes v(c_k) \otimes \text{id}, \quad k = 1, \ldots, n; \quad \zeta \mapsto -\text{id}.
\]

If \( n-l \) is odd, the representations constructed above for \( \epsilon = 1 \) and \( \epsilon = -1 \) are equivalent: by (1.1) the operator \( J \otimes \text{id} \otimes \text{id} \) in \( W_\lambda \) intertwines these representations.

Denote the representation of the group \( D_n \) in the space \( W_\lambda \) constructed above by \( \rho_\lambda \). If \( n-l \) is odd then put \( W_{\lambda,1} = W_\lambda \) and \( \rho_{\lambda,1} = \rho_\lambda \); let us remark that then \( l \) is even, and \( \delta = (\pm 1)^l = 1 \). If \( n-l \) is even then by Proposition 1.1(b) the element \( J \otimes \text{id} \otimes \text{id} \in \text{End}(W_\lambda) \) commutes with the image of \( \rho_\lambda \). In addition, this element is involutive. Therefore the representation \( \rho_\lambda \) splits into two components corresponding to the \( \pm 1 \)-eigenspaces of this element. Denote the components by \( \rho_{\lambda,1} \) and the eigenspaces by \( W_{\lambda,\pm 1} \), respectively. Let us remark that since \( n \) is odd and \( n-l \) is even, \( l \) is odd and \( \delta = \pm 1 \).

Thus for any strict partition \( \lambda \) of \( n \) and \( \delta = (\pm 1)^l \) we have constructed a representation \( \rho_{\lambda,\delta} \) of the group \( D_n \). Denote by \( \mathcal{C}_n \) the set of all such pairs \((\lambda, \delta)\).

**Theorem 1.3.** The representations \( \rho_{\lambda,\delta} \) are pairwise non-equivalent and constitute a complete family of irreducible representations of the group \( D_n \) such that \( \zeta \mapsto -\text{id} \).

**Proof.** It is shown in [Se] that the irreducible representations of the group \( D_n \) such that \( \zeta \mapsto -\text{id} \) can be labelled by elements of the set \( \mathcal{C}_n \). Therefore it suffices to prove that the representations \( \rho_{\lambda,\delta} \) are irreducible and pairwise non-equivalent.
Suppose that $n$ is even. Let us fix two pairs $(\lambda, \delta)$, $(\lambda', \delta') \in \mathcal{C}_n$ and consider an intertwining operator $\xi: W_{\lambda, \delta} \rightarrow W_{\lambda', \delta'}$. We have

$$ W_{\lambda, \delta} = U \otimes U_{\lambda, \delta} \quad \text{and} \quad W_{\lambda', \delta'} = U \otimes U_{\lambda', \delta'} . $$

The operator $\xi$ commutes with each $v(\mathcal{C}_n) \otimes \text{id}$ and the representation $v$ is irreducible. Therefore $\xi$ is of the form $\text{id} \otimes \gamma$ for some operator $\gamma: U_{\lambda, \delta} \rightarrow U_{\lambda', \delta'}$. The latter operator commutes with the action of the group $T_n$ since for each $k = 1, \ldots, n - 1$

$$ \xi p_{\lambda, \delta}(s_k) - p_{\lambda', \delta'}(s_k) \xi = i \cdot v(\mathcal{C}_n) \otimes (\gamma \tau_{\lambda, \delta}(t_k) - \tau_{\lambda', \delta'}(t_k) \gamma) = 0 . $$

Both representations $U_{\lambda, \delta}$ and $U_{\lambda', \delta'}$ are irreducible, and we get $\gamma = 0$ unless $(\lambda, \delta) = (\lambda', \delta')$. In the latter case we get $\gamma \in \mathbb{C} \cdot \text{id}$. This is the required statement.

Now suppose that $n$ is odd. Let us fix two strict partitions $\lambda, \lambda'$ of $n$ and consider an intertwining operator $\xi: W_{\lambda} \rightarrow W_{\lambda'}$. We may assume that

$$ W_j = \mathbb{C}^2 \otimes U \otimes U_{\lambda, 1} \quad \text{and} \quad W_{J_1} = \mathbb{C}^2 \otimes U \otimes U_{\lambda, 1} . $$

The operator $\xi$ commutes with each $J \otimes v(\mathcal{C}_n) \otimes \text{id}$ and the representation $v$ is irreducible. Hence

$$ \xi = E \otimes \text{id} \otimes \gamma_0 + J \otimes \text{id} \otimes \gamma_1 $$

for some operators $\gamma_0, \gamma_1: U_{\lambda, 1} \rightarrow U_{\lambda', 1}$. Also $\xi$ commutes with the action of each $s_k$:

$$ \xi p_{\lambda}(s_k) - p_{\lambda'}(s_k) \xi = J \otimes v(\mathcal{C}_n) \otimes (\gamma_0 \tau_{\lambda, 1}(t_k) - \tau_{\lambda', 1} \gamma_0 ) $$

$$ - iK \otimes v(\mathcal{C}_n) \otimes (\gamma_1 \tau_{\lambda, 1}(t_k) + \tau_{\lambda', 1} \gamma_1 ) = 0 . $$

Therefore for each $k = 1, \ldots, n - 1$ we have

$$ \gamma_0 \tau_{\lambda, 1}(t_k) = \tau_{\lambda', 1}(t_k) \gamma_0 , \quad (1.4) $$

$$ \gamma_1 \tau_{\lambda, 1}(t_k) = - \tau_{\lambda', 1}(t_k) \gamma_1 . \quad (1.5) $$

Both representations $U_{\lambda, 1}$ and $U_{\lambda', 1}$ are irreducible, so from (1.4) we obtain that $\gamma_0 = 0$ unless $\lambda = \lambda'$. In the latter case we get $\gamma_0 \in \mathbb{C} \cdot \text{id}$. It follows from (1.5) that

$$ \gamma_1 \tau_{\lambda, 1}(t) = \tau_{\lambda', 1}(t) \gamma_1 , \quad t \in B_n . \quad (1.6) $$
By Proposition 1.1(c) we obtain that \( \gamma_1 = 0 \) unless \( \lambda = \lambda' \). In the latter case from (1.6), Proposition 1.1(a, b) and from that \( n - l \) and \( l \) are of the opposite parity, we get

\[
\gamma_1 = \begin{cases} 
  u \cdot \text{id} & \text{if } l \text{ is even,} \\
  u \cdot \text{id} + v \cdot \varphi & \text{if } l \text{ is odd}
\end{cases}
\]

for some \( u, v \in \mathbb{C} \). But due to (1.5) we always have \( u = 0 \). It means that the representation \( \rho_\lambda \) is irreducible for even \( l \), while for odd \( l \) it splits into two irreducible components. These components have to be \( \rho_{\lambda, 1} \) and \( \rho_{\lambda, -1} \).

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2. SHIFTED TABLEAUX AND THE WEAK ORDER ON THE GROUP \( S_n \)

Let a strict partition \( \lambda = (\lambda_1, \ldots, \lambda_l) \) of the number \( n \) be fixed; we assume \( \lambda_1 > \cdots > \lambda_l > 0 \). A shifted tableau of shape \( \lambda \) is an array

\[
A = [A(i, j) \mid 1 \leq i \leq l, 1 \leq j \leq \lambda_i + i - 1]
\]

such that each of the numbers \( 1, \ldots, n \) occurs in \( A \) exactly once. When drawing tableaux on the plane, we assume that the coordinate \( i \) increases from the left to the right, while \( j \) increases from the top to the bottom. Denote by \( (A) \) the sequence obtained by reading the tableau \( A \) in the usual way, that is downwards by rows, from the left to the right in each row. Also denote by \( (A)^* \) the sequence obtained from reading \( A \) by columns from the left to the right, downwards in each column. For example, if

\[
A = \begin{array}{ccc}
1 & 2 & 3 \\
 & 6 & 8 \\
7
\end{array}
\]

then \( (A) = (12453687) \) and \( (A)^* = (12346758) \).

Denote by \( \mathcal{T}_\lambda \) the set of all shifted tableaux of the shape \( \lambda \). The symmetric group \( S_n \) acts transitively on the tableaux \( A \in \mathcal{T}_\lambda \) by the permutations of their entries. Define the bijection \( \mathcal{T}_\lambda \to S_n : A \to w_A \) by the equality

\[
(w_A(n) \cdots w_A(1)) = (A)^*.
\]

This bijection has the obvious property

\[
w_{s \cdot A} = s \cdot w_A, \quad s \in S_n.
\]
The tableau $A$ is called standard if $A(i, j) < A(i, j + 1)$ and $A(i, j) < A(i + 1, j)$ for all possible $i$, $j$. Denote by $\mathcal{F}$ the subset of $\mathcal{F}$ consisting of the standard tableaux. There are two distinguished elements in the set $\mathcal{F}$: the row tableau $A'$ and the column tableau $A''$; they can be defined by

$$(A')^* = (A'')^* = (1 \cdots n).$$

For instance, if $\lambda = (4, 3, 1)$ then

$$1 \ 2 \ 3 \ 4 \quad 1 \ 2 \ 4 \ 7$$
$$A' = \quad 5 \ 6 \ 7 \quad A'' = \quad 3 \ 5 \ 8$$
$$8 \quad 6$$

Let $w_0$ be the element of the maximal length in $S_n$: $w_0(k) = n - k + 1$. Consider the element $s_0 \in S_n$ such that $s_0 \colon A \mapsto A'$.

**Lemma 2.1.** We have the equality $w_0 = s_0 w_A$.

**Proof.** By the definition of the tableau $A'$ we have

$$(w_A(n) \cdots w_A(1)) = (A')^* = (1 \cdots n),$$

so $w_{A'} = w_0$. Now by the property (2.2) we get $s_0 w_A = w_{A'} \Rightarrow w_0 = w_{A'}$. 

Endow the set $S_n$ with the partial order known as weak order $[B]$: the element $s$ precedes $s'$ if and only if there are adjacent transpositions $s_{k_1}, \ldots, s_{k_p}$ such that $s' = s_{k_p} \cdots s_{k_1} s$, length($s'$) = length($s$) + $p$.

We shall write $s \preceq s'$ for this relation of precedence. For each $s \in S_n$, denote by $\mathcal{I}(s)$ the set of inversions in the substitution

$$\begin{pmatrix} 1 & \cdots & n \\ s(1) & \cdots & s(n) \end{pmatrix},$$

then

$$\text{length}(s) = \# \mathcal{I}(s).$$

**Lemma 2.2.** Let $A \in \mathcal{F}$ and $k = 1, \ldots, n - 1$. Then

(a) the elements $w_A$ and $w_{s_k A}$ are neighbouring in $S_n$ with respect to the weak order;

(b) $w_A \preceq w_{s_k A}$ if and only if the tableau $s_k \colon A$ is standard and $k$ stands in $A$ higher than $k + 1$. 

Proof. The statement (a) follows from the property (2.2) and from the equality
\[
\text{length}(s_k s) = \text{length}(s) + 1, \quad s \in S_n.
\]

Now we will verify the statement (b). Let \( k = A(i, j) \) and \( k + 1 = A(i', j') \). Since the tableau \( A \) is standard, only the following four cases are possible:
\[
\begin{align*}
i' &= i & \text{and} & \quad j' &= j + 1; \\
i' &= i + 1 & \text{and} & \quad j' &= j; \\
i' &= i & \text{and} & \quad j' &= j + 1; \\
i' &= i + 1 & \text{and} & \quad j' &= j - 1.
\end{align*}
\]

Exactly in the fourth case the number \( k + 1 \) occurs before \( k \) in the sequence \((A)^*\). By definition (2.1), it means that the set \( \mathcal{I}(w_{sk}, A) = \mathcal{I}(s_k w_{A}) \) is greater than \( \mathcal{I}(w_{A}) \) by the inversion of \( k \) and \( k + 1 \). Also only in the fourth case both conditions of (b) are satisfied: the tableau \( s_k \cdot A \) is standard and \( k \) stands in \( A \) higher than \( k + 1 \). Thus (b) is also verified.

Put \( w_\beta = w_{A'} \). The set \( \mathcal{S}_\beta \) can be alternatively described by the following

**Proposition 2.3.** We have the equality \( \{ w_\beta \mid A \in \mathcal{S}_\beta \} = \{ s \in S_n \mid s \geq w_{A} \} \).

**Proof.** Suppose that \( s \geq w_{A} \). Then there exist indices \( k_1, \ldots, k_p \) such that \( s = s_{k_1} \cdots s_{k_p} w_{A'} \) and
\[
s_{k_p} \cdots s_{k_1} w_{A'} \prec s_{k_p} s_{k_p-1} \cdots s_{k_1} w_{A'}; \quad q = 1, \ldots, p.
\]
Consider the tableaux \( A_q = s_{k_q} \cdots s_{k_1} A' \). We have \( w_{A_q} = s_{k_q} \cdots s_{k_1} w_{A'} \) by (2.2). So
\[
A_q = s_{q} \cdot A_{q-1}, \quad w_{A_q} \succ w_{A_{q-1}}
\]
where \( A_0 = A' \). Since the tableau \( A' \) is standard, applying Lemma 2.2(b) consecutively to
\[
A = A', A_1, \ldots, A_{p-1} \quad \text{and} \quad k = k_1, k_2, \ldots, k_p
\]
we prove that the tableaux \( A_1, \ldots, A_p \) are also standard. But \( s = w_{A_q} \), and we have established the inclusion
\[
\{ w_\beta \mid A \in \mathcal{S}_\beta \} \supset \{ s \in S_n \mid s \geq w_{A} \}.
\]
Now let us fix a tableau $A \in \mathcal{S}_k^*$; we assume that $A \neq A'$. Suppose there exists a number $k$ such that the tableau $s_k \cdot A$ is standard and $k + 1$ stands in $A$ higher than $k$. Put $A' = s_k \cdot A$, then by (2.2) and Lemma 2.2(b) we have $w_A = s_k \cdot A'$ and $w_A > w_{A'}$. By repeating this argument, we get a sequence of indices $k_0, \ldots, k_1$ and a standard tableau $A_0$ such that

$$s_{k_0-1} \cdots s_{k_1} w_{A_0} \prec s_{k_0} s_{k_0-1} \cdots s_{k_1} w_{A_0}; \quad q = 1, \ldots, p$$

and for each $k = 1, \ldots, n - 1$ either the tableau $s_k \cdot A_0$ is not standard, or it is but $k$ stands in $A_0$ higher than $k + 1$. This property of $A_0$ implies that $A_0 = A'$. Indeed, suppose that $A_0 \neq A'$. Let $k + 1$ be the least of the numbers which have different positions in the tableaux $A_0$ and $A'$. Let $k = A_0(i, j)$, then $j < i + 1$. Define the number $k' = A_0(i', j')$ by the equality $k' + 1 = A_0(i, j + 1)$.

Since the tableau $A_0$ is standard, we have $k' + 1 > k$. But $k + 1$ has different positions in $A_0$ and $A'$, that is $k + 1 \neq k' + 1$. Therefore $k' > k$. Now it follows from the definition of $k + 1$ that either $i' > i$, or $i' = i$ and $j' > j$. The latter case is impossible since $A_0(i, j + 1) = k' + 1$ and $A_0$ is standard. Furthermore, the condition $A_0 \in \mathcal{S}_k^*$ implies that $j' < j + 1$. Therefore the tableau $s_{k'} \cdot A_0$ is also standard, and the number $k' + 1$ stands in $A_0$ higher than $k'$. This contradiction proves that $A_0 = A'$. Thus we have established the inclusion

$$\{w_A \mid A \in \mathcal{S}_k^*\} \subset \{s \in S_n \mid s \succ w_{A'}\}.$$

Let us now provide some reduced decompositions for the elements $w_A$ and $s_A$, where $A \in \mathcal{S}_k^*$. Let a standard shifted tableau $A$ of the form $\lambda$ be fixed. For each $k$ denote by $\mathcal{B}_k$ and $\mathcal{B}'_k$ the subsequences of $(\lambda)$ consisting of all the numbers $k' < k$ which occur respectively before and after $k$ in that sequence. Let $b_k$ and $b'_k$ denote the lengths of the sequences $\mathcal{B}_k$ and $\mathcal{B}'_k$ respectively.

**Lemma 2.4.** There are reduced decompositions

$$w_A = \prod_{k=2}^{\infty} (s_{k-1} \cdots s_k - b_k), \quad s_A = \prod_{k=2}^{\infty} (s_{k-1} - b'_k \cdots s_k - 1).$$

**Proof.** We will employ the induction on $n$. If $n = 1$ then $w_A = s_A = 1$ while there is no factors in the above products, so Lemma 2.4 becomes trivial.

Suppose that $n > 1$. Observe that the total number of factors in the above products is

$$\sum_{k=2}^{n} (b_k + b'_k) = \sum_{k=2}^{n} (k - 1) = \text{length}(w_A).$$
therefore if we verify both decompositions, we shall obtain from Lemma 2.1 that they are reduced. Moreover, then we will get

\[ \text{length}(w_0) = \text{length}(w_4) + \text{length}(s_i). \quad (2.3) \]

Let \( \Omega \) be the array obtained by taking the number \( n \) out of \( A \). Evidently, it is a standard shifted tableau for a certain strict partition of \( n-1 \). Moreover, for each \( k < n \) the subsequences of \( (\Omega)^* \) consisting of all the numbers \( k' < k \) which occur before and after \( k \) in that sequence, coincide with \( \theta_k \) and \( \theta'_k \), respectively. Therefore by the inductive assumption we get the decompositions

\[ w_\Omega = \prod_{k=2}^{n-1} (s_{k-1} \cdots s_k), \quad s_\Omega = \prod_{k=2}^{n-1} (s_{k-1} \cdots s_{k-1}). \]

The sequence \((\Omega)^*\) is obtained by taking the number \( n \) out of \((A)^*\), and \( n \) occupies the place \( b_n+1 \) in the latter sequence. But by the definition (2.1) we have

\[ (w_\Omega(n) \cdots w_\Omega(1)) = (A)^*, \quad (w_\Omega(n-1) \cdots w_\Omega(1)) = (\Omega)^*, \]

therefore \( w_\Omega = w_{\Omega} s_{n-1} \cdots s_{n-b_n} \). Thus the first equality of Lemma 2.4 is proved.

Since \( s_A \cdot A = A' \), acting by the permutation \( s_A \) on the entries of the sequence \((A)^*\) one obtains the sequence \((1 \cdots n)\). Acting by \( s_\Omega \) on the same entries, one gets the sequence \((1 \cdots b_n, n, n-b'_n, \ldots n-1)\). Therefore here \( s_A = s_{n-b'_n} \cdots s_{n-1} s_\Omega \). So the second equality of Lemma 2.4 is proved.

Due to (2.3) Lemma 2.1 and Lemma 2.4 provide a reduced decomposition of the element \( w_0 \). In particular, for \( A = A' \) we get the obvious reduced decomposition

\[ w_{\cdot A} = w_0 = \prod_{k=2}^{n} (s_{k-1} \cdots s_k). \]

We will also use decompositions of the element \( w_0 \) of another kind. Let us define the bijection \( \mathcal{F}_A : S_n \rightarrow S_n \): \( A \mapsto w^*_A \) in the way parallel to (2.1):

\[ (w^*_A(n) \cdots w^*_A(1)) = (A). \quad (2.4) \]

As well as \( A \mapsto w_A \), this bijection commutes with the action of the group \( S_n \):

\[ w^*_A \cdot s = s w^*_A, \quad s \in S_n. \quad (2.5) \]

Put \( s_A = s_{\cdot A} \); that is \( s_A : A' \mapsto A' \).
**Lemma 2.5.** We have the equality \( s_2 w_0 = s_A w_A^* \).

*Proof.* It is quite similar to that of Lemma 2.1. By the definition of the tableau \( A' \) we have

\[
(w_A^*(n) \cdots w_A^*(1)) = (A') = (1 \cdots n),
\]

therefore \( w_A^* = w_0 \). Now by the property (2.5) we get

\[
s_2 w_0 = s_2 w_A^* = w_A^* A' = w_A^* A = s_A w_A^*.
\]

Now define the element \( s_A^* \in S_n \) by the equality

\[
w_0 = w_A s_A^*,
\]

then due to Lemma 2.5 and to the equality \( w_0^2 = 1 \) we get

\[
s_A^* = (w_A^*)^{-1} w_0 = w_0 s_A^{-1} s_A w_0.
\]

Again, let a standard shifted tableau \( A \) of the form \( \lambda \) be fixed. For each \( k \) denote by \( \alpha_k \) and \( \alpha'_k \) the subsequences of \( (A) \) consisting of all the numbers \( k' < k \) which occur respectively before and after \( k \) in that sequence. Let \( a_k \) and \( a'_k \) denote the lengths of the sequences \( \alpha_k \) and \( \alpha'_k \) respectively.

**Lemma 2.6.** There are reduced decompositions

\[
w_A^* = \prod_{k=2}^{\infty} (s_{k-1} \cdots s_{k-a_k}), \quad s_A^* = \prod_{k=2}^{\infty} (s_{n-k+a'_k} \cdots s_{n-k+1}).
\]

*Proof.* As well as in the proof of Lemma 2.4, we will use the induction on \( n \). If \( n = 1 \), then \( w_A^* = s_A^* = 1 \) while there is no factors in the above products, so Lemma 2.6 becomes trivial. Suppose that \( n > 1 \). Observe that the total number of factors in the above products is equal to

\[
\sum_{k=2}^{n} (a_k + a'_k) = \sum_{k=2}^{n} (k - 1) = \text{length}(w_0),
\]

therefore if we verify both decompositions, we will immediately obtain from (2.6) that they are reduced. Moreover, then we will get

\[
\text{length}(w_0) = \text{length}(w_A^*) + \text{length}(s_A^*).
\]

Consider the tableau \( \Omega \) obtained by taking the number \( n \) out of \( A \). Evidently, it is a column tableau for a certain strict partition of \( n-1 \).
Moreover, for each $k < n$ the subsequences of $(\Omega)$ consisting of all the numbers $k' < k$ which occur before and after $k$ in that sequence, coincide respectively with $s_k$ and $s'_k$. Therefore by the inductive assumption we get the decomposition

$$w_n^* = \prod_{k=2}^{n-1} (s_{k-1} \cdots s_{k-a})_n.$$  

The sequence $(\Omega)$ is obtained by taking the number $n$ out of $(A)$, and $n$ occupies the place $a_n+1$ in the latter sequence. But by the definition (2.4) we have

$$(w^*_n(n) \cdots w^*_n(1)) = (A), \quad (w^*_n(n-1) \cdots w^*_n(1)) = (\Omega);$$

therefore $w_n^* = w_n^*_{n-1} \cdots s_{n-a}$. The first equality of Lemma 2.6 is proved.

Let us verify the second equality. By applying the definition (2.6) to the group $S_{n-1}$ and making use of the inductive assumption, we obtain

$$w_0 = w_n^*_2 \cdot \prod_{k=2}^{n-1} (s_{n-k+a'_k-1} \cdots s_{n-k})_n \cdot (s_{n-1} \cdots s_1)$$

$$= w_n^*_2(s_{n-1} \cdots s_1) \cdot \prod_{k=2}^{n-1} (s_{n-k+a'_k} \cdots s_{n-k+1})$$

$$= w_n^*(s_{n-1} \cdots s_{n-a}) \cdot (s_{a'-1} \cdots s_1) \cdot \prod_{k=2}^{n-1} (s_{n-k+a'_k} \cdots s_{n-k+1})$$

$$= w_n^*(\prod_{k=2}^{n} (s_{n-k+a'_k} \cdots s_{n-k+1});$$

here we have used the equality $a_n + a'_n = n-1$ and the first decomposition of Lemma 2.6. Now the second decomposition follows from the definition (2.6).

Due to (2.8) the equality (2.6) along with Lemma 2.6 provides a reduced decomposition of the element $w_0$.

Thus for each tableau $A \in \mathcal{S}$ we have defined four elements $s_A$, $w_A$ and $s'_A$, $w'_A$ of the group $S_n$. We will assign to each $A \in \mathcal{S}$ one more element $g_A \in S_n$. Consider the sequence

$$(A) = (A(1,1) \cdots A(l,l)).$$
Denote by $\langle A \rangle'$ the sequence obtained by taking all the numbers $A(i, i)$ out of $(1 \ldots n)$. Let

$$g_A: \langle A \rangle \mapsto \langle A' \rangle, \langle A \rangle' \mapsto \langle A' \rangle'.$$

(2.9)

By this definition, we have $g_{A'} = 1$.

**Lemma 2.7.** Let $w_A < w_{A'}$. Then $s_A: A \in S_n$ and

$$g_A s_A = \begin{cases} g_A & \text{if } k + 1 \in \langle A \rangle; \\ g_{A'} & \text{otherwise.} \end{cases}$$

**Proof.** Thanks to Lemma 2.2(b), the assumption $w_A < w_{A'}$ implies that $s_A: A \in S_n$. Moreover, the number $k$ stands in $A$ higher than $k + 1$. Since the tableau $A$ is standard, only the two cases are possible: either $k \in \langle A \rangle'$, $k + 1 \in \langle A \rangle'$ or $k, k + 1 \in \langle A \rangle'$. Now the statement of Lemma 2.7 becomes evident.

3. DEGENERATE AFFINE SERGEEV ALGEBRA

In this section we introduce the main underlying object of the present article. It is the associative algebra generated by the factor-algebra $M_n = \mathbb{C}[D_n]/\langle \zeta = -1 \rangle$ and the pairwise commuting elements $x_1, \ldots, x_n$ subject to the relations

$$x_k s_k = s_k x_k \quad \text{if } k' \neq k, k + 1, \quad x_k s_k - x_k x_{k+1} = -1 - c_k c_{k+1};$$

(3.1)

$$x_k c_k = c_k x_k \quad \text{if } k' \neq k, \quad x_k c_k = -c_k x_k.$$  

(3.2)

This is an analogue of the degenerate affine Hecke algebra $He_n$, which was considered in [Ch1] and [D], [L]. We will denote the above defined algebra by $Se_n$ and call it the *degenerate affine Sergeev algebra* in honour of A. Sergeev who has used the group $D_n$ to describe the irreducible polynomial representations of the queer classical Lie superalgebra [Se].

**Proposition 3.1.** (a) The subalgebra $\mathbb{C}[x_1, \ldots, x_n] \subset Se_n$ is maximal commutative.

(b) The centre of $Se_n$ consists of all symmetric polynomials in $x_1^2, \ldots, x_n^2$.

**Proof (cf. [Ch3]).** For each $h \in \mathbb{C}$ consider the algebra $Se_n(h)$ obtained from $Se_n$ by replacing the second relation in (3.1) by

$$x_k s_k - x_k x_{k+1} = h(-1 - c_k c_{k+1}).$$
Evidently $S_{e_n} = S_{e_n}(1)$, so the algebra $S_{e_n}(h)$ is a deformation of $S_{e_n}$. Moreover, this deformation is trivial for $h \neq 0$: an algebra isomorphism $S_{e_n} \to S_{e_n}(h)$ can be defined by the maps

$$x_k \mapsto h^{-1}x_k, \quad k = 1, \ldots, n.$$ 

The algebra $S_{e_n}(0)$ is the factor by the relation $\zeta = -1$ of the semi-direct product of the group $D_n$ and the commutative algebra $C[x_1, \ldots, x_n]$. Here the group $D_n$ acts on $x_1, \ldots, x_n$ through its factor-group $C_{n}$:

$$sx_k = x_{\rho(k)}s, \quad s \in S_n; \quad \zeta x_k = x_k \zeta; \quad \zeta_s x_k = x_k \zeta_s; \quad c_k x_k = x_k c_k \quad \text{if} \quad k' \neq k; \quad c_k x_k = -x_k c_k.$$

In particular, the subalgebra $C[x_1, \ldots, x_n] \subset S_{e_n}(0)$ is maximal commutative.

For any $h$ each element of the algebra $S_{e_n}(h)$ can be expressed as a sum of products of the form $m x_1^d \cdots x_n^d$, where $m \in M_n$. For each $d = 0, 1, 2, \ldots$ consider the linear subspace in $S_{e_n}(h)$

$$M_n \cdot \{ x_1^{d_1} \cdots x_n^{d_n} | d_1 + \cdots + d_n \leq d \}.$$

These subspaces are finite-dimensional and exhaust the algebra $S_{e_n}(h)$. On the other hand, the relations in the algebra $S_{e_n}(h)$ depend on $h$ continuously. Therefore $C[x_1, \ldots, x_n]$ is a maximal commutative subalgebra in $S_{e_n}(h)$ if $h$ is close enough to 0. But the algebras $S_{e_n}(h)$ for all $h \neq 0$ are all isomorphic, and the statement (a) follows.

To prove (b) let us first check that the symmetric polynomials in $x_1^2, \ldots, x_n^2$ belong to the centre of the algebra $S_{e_n}$. In the latter algebra we have

$$s_k(x_k + x_{k+1})x_k = x_k + x_{k+1} - 2h c_k c_{k+1};$$

$$s_k(x_k + x_{k+1})^2 x_k = (x_k + x_{k+1} - 2h c_k c_{k+1})^2 = (x_k + x_{k+1})^2;$$

$$s_k(x_k x_{k+1})x_k = (x_{k+1} - s_k(1 + c_k c_{k+1}))(x_k + s_k (1 - c_k c_{k+1}))$$

$$= x_k x_{k+1}.$$ 

It follows that $s_k$ commutes with $x_1^2 + x_2^2, \ldots, x_k^2 + x_{k+1}^2$, ... in the algebra $S_{e_n}$. By the first relation in (3.1) this fact implies that $s_1, \ldots, s_{n-1}$ commute with the elements

$$x_1^2 + \cdots + x_n^2, \quad x_1^4 + \cdots + x_n^4,$$

in the algebra $S_{e_n}$. The latter elements also commute with $c_1, \ldots, c_n$ due to (3.2). Therefore these elements belong to the centre of $S_{e_n}$. 

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Thus we have verified that all symmetric polynomials in $x_1^2, \ldots, x_n^2$ belong to the centre of the algebra $S_{e_n}$. As an argument similar to that used in (a) shows, these polynomials exhaust the centre.

For each $k = 1, \ldots, n-1$ define the element of the algebra $S_{e_n}$

$$\Phi_k = s_k(x_k^2 - x_{k+1}^2) + (x_k + x_{k+1}) - c_k c_{k+1}(x_k - x_{k+1}).$$

It follows directly from the relations (3.1), (3.2) that

$$\Phi_k x_{k'} = x_{k'} \Phi_k \quad \text{if} \quad k' \neq k, k+1; \quad (3.3)$$

$$\Phi_k x_{k+1} = x_{k+1} \Phi_k; \quad \Phi_k x_k = x_k \Phi_k. \quad (3.4)$$

Now we are able to prove the next proposition.

**Proposition 3.2.** The following relations hold in the algebra $S_{e_n}$:

$$\Phi_k^2 = -(x_k^2 - x_{k+1}^2)^2 + (x_k + x_{k+1})^2 + (x_k - x_{k+1})^2;$$

$$\Phi_k \Phi_{k'} = \Phi_{k'} \Phi_k \quad \text{if} \quad k' - k > 1,$$

$$\Phi_k \Phi_{k+1} = \Phi_{k+1} \Phi_k. \quad (3.5)$$

**Proof.** The second equality of Proposition 3.2 is provided directly by (3.1), (3.2). Let us verify the first equality. Due to (3.4) we have

$$\Phi_k^2 = (s_k(x_k^2 - x_{k+1}^2) + (x_k + x_{k+1}) - c_k c_{k+1}(x_k - x_{k+1})) \Phi_k$$

$$= -s_k \Phi_k(x_k^2 - x_{k+1}^2) + \Phi_k(x_k + x_{k+1}) + c_k c_{k+1} \Phi_k(x_k - x_{k+1})$$

$$= -(x_k^2 - x_{k+1}^2)^2 + (x_k + x_{k+1})^2 + (x_k - x_{k+1})^2.$$

The verification of the third relation in Proposition 3.2 is also straightforward, but rather lengthy. Here we will omit that calculation.

Let us fix an element $s \in S_n$ together with its reduced decomposition $s = s_{s_1} \cdots s_{s_1}$. Define

$$\Phi_s = \Phi_{s_1} \cdots \Phi_{s_1},$$

by the second and the third equalities of Proposition 3.2 this definition does not depend on the choice of the reduced decomposition. It follows from (3.4) and from this definition that

$$\Phi_s x_k = x_k \Phi_s; \quad k = 1, \ldots, n. \quad (3.5)$$
The elements $\Phi_s \in S_n$ will play the key role in the present paper. Let us fix a character $\chi$ of the algebra $\mathbb{C}[x_1, \ldots, x_n]$. Consider the representation $\pi_\chi$ of the algebra $S_n$ induced from the character $\chi$. Let us identify the space $M_\pi$ of the representation $\pi_\chi$ with the algebra $M_n$. The elements $s_k$ and $c_k$ act in $M_\pi$ via left multiplication, while the action of $x_1, \ldots, x_n$ is determined through (3.1) by

$$x_k \cdot m = (x_k m) \cdot 1, \quad m \in M_n; \quad x_k \cdot 1 = \chi(x_k). \quad (3.6)$$

Consider the standard action of the group $S_n$ on the characters $\chi$:

$$s \cdot \chi(x_k) = \chi(x_{s^{-1} k}), \quad s \in S_n.$$

**Proposition 3.3.** The operator of the right multiplication in $M_n$ by $\pi_\chi(\Phi_s)(1)$ is an intertwining operator $M_n \to M_n$.

**Proof.** Let $\mu$ denote the operator of the right multiplication in $M_n$ by $\pi_\chi(\Phi_s)(1)$. The action of $s_1, \ldots, s_{n-1}$ and $c_1, \ldots, c_n$ in the vector space $M_\pi = M_n = M_n$ is the left multiplication and commutes with $\mu$. Since the vector $1 \in M_n$ is cyclic for this action, it suffices to check that the actions of $\pi_\chi(x_k)\mu$ and $\mu, \pi_\chi(x_k)$ on $1 \in M_n$ coincide for any $k$. Thanks to (3.5) and (3.6) we have

$$\pi_\chi(x_k)\mu(1) = \pi_\chi(x_k)(\pi_\chi(\Phi_s)(1)) = \pi_\chi(x_k \Phi_s)(1) = \pi_\chi(\Phi_s x_{s^{-1} k})(1) = \chi(x_{s^{-1} k}) \pi_\chi(\Phi_s)(1) = \mu \pi_\chi(x_k)(1),$$

and Proposition 3.3 follows. □

By (3.6) and Proposition 3.3, the element $\pi_\chi(\Phi_s)(1) \in M_n$ is an eigenvector for the operators $\pi_\chi(x_k); k = 1, \ldots, n$. Namely, we have the following

**Corollary 3.4.** For each $k = 1, \ldots, n$ and any $s \in S_n$ we have

$$\pi_\chi(x_k)\pi_\chi(\Phi_s)(1) = s \cdot \chi(x_k) \pi_\chi(\Phi_s)(1).$$

**Proposition 3.5.** One can define a homomorphism $i : S_n \to M_n$ identical on $M_n$ by $x_k \mapsto 0$. Then

$$i(x_k) = \sum_{k' = 1}^{k-1} (1 + c_k c_{k'}) \cdot (k, k'); \quad k = 1, \ldots, n \quad (3.7)$$

where $(k, k')$ denotes the transposition of $k$ and $k'$ in $S_n$. 
Proof. Let \( \iota \colon \mathfrak{Se}_n \to \mathfrak{M}_n \) be a homomorphism identical on the subalgebra \( \mathfrak{M}_n \subseteq \mathfrak{Se}_n \). One can rewrite the second relation in (3.1) as

\[ x_{k+1} = s_k x_k s_k + (1 + c_k + c_k) s_k. \]

Therefore if \( \iota(x_k) = 0 \) then by applying the latter relation consecutively to \( k = 1, 2, \ldots, n - 1 \) we obtain all the equalities (3.7). To complete the proof of Proposition 3.5 we have to verify that the elements \( \iota(x_k) \in \mathfrak{M}_n \) obey the defining relations for the generators \( x_k \in \mathfrak{Se}_n \).

The relations (3.1), (3.2) for the elements \( \iota(x_k) \) instead of \( x_k \) are verified directly. Furthermore, if \( k' < k \) then the commutator \( [\iota(x_k), \iota(x_{k'})] \) equals the sum over \( k = 1, \ldots, k' - 1 \) of the commutators

\[ [(1 + c_k c_k') \cdot (k, k') + (1 + c_k c_k') \cdot (k, k') + (1 + c_k' c_k') \cdot (k', k')]. \]

Each of the latter commutators is equal to zero.

4. THE ELEMENTS \( \pi_j(\Phi_{\alpha_j})(1) \)

As well as in Section 2, let a strict partition \( \lambda \) of \( n \) be fixed. Then we have two bijections \( \mathcal{F}_\lambda \to \mathfrak{S}_n \) defined by the formulas (2.1) and (2.4). Let us also fix an array of the shape \( \lambda \) with arbitrary complex entries

\[ u(i, j) \mid 1 \leq i \leq l, 1 \leq j \leq \lambda_i + i - 1 \].

Define the character \( \chi \) by the equality \( w_{\lambda'} \cdot \chi(x_k) = u(i, j) \), where \( k = A'(i, j) \). Then for any \( A \in \mathcal{F}_\lambda \) and each \( k = 1, \ldots, n \) we have

\[ w_A \cdot \chi(x_k) = u(i, j), \quad A(i, j) = k. \quad (4.1) \]

Indeed, let \( A = s \cdot \lambda' \), then \( A'(i, j) = s^{-1}(k) \) and due to (2.2) we get

\[ w_A \cdot \chi(x_k) = w_{s \cdot \lambda'} \cdot \chi(x_k) = w_{\lambda'} \cdot \chi(x_s^{-1}(k)) = u(i, j). \]

We will consider the family of the elements \( \pi_j(\Phi_{\alpha_j})(1) \in \mathfrak{M}_n \), where \( A \) runs through the set \( \mathcal{F}_\lambda \). Let us write down the explicit formulas for these elements. Put

\[ \varphi_k(u, u') = s_k(u^2 - u'^2) + (u + u') - c_k c_{k+1}(u - u'). \]
It follows from (3.6) that if \( \chi(x_k) = u \) and \( \chi(x_{k+1}) = u' \), then
\[
\pi_{\chi}(\Phi_{\chi})(1) = \varphi_{\chi}(u, u').
\] (4.2)

Let a tableau \( A \in \mathcal{F}_2 \) be fixed. Then we have defined the sequences
\[
\mathcal{B}_k = (\mathcal{B}_k(p) \mid 1 \leq p \leq b_k), \quad \mathcal{B}_k' = (\mathcal{B}_k'(q) \mid 1 \leq q \leq b_k').
\]

We will use the reduced decomposition for the elements \( w_k, s_k \in S_n \) provided by Lemma 2.4. Thanks to Lemma 2.1, (2.3) and to Proposition 3.3 we have
\[
\pi_{\chi}(\Phi_{w_k})(1) = \pi_{\chi}(\Phi_{s_k})(1) \cdot \pi_{\chi}(\Phi_{s_k})(1). \quad (4.3)
\]

**Proposition 4.1.** Put \( u_k = w_{k+1}(x_k) \). Then
\[
\pi_{\chi}(\Phi_{w_k})(1) = \prod_{k=2}^{n} \left( \prod_{p=1}^{b_k} \varphi_{k-p}(u_k, u_{s_k(p)}) \right).
\]
\[
\pi_{\chi}(\Phi_{s_k})(1) = \prod_{k=2}^{n} \left( \prod_{q=1}^{b_k'} \varphi_{k-q}(u_{s_k(q)}, u_{s_k(q+1)}, u_k) \right).
\]

**Proof.** Let an index \( k \geq 2 \) be fixed. Consider the array \( \Omega \) obtained by taking the numbers \( k, \ldots, n \) out of \( A \). Evidently, it is a standard shifted tableau for a certain strict partition of \( k - 1 \). For each \( k' \leq k \) the subsequences of \( (\Omega)^* \) consisting of all the numbers \( k'' < k' \) which occur before and after \( k' \) in that sequence, coincide respectively with \( \mathcal{B}_k \) and \( \mathcal{B}_k' \). In particular, when \( k' = k \) we get
\[
\mathcal{B}_k = (\Omega)^* .
\] (4.4)

Furthermore, by the definition of the element \( s_k \in S_{k-1} \) we have
\[
\mathcal{B}_k = (s_{k-1}(k-b_k') \cdots s_{k-1}(k-1)).
\] (4.5)

We will verify the formula for the element \( \pi_{\chi}(\Phi_{w_k})(1) \) first. Let an index \( p \leq b_k \) be fixed. By Lemma 2.4 we get the decomposition \( w_k = w_2 s_{k-1} \cdots s_{k-p} \) for some \( s \in S_n \) such that
\[
\text{length}(w_k) = \text{length}(w_2) + p + \text{length}(s).
\]
To get the required formula it suffices to prove that
\[
\pi_{\chi}(\Phi_{k-p} \Phi_{s})(1) = \varphi_{k-p}(u_k, u_{s(p)}) \cdot \pi_{\chi}(\Phi_{s})(1).
\]
Put \( w = w_{\Omega} s_{k-1} \cdots s_{k-p} \), then we have \( w s = w s_{A} \) and
\[
w(k - p) = w_{\Omega}(k) = k,
\]
\[
w(k - p + 1) = w_{\Omega} s_{k-1} \cdots s_{k-p+1}(k - p) = w_{\Omega}(k - p) = K(k(p));
\]
where the last equality follows from (2.1) and (4.4). Therefore by the definition of the numbers \( u_{1}, \ldots, u_{n} \) we have
\[
s \cdot x k_{(k-1)} = w s \cdot x (k_{(k-1)}) = w_{A} \cdot x k_{(k)} = u_{k},
\]
\[
s \cdot x (k_{(k-1)+1}) = w s \cdot x (k_{(k-1)+1}) = w_{A} \cdot x (k_{(k-1)+1}) = u_{k}.
\]
Now due to Corollary 3.4 and to (4.2) we get
\[
\pi_{s}(\Phi_{k-\Omega})_{(1)} = \pi_{s} x_{(k-\Omega)}(1) \pi_{s} x_{(k)}(1) = \psi_{k-\Omega}(u_{q}, u_{k}).
\]
Thus the first formula of Proposition 4.1 is verified.

Verification of the formula for \( \pi_{s} x_{(k-\Omega)}(1) \) is similar. Let an index \( q \leq b'_{k} \) be fixed. By Lemma 2.4 we have the decomposition \( s_{A} = s_{s_{k-1}} \cdots s_{k-1} s_{\Omega} \) for some \( s' \in S_{n} \) such that
\[
\text{length}(s_{A}) = \text{length}(s') + q + \text{length}(s_{\Omega}).
\]
Put
\[
t = s_{k-1} \cdots s_{k-1} s_{\Omega}, \quad q' = b'_{k} - q + 1.
\]
Due to Corollary 3.4 to get the required formula it suffices to verify that
\[
\pi_{s} x_{(k-\Omega)}(1) = \psi_{k-\Omega}(u_{q'}, u_{k}). \tag{4.6}
\]
By the equality (4.5) we have
\[
t^{-1}(k - q) = s_{\Omega}^{-1}(k - q) = q', \quad t^{-1}(k - q + 1) = s_{\Omega}^{-1}(k) = k;
\]
therefore by the definition of the numbers \( u_{1}, \ldots, u_{n} \) we get
\[
w_{A} \cdot x (k_{q-q}) = w_{A} \cdot x (k_{q-q+1}) = w_{A} \cdot x k_{q} = u_{q'},
\]
\[
w_{A} \cdot x (k_{q+1}) = w_{A} \cdot x (k_{q+1}) = w_{A} \cdot x k_{q} = u_{k}.
\]
Now the equality (4.6) follows from (4.2).

Let a tableau \( A \in S'_{k} \) still be fixed. Then we have defined the sequences
\[
A_{k} = (A_{k}(p) \mid 1 \leq p \leq a_{k}), \quad A'_{k} = (A'_{k}(q) \mid 1 \leq q \leq a'_{k}).
\]
Thanks to (2.6), (2.8) and to Proposition 3.3 we have
\[
\pi_x(\Phi_{w_0})(1) = \pi_{x_1}(\Phi_{x_1})(1) \cdot \pi_x(\Phi_{x_2})(1). \tag{4.7}
\]
Note that in the next proposition the numbers \(u_1, \ldots, u_n\) are determined through the tableau \(A'\). This proposition will be used for \(A = A'\) only. But to draw further the parallel between the bijections \(A \mapsto w_A\) and \(A \mapsto w_A^*\), we prove it for an arbitrary \(A \in \mathcal{G}_i\).

**Proposition 4.2.** Let \(u_k = w_{x_k} \cdot x(x_k)\), then
\[
\begin{align*}
\pi_{x_1}(\Phi_{x_1})(1) &= \prod_{k=2}^{n} \left( \prod_{p=1}^{a_{k-1}} \varphi_{k-p}(u_k, u_{s_0(p)}) \right), \\
\pi_{x_2}(\Phi_{x_2})(1) &= \prod_{k=2}^{n} \left( \prod_{q=1}^{a_{k-1}} \varphi_{n-k+q}(u_k, u_{s_2(k-1-q)}) \right).
\end{align*}
\]

**Proof.** Let an index \(k \geq 2\) be fixed. Consider the array \(\Omega\) obtained by taking the numbers \(k, \ldots, n\) out of \(A\). Evidently, it is a standard shifted tableau for a certain strict partition \(\omega\) of \(k-1\). For each \(k' \leq k\) the sub-sequences of \((\Omega)\) consisting of all the numbers \(k'' < k'\) which occur before and after \(k'\) in that sequence, coincide respectively with \(\mathscr{A}_{k'}\) and \(\mathscr{A}_{k''}\). In particular, when \(k' = k\) we get
\[
\mathscr{A}_k = (\Omega). \tag{4.8}
\]
Furthermore, since \(s_{\Omega}^{-1} s_\omega \cdot \Omega = \Omega\) we have
\[
\mathscr{A}_k = (s_{\Omega}^{-1} s_\omega(k - a_k') \cdots s_{\Omega}^{-1} s_\omega(k - 1)). \tag{4.9}
\]
We will verify the formula for the element \(\pi_{x_1}(\Phi_{x_1})(1)\) first. Let an index \(p \leq a_k\) be fixed. Due to Lemma 2.6 we have the decomposition \(w_A^* = w_{s_{\Omega}^{-1} s_\omega(k - a_k') \cdots s_{\Omega}^{-1} s_\omega(k - 1)}^* t\) for some \(t \in S_n\) such that
\[
\text{length}(w_A^*) = \text{length}(w_{s_{\Omega}^{-1} s_\omega(k - a_k') \cdots s_{\Omega}^{-1} s_\omega(k - 1)}) + p + \text{length}(t).
\]
To get the required formula it suffices to prove that
\[
\pi_{x_1}(\Phi_{k-p})(1) = \varphi_{k-p}(u_k, u_{s_0(p)}). \tag{4.10}
\]
Put \(w = w_{s_{\Omega}^{-1} s_\omega(k - a_k') \cdots s_{\Omega}^{-1} s_\omega(k - 1)}^* t\), then due to (2.6) we have \(w_0 = w_{s_{\Omega}^{-1} s_\omega(k - a_k') \cdots s_{\Omega}^{-1} s_\omega(k - 1)}^*\) while
\[
w(k - p) = w_{s_{\Omega}^{-1} s_\omega(k - a_k') \cdots s_{\Omega}^{-1} s_\omega(k - 1)}^*(k) = k, \quad w(k - p + 1) = w_{s_{\Omega}^{-1} s_\omega(k - a_k') \cdots s_{\Omega}^{-1} s_\omega(k - 1)}^*(k + 1) = \mathscr{A}_k(p).
\]
where the last equality is provided by (2.4) and (4.8). Therefore by the definition of the numbers $u_1, \ldots, u_n$ we have

$$\ell s^* \cdot \mathcal{Z}(x_{k-p}) = w s^* \cdot \mathcal{Z}(x_{n(k-p)}) = w_0 \cdot \mathcal{Z}(x_k) = u_k,$$

$$\ell s^* \cdot \mathcal{Z}(x_{k-p+1}) = w s^* \cdot \mathcal{Z}(x_{n(k-p+1)}) = w_0 \cdot \mathcal{Z}(x_{\omega(k)}) = u_{\omega(k)}.$$

Now the equality (4.10) follows directly from (4.2). The first formula of Proposition 4.2 is verified.

Verification of the formula for the element $\pi_{s^*}((\Phi_{s^*})(1))$ is quite similar. Let an index $q \leq a^*_k$ be fixed. Put

$$q' = \omega(k) - q + 1; \quad s = \prod_{k' = a}^{k-1} (s_{n-k'} + \omega_k \cdots s_{n-k'+1}).$$

Since $n - k' + 1 > n - k + 1$ in the last expression, we get

$$s^{-1}(n - k + 1) = n - k + 1,$$

$$s^{-1}(n - k + q + 1) = (s_{n-k+q}^{-1})^{-1}(q) + n - k + 1 = k - s^{-1}_{\omega}(k - q) + n - k + 1 = n + 1 - q';$$

here we have applied the second equality of Lemma 2.6 and (2.7) to the group $S_{k-1}$, and made use of (4.9).

Due to Lemma 2.6 and to (2.8) we have $s^*_n = w s_{n-k+q} \cdots s_{n-k+1} s$ for some $w \in S_n$ such that

$$\text{length}(s^*_n) = \text{length}(w') + q + \text{length}(s).$$

Put $s' = s_{n-k+q} \cdots s_{n-k+1} s$; to get the second formula of Proposition 4.2 it suffices to verify that

$$\pi_{s'} ((\Phi_{s_{n-k+q}})(1)) = \varphi_{n-k+q}(u_k, u_q).$$

But due to (4.2) the last formula follows from the equalities

$$s' \cdot \mathcal{Z}(x_{n-k+q}) = \mathcal{Z}(x_{s'}^{-1}(n-k+q))$$

$$= \mathcal{Z}(x_{s^{-1}(n-k+q)}) = \mathcal{Z}(x_{n-k+1}) = w_0 \cdot \mathcal{Z}(x_k) = u_k;$$

$$s' \cdot \mathcal{Z}(x_{n-k+q+1}) = \mathcal{Z}(x_{s'}^{-1}(n-k+q+1)) = \mathcal{Z}(x_{s^{-1}(n-k+q+1)})$$

$$= \mathcal{Z}(x_{n+1-q'}) = w_0 \cdot \mathcal{Z}(x_{q'}) = u_{q'}. \quad \Box$$
Proposition 4.3. Let the number \( A'(i, j) \) occupy the place \( p \) in the sequence \( (A') \). Suppose that \( j < l + i - 1 \). Then

\[
\pi_{x_{i,j}}(\Phi_{n-p})(1) = \pi_{x_{i,j}}(\Phi_{n-p})(1) \cdot \varphi_{n-p}(u(i, j), u(i, j)).
\]

Proof. The place \( p + 1 \) in the sequence \( (A') \) is occupied by the number \( A'(i, j) \). By the definition (2.4) we have

\[
w_{n-p}(n-p+1) = A'(i, j) \quad \text{and} \quad w_{n-p}(n-p) = A'(i, j+1).
\]

Evidently, \( A'(i, j) < A'(i, j+1) \); so the set \( \mathcal{A}(w_{n-p}) \) is greater than \( \mathcal{A}(w_{n-p} \cdot s_n) \) by the inversion of \( A'(i, j) \) and \( A'(i, j+1) \). Thus

\[
\text{length}(w_{n-p}) = \text{length}(w_{n-p} \cdot s_n) + 1.
\]

Due to (2.6) and (4.1) we have

\[
s_{n-p} \cdot \mathcal{Z}(x_{n-p}) = w_{n-p} \varphi_{n-p} \cdot \mathcal{Z}(x_{n-p}(n-p)) = w_{n-p} \cdot \mathcal{Z}(x_{n-p}(i, j+1)) = u(i, j+1),
\]

\[
s_{n-p} \cdot \mathcal{Z}(x_{n-p+1}) = w_{n-p} \varphi_{n-p} \cdot \mathcal{Z}(x_{n-p+1}(n-p+1)) = w_{n-p} \cdot \mathcal{Z}(x_{n-p+1}(i, j)) = u(i, j).
\]

Now by Proposition 3.3 we obtain that

\[
\pi_{x_{i,j}}(\Phi_{n-p})(1) = \pi_{x_{i,j}}(\Phi_{n-p})(1) \cdot \varphi_{n-p}(u(i, j), u(i, j)).
\]

The remainder of this section is devoted to the properties of the elements \( \varphi_k(u, u') \).

Lemma 4.4. The following relations hold in the algebra \( M_2 \):

\[
\varphi_k(u', u) \varphi_k(u, u') = -(u'^2 - u^2) + (u + u')^2 + (u - u')^2;
\]

\[
\varphi_k(u, v) \varphi_k(u', v') = \varphi_k(u', v') \varphi_k(u, v) \quad \text{if} \quad k' - k > 1;
\]

\[
\varphi_k(u, u') \varphi_{k+1}(u'^n, u) = \varphi_{k+1}(u'^n, u) \varphi_k(u', u) \varphi_{k+1}(u, u').
\]

Proof. It is based on the Proposition 3.2. Put \( \mathcal{Z}(x_k) = u \) and \( \mathcal{Z}(x_{k+1}) = u' \). Then due to Proposition 3.3 and to (4.2) we have

\[
\pi_{x_k}(\Phi_{x_k})(1) = \pi_{x_k}(\Phi_{x_k})(1) \cdot \pi_{x_k}(\Phi_{x_k})(1) = \varphi_k(u', u) \varphi_k(u, u').
\]
On the other hand, by (3.6) we get the equality
\[\pi_x (- (x_k^2 - x_{k+1}^2)^2 + (x_k + x_{k+1})^2 + (x_k - x_{k+1})^2)(1)\]
\[= -(u^2 - u')^2 + (u + u')^2 + (u - u')^2.\]

Then the first relation of Lemma 4.4 follows from that of Proposition 3.2. Now let \(x(x_k) = u^2, x(x_{k+1}) = u\) and \(x(x_{k+2}) = u'.\) By the second relation of Proposition 3.2 and by Proposition 3.3 we have the equality
\[\pi_{n+1} x(\Phi_k(1) - \pi_{n+1} x(\Phi_{k+1})(1) - \pi_x(\Phi_k)(1)\]
\[\pi_{n+1} x(\Phi_{k+1})(1) - \pi_{n+1} x(\Phi_k)(1),\]
which is none other than the third relation of Lemma 4.4. The second relation follows directly from the definition of \(\varphi_k(u, u').\]

**Lemma 4.5.** We have
\[\varphi_k(u, u')^2 = 2(u + u') \cdot \varphi_k(u, u') + ((u^2 - u')^2 - (u + u')^2 - (u - u')^2).\]

**Proof.** It is a straightforward calculation.

The first relation of Lemma 4.4 implies that the element \(\varphi_k(u, u') \in M_n\) is invertible unless
\[(u + u')^2 + (u - u')^2 = (u^2 - u')^2.\]
(4.11)

In the latter case due to Lemma 4.5 the element \(\varphi_k(u, u')\) is an idempotent up to a scalar factor:
\[\varphi_k(u, u')^2 = 2(u + u') \cdot \varphi_k(u, u')\]

Both sides of the relation (4.11) depend only on \(u^2\) and \(u'^2.\) Let us perform the substitution
\[u^2 = v(v + 1), \quad u'^2 = v'(v' + 1);\]
then (4.11) takes the form
\[((v - v')^2 - 1)((v + v' + 1)^2 - 1) = 0.\]

The replacement of \(v\) by \(-v - 1\) interchanges the factors at the left hand side of the last relation, but does not change \(u^2.\) Therefore each pair

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((u', u'')) satisfying (4.11) may be obtained by the above substitution from a pair (v, v') such that \( v - v' = \pm 1 \). We will make use of this observation in the next section.

5. FUSION PROCEDURE

If \( u + u' \neq 0 \) then for \( k = 1, \ldots, n - 1 \) one can define an element of the algebra \( M_n \)

\[
\psi_k(u, u') = \frac{1}{u - u'}; \quad \varphi_k(u, u') = s_k + \frac{1}{u - u'} - \frac{c_k c_{k+1}}{u + u'}
\]

When defined, these elements by Lemma 4.4 obey the relations

\[
\psi_k(u', u) \psi_k(u, u') = 1 - \frac{1}{(u - u')^2} - \frac{1}{(u + u')^2};
\]

\[
\psi_k(u, v) \psi_k(u', v') = \psi_k(u', v') \psi_k(u, v) \quad \text{if} \quad k' - k > 1;
\]

\[
\psi_k(u, u') \psi_{k+1}(u'', u') \psi_k(u'', u) = \psi_{k+1}(u'', u) \psi_k(u', u) \psi_{k+1}(u, u').
\]

The last relation is none other else than the Yang-Baxter equation with the spectral parameters \( u, u', u'' \); see [KS]. The solution \( \psi_k(u, u') \) was introduced in [N2] in order to define the Yangian of the queer classical Lie superalgebra; cf. [D], [KR]. Consider the product

\[
\psi_k(u, u') \psi_{k+1}(u'', u') \psi_k(u'', u);
\]

it is a function of \( u, u', u'' \) defined only when \( (u \pm u')(u \pm u'')(u' \pm u'') \neq 0 \). Nevertheless we have the following lemma. Put

\[
\theta_k(u, u') = \psi_k(u, u') s_{k+1} \psi_k(u', u)
\]

\[
\quad + \psi_k(u, u') \left( - \frac{1}{(u - u')^2} + \frac{c_k c_{k+1}}{(u + u')^2} + \frac{c_{k+1} c_{k+2}}{u^2 - u'^2} + \frac{c_{k+2} c_k}{u^2 - u'^2} \right).
\]

**Lemma 5.1.** The restriction of the function (5.2) onto \((u, u', u'')\) such that \((u, u')\) satisfies the condition (4.11) and \( u + u' \neq 0 \), has a limit at \( u'' = u' \) equal to \( \theta_k(u, u') \).
Proof. Using only the definition of the factors in the product (5.2), one can bring that product to the form
\[
\psi_e(u, u') \left( s_{k+1} \psi_e(u'', u) - \frac{1}{(u-u')(u-u'')} \right)
+ \frac{c_k c_{k+1}}{(u+u')(u+u'')} + \frac{c_{k+1} c_{k+2}}{(u+u')(u-u'')} + \frac{c_{k+2} c_k}{(u-u')(u-u'')}
= \left(1 - \frac{1}{(u-u')^2} \right) - \frac{1}{(u-u')^2} \frac{1}{(u-u'')} \left( c_k c_{k+2} + c_{k+2} c_k \right).
\]
Now let the pair \((u, u')\) satisfy the condition \((4.11)\) and \(u \pm u' \neq 0\). Then the last line of the above expression vanishes, while the remaining expression is continuous at \(u'' = u'\) and takes the value \(\theta_e(u, u')\).

Lemma 5.2. Suppose that the pair \((u, u')\) satisfies the condition \((4.11)\) and \(u \pm u' \neq 0\). Then
\[
\theta_e(u, u') \psi_e(u, u') = \psi_e(u, u') \left( \frac{2}{(u+u')^3} - \frac{2}{(u-u')^3} \right).
\]

Proof. We shall prove that the expression \(\theta_e(u, u')\) can be rewritten as
\[
\psi_e(u, u') \left( s_{k+1} - \frac{u-u'}{2(u+u')^2} c_k c_{k+1} - \frac{c_{k+1} c_{k+2} + c_{k+2} c_k}{2(u+u')} \right) \psi_e(u', u)
+ \psi_e(u, u') \left( \frac{u-u'}{(u+u')^3} - \frac{1}{(u-u')^2} \right).
\]
Since \(\psi_e(u', u) \psi_e(u, u') = 0\) by the first equality in (5.1) and
\[
\psi_e(u, u')^2 = \frac{2}{u-u} \psi_e(u, u')
\]
by Lemma 4.5, the statement of Lemma 5.2 will follow. By the definition of \(\psi_e(u', u)\) and \(\psi_e(u, u')\) we get the equalities
\[
\psi_e(u, u') \left( - \frac{u-u'}{2(u+u')^2} c_k c_{k+1} - \frac{c_{k+1} c_{k+2} + c_{k+2} c_k}{2(u+u')} \right) \psi_e(u', u)
= \psi_e(u, u')^2 \left( \frac{u-u'}{2(u+u')^2} c_k c_{k+1} + \frac{c_{k+1} c_{k+2} + c_{k+2} c_k}{2(u+u')} \right) - \psi_e(u, u') \frac{u-u'}{(u+u')^3}
= \psi_e(u, u') \left( - \frac{u-u'}{(u+u')} + \frac{c_k c_{k+1}}{(u+u')^2} + \frac{c_{k+1} c_{k+2} + c_{k+2} c_k}{u^2-u''^2} \right).
\]
By comparing the last line of these equalities with the definition of \( \theta_k(u, u') \) we prove that (5.3) equals \( \theta_k(u, u') \).

**Corollary 5.3.** We have the equality \( \theta_k(\sqrt{2}, 0) \psi_k(\sqrt{2}, 0) = 0 \).

**Lemma 5.4.** Suppose the pair \((u, u')\) satisfies the condition (4.11) and \( u \pm u' \neq 0 \). Then

\[
\psi_{k+1}(u, u') \theta_k(u, u') = \psi_{k+1}(u, u') \left( \frac{2}{(u + u')^2} - \frac{2}{(u - u')^2} \right).
\]

**Proof.** Consider the subalgebra in \( M_n \) generated by the elements \( s_k, s_{k+1}, c_k, c_{k+1}, c_{k+2} \). One can define an involutive antiautomorphism of this subalgebra by

\[
s_k \mapsto s_{k+1}, \quad s_{k+1} \mapsto s_k; \quad c_k \mapsto c_{k+2}, \quad c_{k+1} \mapsto c_{k+1}, \quad c_{k+2} \mapsto c_k;
\]

it follows immediately from the relations for the above generators. Obviously, under this antiautomorphism \( \psi_k(u, u') \mapsto \psi_{k+1}(u, u') \). By the third relation in (5.1) and by Lemma 5.1, the element \( \theta_k(u, u') \) is a fixed point of this antiautomorphism. Applying it to the equality provided by Lemma 5.2, we get Lemma 5.4.

**Corollary 5.5.** We have the equality \( \psi_{k+1}(\sqrt{2}, 0) \theta_k(\sqrt{2}, 0) = 0 \).

In the previous section we defined the character \( \chi \) by using an array of the form \( \lambda \) with arbitrary complex entries \( u(i, j) \). Now let us introduce the \( \ell \)-tuple \( r=(r_1, \ldots, r_\ell) \) of auxiliary real parameters such that \( 0 \leq r_i < 1 \), and specify

\[
u(i, j) = \sqrt{v(i, j)(v(i, j) + 1)}, \quad v(i, j) = j - i + r_i. \tag{5.4}\]

If the entries \( u(i, j) \) and \( u(i', j') \) are neighbouring in the same row, that is if

\[i' = i \quad \text{and} \quad j' = j \pm 1,\]

then \( v(i, j) - v(i', j') = \pm 1 \) and the pair \((u, u')=(u(i, j), u(i', j'))\) satisfies the relation (4.11). Furthermore, since

\[u(i, j)^2 - u(i', j')^2 = (v(i, j) - v(i', j'))(v(i, j) + v(i', j') + 1)\]
and \( j - i, j' - i' \) are non-negative integers, we have
\[
u(i, j)^2 - \nu(i', j')^2 = 0 \iff j = i' = j', \quad r_i = r_{j'}.
\]

(5.5)

Let a standard shifted tableau \( A \) of the form \( \lambda \) be fixed. Put \( u_k = \nu(i, j) \) if \( k = A(i, j) \). Introduce the set
\[
A = \left\{ (r_1, \ldots, r_i) \big| \prod_{i < i'} (r_i - r_{i'}) = 0; 0 \leq r_i < 1 \right\}.
\]

We shall always assume that \( r = (r_1, \ldots, r_i) \notin A \), then \( u_k \pm u_k \notin A \) for all \( k \neq \lambda' \). Therefore one can define the element of the algebra \( M_n \)
\[
\psi_A(r) = \prod_{k=2}^{\lambda} \prod_{p=1}^{n} \psi_{k-p}(u_k, u_{\lambda(p)})
\]

(5.6)

By the first formula of Proposition 4.1 and by the definition of \( \psi_A(u, u') \), the elements \( \psi_A(r) \) and \( \pi_A(\Phi_u)(1) \) differ only by a scalar factor. Consider the former element of \( M_n \) as a function of \( r \). This function may have singularities in \( A \). However, as the following theorem shows, this is not the case. The process of continuation of the function \( \psi_A(r) \) to the set \( A \) is called fusion procedure [Ch2].

**Theorem 5.6.** At any point of \( A \) the function \( \psi_A(r) \) has a non-zero limit.

**Proof.** By Lemma 2.4 we have the reduced decomposition
\[
w_A = \prod_{k=2}^{\lambda} (s_{k-1} \cdots s_{k-\lambda_0}).
\]

Thus if we expand the product (5.6) using the definition of \( \psi_A(u, u') \) only, we will get a sum of the form
\[
w_A + \sum_{s \in S_n} z_s s \quad z_s \in \mathbb{Z}_n;
\]

where \( z_s \neq 0 \) only if \( \text{length}(s) < \text{length}(w_A) \). Therefore if the function \( \psi_A(r) \) has a limit at a certain point of \( A \), this limit is non-zero.

By the definition of the sequences \( \mathcal{B}_k \) and \( \mathcal{B}'_k \) we have
\[
\mathcal{B}_k \cup \mathcal{B}'_k = \{ 1, \ldots, k - 1 \}; \quad k = 2, \ldots, n.
\]
Put $b = b'_1 + \cdots + b'_r$, then by Lemma 2.4 $\text{length}(s_A) = b$. The formula (4.3) and Proposition 4.1 imply that

$$
\prod_{k=2}^{n} \left( \prod_{q=1}^{k-1} \psi_{k-q} \left( u_{n-q+1}, u_k \right) \right) \cdot \phi_A(r) \cdot \prod_{k=2}^{n} \left( \prod_{q=1}^{k-1} \psi_{n-k-q+1} \left( u_k, u_{n-q+1} \right) \right) = (-1)^b \cdot \prod_{1 \leq k' < k \leq n} \left( u_{k'}^2 - u_k^2 \right)^{-1} \cdot \pi_A(d_n)(1) = \phi_A(r). \tag{5.7}
$$

Here the last equality follows from Proposition 4.1 applied to the tableau $A'$, since $w_0 = w_A'$ and

$$u_{s_A^{-1}(k)} = u(i, j) \iff s_A^{-1}(k) = A(i, j) \iff k = A'(i, j).$$

Let $k = A(i, j)$ and $A'(b'_k - q + 1) = A(i', j')$, where $k \geq 2$ and $q \leq b'_k$. Then by the definition of the numbers $u_1, \ldots, u_n$ we have

$$u_k = u(i, j) \quad \text{and} \quad u_{s_A(i', j')} = u(i', j').$$

Since the tableau $A$ is standard, it follows from the definition of the sequence $s_A$ that $i' \leq i - 1$ and $j' \geq j + 1$. Then for any $0 \leq r_i, r_j < 1$

$$v(i', j') - v(i, j) = r_j - r_i - j + i - r_j \geq 2 + r_j - r_i > 1,$$

$u(i, j) \neq d(i', j')$ and the pair $(u(i', j'), u(i, j))$ does not satisfy the condition (4.11). Thus each factor $\psi_{k-q}(u_{n-q+1}, u_k)$ in (5.7) is continuous and invertible for any $r$ such that $0 \leq r < 1$. Therefore it suffices to prove Theorem 5.6 only for $A = A'$. Assume that $A = A'$. Since $s_A = 1$ and

$$s_A \cup A' = \{1, \ldots, n \}; \quad k = 2, \ldots, n;$$

due to (4.7) and to Proposition 4.2 we have

$$\phi_A(r) = \prod_{1 \leq k' < k \leq n} \left( u_{k'}^2 - u_k^2 \right)^{-1} \cdot \pi_A(d_n)(1) = \phi_A(r) \cdot \prod_{k=2}^{n} \left( \prod_{q=1}^{k-1} \psi_{n-k-q+1} \left( u_k, u_{n-q+1} \right) \right), \tag{5.8}$$
where
\[
\theta_{j}(r) = \prod_{k=2}^{n} \left( \prod_{p=1}^{q} \psi_{k-p}(u_{k}, u_{\delta(k)}) \right).
\] (5.9)

Let \( k = A'(\bar{i}, \bar{j}) \) and \( \delta'(\delta'_{k} - q + 1) = A'(\bar{i}', \bar{j}') \), where \( k \geq 2 \) and \( q \leq \delta'_{k} \). Then by definition of \( u_{1}, \ldots, u_{n} \) we have \( u_{k} = u(\bar{i}, \bar{j}) \) and \( u_{\delta'(\delta'_{k} - q + 1)} = u(\bar{i}', \bar{j}') \). It follows from the definition of \( \delta'_{k} \) that \( \bar{i}' \geq \bar{i} + 1 \) and \( \bar{j}' \leq \bar{j} - 1 \). Then for any \( 0 \leq r_{i}, r_{j}, < 1 \)
\[
v(i, j) - v(i', j') = j - i + r_{j} - j' + i' - r_{i} \geq 2 + r_{i} - r_{j}, > 1,
\]
and the pair \((u(i, j), u(i', j'))\) does not satisfy (4.11).

Thus each factor \( \psi_{n-k+q}(u_{k}, u_{\delta'(\delta'_{k} - q + 1)}) \) in (5.8) is continuous and invertible for any \( r \) such that \( 0 \leq r < 1 \). Therefore it suffices to prove that at any point of \( A \) the function \( \theta_{j}(r) \) has a limit.

Let \( k = A'(\bar{i}, \bar{j}) \) and \( \delta'_{k}(p) = A'(\bar{i}', \bar{j}') \), where \( k \geq 2 \) and \( p \leq \delta'_{k} \). Then by the definition of the numbers \( u_{1}, \ldots, u_{p} \) we have \( u_{k} = u(\bar{i}, \bar{j}) \) and \( u_{\delta'(p)} = u(\bar{i}', \bar{j}') \). Thanks to (5.5) the factor \( \psi_{k-p}(u_{k}, u_{\delta'(p)}) \) in (5.9) has a singularity only at \( r_{j} = r_{j} \), and only if \( j' - i' \) that is if \( j' \leq j \).

It follows from the definition of the sequence \( \delta'_{k} \) that \( i' \leq i \) and \( j' \leq j \), at least one if these inequalities being strict. If \( j - i = j' - i' \) then both of these inequalities are strict, \( p < \delta'_{k} \) and
\[
\delta'_{k}(p + 1) = A'(\bar{i}', \bar{j}' + 1), \quad u_{\delta'(p + 1)} = u(\bar{i}', \bar{j}' + 1).
\]
By the definition (5.4) the pair \((u(\bar{i}', \bar{j}' + 1), u(\bar{i}', \bar{j}'))\) satisfies the condition (4.11) and
\[
u(\bar{i}', \bar{j}')^{2} - u(\bar{i}', \bar{j}' + 1)^{2} \neq 0.
\]

Denote by \( \hat{\theta}_{j}(r) \) the product obtained from (5.9) by inserting before each factor \( \psi_{k-p}(u_{k}, u_{\delta'(p)}) \) such that \( j - i = j' - i' \), the expression
\[
\psi_{k-p-1}(u_{\delta'(p + 1)}), u_{\delta'(p)} \cdot (u_{\delta'(p + 1)} - u_{\delta'(p)})/2.
\] (5.10)

If \( j - i = j' - i' \), then by Lemma 5.1 the product
\[
\psi_{k-p-1}(u_{\delta'(p + 1)}), u_{\delta'(p)} \psi_{k-p}(u_{k}, u_{\delta'(p)}) \psi_{k-p-1}(u_{k}, u_{\delta'(p + 1)})
\]
has the limit equal to \( \hat{\theta}_{k-p-1}(u_{\delta'(p + 1)}), u_{\delta'(p)} \) at \( u_{k} = u_{\delta'(p)} \), that is when \( r_{j} = r_{j} \). Thus to get Theorem 5.6 it suffices to prove that
\[
\hat{\theta}_{j}(r) = \theta_{j}(r).
\] (5.11)
Let $k \geq 2$ and $p \leq a_k$ such that $j - i = j' - i'$, be fixed. Put $u = u_{\alpha j(p+1)}$ and $u' = u_{\alpha j'(p')}$. The product of the factors which occur in (5.9) before $\psi_{k-p}(u_k, u')$, is

$$\prod_{k' = 2}^{2} \prod_{q = 1}^{\alpha j(q)} \psi_{k' - q}(u_k, u_{\alpha j(q)}) \cdot \prod_{q = 1}^{p - 1} \psi_{k - q}(u_k, u_{\alpha j(q)}).$$

Denote this product by $\sigma(r)$. It is enough to verify the equality

$$\sigma(r) \cdot \psi_{k-p}(u, u') \cdot (u - u')/2 = \sigma(r). \quad (5.12)$$

Indeed, let us eliminate the expressions (5.10) inserted into the product (5.9), successively from the left to the right. Using the last equality on each step, we will arrive at (5.11).

The verification of (5.12) is quite similar to that of the first equality in Proposition 4.2. Consider the array $Q'$ obtained by taking the numbers $k, \ldots, n$ out of $Q$. Evidently, it is the column tableau for a certain strict partition $\omega$ of $k - 1$. For each $k' \leq k$ the subsequence of $(Q)$ consisting of all the numbers $k'' < k'$ which occur before $k'$ in that sequence, coincides with $\alpha_{k'}$. Put $\tilde{\chi} = w_0^1 w_0^1 \chi$, then by the definition of $u_1, \ldots, u_n$ we have

$$w_{Q'} \cdot \tilde{\chi}(x_k) = u_{k'}, \quad k' = 1 \cdots k - 1.$$

It follows from Proposition 4.2 applied to the tableau $Q'$ that

$$\prod_{k' = 2}^{2} \prod_{q = 1}^{\alpha j(q)} \psi_{k' - q}(u_k, u_{\alpha j(q)}) = \prod_{k' = 2}^{2} \prod_{q = 1}^{\alpha j(q)} \frac{\varphi_{k' - q}(u_k, u_{\alpha j(q)})}{u_k^2 - u_{\alpha j(q)}^2} = \pi_{\tilde{\chi}}(\Phi_{w_{Q'}}(1) \cdot f(r)), \quad (5.13)$$

where $\tilde{\chi} = w_0^1 \cdot \tilde{\chi}$ and

$$f(r) = \prod_{1 \leq q \leq k} (u_k^2 - u_{\alpha j(q)}^2)^{-1}.$$

Since $(Q') = \alpha_{k'}, \quad \alpha_{k'}(p) = \alpha'(i', j')$ and $\alpha_{k'}(p+1) = \alpha'(i', j'+1)$ the application of Proposition 4.3 to the partition $\omega$ and the character $\tilde{\chi}$ provides the equality

$$\pi_{\tilde{\chi}}(\Phi_{w_{Q'}})(1) = \pi_{\alpha_{k'-1}} \cdot \tilde{\chi}(\Phi_{w_{Q'}}(1)) \cdot \varphi_{k'-1}(u, u'). \quad (5.14)$$
Now observe that due to Lemma 4.5 we have
\[ \varphi_{k-p-1}(u, u') \cdot \psi_{k-p-1}(u, u') = \frac{2}{u - u'} \cdot \varphi_{k-p-1}(u, u'). \]

Now by gathering (5.13) and (5.14) together, we get (5.12). Indeed, we now have the equalities
\[ \varphi_{k-p-1}(u, u') \cdot (u - u')/2 \]
\[ = \pi_j(\Phi_{u, u'}) (1) \cdot f(r) \cdot \prod_{q=1}^{p-1} \psi_{k-p}(u_k, u_{\delta(q)}) \]
\[ \times \psi_{k-p-1}(u, u') \cdot (u - u')/2 \]
\[ = \pi_j(\Phi_{u, u'}) (1) \cdot \psi_{k-p-1}(u, u') \cdot \prod_{q=1}^{p-1} \psi_{k-p}(u_k, u_{\delta(q)}) \]
\[ \times (u - u') \cdot f(r)/2 \]
\[ = \pi_j(\Phi_{u, u'}) (1) \cdot \varphi_{k-p-1}(u, u') \cdot \psi_{k-p-1}(u, u') \]
\[ \times \prod_{q=1}^{p-1} \psi_{k-p}(u_k, u_{\delta(q)}) \cdot (u - u') \cdot f(r)/2 \]
\[ = \pi_j(\Phi_{u, u'}) (1) \cdot \varphi_{k-p-1}(u, u') \cdot f(r) \]
\[ \times \prod_{q=1}^{p-1} \psi_{k-p}(u_k, u_{\delta(q)}) \]
\[ = \pi_j(\Phi_{u, u'}) (1) \cdot f(r) \cdot \prod_{q=1}^{p-1} \psi_{k-p}(u_k, u_{\delta(q)}) = \varpi(r). \]

Thus the equality (5.11) is verified. \[\qed\]

**Proposition 5.7.** Suppose that \( k = A(i, j - 1) \) and \( k + 1 = A(i, j) \). Then
\[ \psi_A(u(i, j - 1), u(i, j)) \cdot \psi_A(r) = 0. \]

**Proof.** By Proposition 4.1 and by the definition (5.6) we have
\[ \psi_A(r) = \pi_j(\Phi_{u, u'}) (1) \cdot \prod_{k' = 2}^{n} \prod_{p=1}^{n_{b_{k'}}} (u_{k'}^2 - u_{2b_{k'}}^2(1))^{-1}. \]
The number \( k \) occurs before \( k + 1 \) in the sequence \((A)^*\). Thanks to the definition (2.1), the set \( \mathcal{I}(w) \) is greater than \( \mathcal{I}(s_kw) \) by the inversion of \( k \) and \( k + 1 \), that is

\[
\text{length}(w) = \text{length}(s_kw) + 1.
\]

By Corollary 3.4 we get

\[
\pi_\lambda(p_{xw})(1) = \pi_{s_kxw}(1) \cdot \pi_\lambda(p_{xw})(1)
\]

\[
= \varphi_\lambda(u(i, j), u(i, j - 1)) \cdot \varphi_\lambda(u(i, j) - 1),
\]

(5.15)

since

\[
s_kw \cdot \mathcal{I}(x) = w \cdot \mathcal{I}(x_k + 1) = u(i, j),
\]

\[
s_kw \cdot \mathcal{I}(x_k + 1) = w \cdot \mathcal{I}(x_k) = u(i, j - 1)
\]

by (4.1). But the pair \((u(i, j), u(i, j - 1))\) satisfies the condition (4.11), and

\[
\psi_\lambda(u(i, j - 1), u(i, j)) \cdot \varphi_\lambda(u(i, j), u(i, j - 1)) = 0
\]

by the first relation of Lemma 4.4. Now Proposition 5.7 follows from (5.15).

6. TWO PROPERTIES OF THE ELEMENT \( \psi_A(r) \)

In this section we will assume that the sequences \( A_k, A'_k \) correspond to the tableau \( A = A' \). We will also assume that \( u_k = u(i, j) \) if \( k = A'(i, j) \). Then \( w_0 \cdot \mathcal{I}(x_k) = u_k \) and the element \( \theta_j(r) \in M_n \) is defined by (5.9).

**Proposition 6.1.** Suppose that \( n = A'(i, j) \) and \( n - 1 = A'(i - 1, j) \). Then

\[
\lim_{r \to 0} \left( \theta_j(r) \cdot \prod_{j = i + 1} \psi_{j - 1 + n - a_n}(u(i - 1, j), u(i, j)) \right) = 0.
\]

**Proof.** We will use the induction on \( j - i \). We will employ several facts established in the proof of Theorem 5.6. Suppose that \( j = i \), that is the number \( n \) stands on the main diagonal of the tableau \( A' \). Then \( a_n = n - 1 \) and

\[
u_{A_n(n - 2)} = u(i - 1, j - 1), \quad u_{A_n(n - 1)} = u(i - 1, j).
\]
Therefore the product (5.9) has the form
\[
\psi(r) \cdot \psi_j(u(i,j), u(i-1, j-1)) \psi_j(u(i,j), u(i-1, j))
\]
where \( \psi(r) \) has a limit at \( r = 0 \). Moreover, applying the equality (5.12)
when \( k = n \) and \( p = n - 2 \), we can bring (5.9) to the form
\[
\psi(r) \cdot \psi_j(u(i-1,j), u(i-1, j-1)) \psi_j(u(i,j), u(i-1, j-1))
\times \psi_j(u(i,j), u(i-1, j)) \cdot (u(i, j) - u(i-1, j-1))/2.
\]
Then by Lemma 5.1 and Corollary 5.3 we get
\[
\lim_{r \to 0} (\psi(r) \cdot \psi_j(u(i-1,j), u(i,j))) = \lim_{r \to 0} \psi_j(u(i-1,j), u(i,j))
\]
\[
= \lim_{r \to 0} \psi(r) \cdot \psi_j(\sqrt{2}, 0) \psi_j(\sqrt{2}, 0)/\sqrt{2} = 0.
\]
Thus Proposition 6.1 is verified for \( j = i \).
Now suppose that \( j - i \geq 1 \). By the definitions of the sequence \( \mathcal{N}_n \) and of
the numbers \( u_k \) we have \( u_k = u(i,j) \) and
\[
\begin{align*}
\psi_{i+1}^{a_n}(u_{i+1}) & = u(i-1, j), \\
\psi_{i+1}^{a_n}(u_{i+1}) & = u(i, j), \quad j = i, \ldots, j-1.
\end{align*}
\]
For short denote \( j+n-a_n = a \) and
\[
\psi_{i+1}^{a_n}(u_{i+1}) = u, \quad u(i-1, j-1) = u', \quad u(i, j) = u''.
\]
Then the product (5.9) takes the form
\[
\psi(r) \cdot \psi_{i+1}^{a_n}(u', u'') \psi_{i+1}^{a_n}(u', u) \cdot \prod_{j=i+1}^{j-1} \psi_{i+1}^{a_n}(u', u)
\]
where \( \psi(r) \) has a limit at \( r = 0 \). Since \( a_n = a_n - j + i - 1 \), \( u_{a_n} = u \) and \( u_{a_n-1} = u' \), by the definition of the product \( \psi(r) \) we have
\[
\begin{align*}
\psi(r) &= \prod_{k=2}^{k=n-2} \left( \prod_{p=1}^{p=a_k} \psi_{i+1}^{a_n}(u_k, u_{a_k}(p)) \right) \\
&\times \prod_{j=i+1}^{j-1} \psi_{i+1}^{a_n}(u', u') \cdot \prod_{p=1}^{p=a_n-i-2} \psi_{i+1}^{a_n}(u, u).
\end{align*}
\]
If \( p \leq a_n - j + i - 2 \), then \( n - p \geq a - i + 2 \) and \( \psi_{a-n}(u, u') \) commutes with each of the factors \( \psi_{a-n}(u, u_{\alpha k}(p)) \) in the latter product. Moreover, if \( u_{\alpha k}(p) = u(i^*, j^*) \) for some \( i^*, j^* \) such that \( j-i = j'-i' \), then

\[
p \leq a_n - j + i - 3, \quad n - p - 1 \geq a - i + 2
\]

and \( \psi_{a-n}(u, u') \) also commutes with \( \psi_{n-p-1}(u, u_{\alpha k}(p+1), u_{\alpha k}(p)) \). Therefore we obtain the equality \( \psi(r) = \bar{\psi}(r) \cdot \psi_{a-n}(u, u') \) where \( \bar{\psi}(r) \) has a limit at \( r = 0 \):

\[
\bar{\psi}(r) = \prod_{k=2}^{n-2} \left( \prod_{p=1}^{a_k} \psi_{a-p}(u_k, u_{\alpha k}(p)) \right) \times \prod_{p=1}^{a_n-1} \psi_{a-n-1-p}(u_n-1, u_{\alpha n-1}(p)) \times \prod_{p=1}^{a_n-j-i-2} \psi_{n-p}(u, u_{\alpha n}(p)). \tag{6.1}
\]

Thanks to the second and the third relations of (5.1), we have

\[
\prod_{j=i-j-1}^{j=i-j} \psi_{a-j}(u, u_{i,j}) \cdot \prod_{j=i-j} \psi_{a-j}(u, u_{i,j}) = \psi_{a-i}(u, u') \cdot \prod_{j=i-j-1}^{j=i-j} \psi_{a-j}(u, u_{i,j}) \cdot \prod_{j=i-j} \psi_{a-j}(u, u_{i,j}).
\]

Thus we get the equality

\[
\theta_j(r) \cdot \prod_{j=i-j} \psi_{a-j}(u, u_{i,j}) = \bar{\psi}(r) \psi_{a-j}(u, u') \psi_{a-i}(u', u') \psi_{a-i}(u', u) \cdot \psi_{a-j}(u, u') \times \prod_{j=i-j-1}^{j=i-j} \psi_{a-j}(u, u_{i,j}) \cdot \prod_{j=i-j} \psi_{a-j}(u, u_{i,j}) \tag{6.2}
\]

Put

\[
\psi(r) = \prod_{j=i-j-1}^{j=i-j} \psi_{a-j-1}(u', u_{i,j}). \tag{6.3}
\]
The products in every line of the right hand side of the equality (6.2) have limits at \( r = 0 \). Moreover,

\[
\lim_{r \to 0} \psi_{a \ldots}(u, u^\prime) = \lim_{r \to 0} \psi_{a \ldots}(u, u^\prime)
\]

and

\[
\lim_{r \to 0} \psi_{a \ldots}(u^\prime, u(i, j)) = \lim_{r \to 0} \psi_{a \ldots}(u(i, j))
\]

for \( j = i, \ldots, j - 1 \) by the definition (5.4). Therefore the limit at \( r = 0 \) of the right hand side of (6.2) is equal to that of

\[
\bar{\theta}(r) \cdot \theta_{a \ldots}(u, u^\prime) \cdot \psi_{a \ldots}(u, u^\prime)
\]

\[
\times \prod_{j=i}^{j-1} \psi_{a \ldots}(u, u(i, j)) \cdot \prod_{j=i}^{j-1} \psi_{a \ldots}(u^\prime, u(i, j))
\]

\[
= \bar{\theta}(r) \cdot \psi_{a \ldots}(u, u^\prime) \cdot \left( \frac{2}{(u + u^\prime)^{\frac{2}{3}}} - \frac{2}{(u - u^\prime)^{\frac{2}{3}}} \right)
\]

\[
\times \prod_{j=i}^{j-1} \psi_{a \ldots}(u, u(i, j)) \cdot \prod_{j=i}^{j-1} \psi_{a \ldots}(u^\prime, u(i, j))
\]

\[
= \bar{\theta}(r) \cdot \psi(r) \cdot \left( \frac{2}{(u + u^\prime)^{\frac{2}{3}}} - \frac{2}{(u - u^\prime)^{\frac{2}{3}}} \right)
\]

\[
\times \prod_{j=i}^{j-1} \psi_{a \ldots}(u, u(i, j)) \cdot \psi_{a \ldots}(u, u^\prime).
\]

Here the first equality is provided by Lemma 5.2, while the second follows from the second and the third relations of (5.1). Since all the factors in the last line of these two equalities have limits at \( r = 0 \), to get Proposition 6.1 it suffices to prove that

\[
\lim_{r \to 0} (\bar{\theta}(r) \cdot \psi(r)) = 0.
\]

Let \( A^c(i, j - 1) = n - m \). Let \( \Omega^c \) be the array obtained by taking the numbers \( n - m + 1, \ldots, n \) out of \( A^c \). Evidently, it is the column tableau for a certain strict partition \( \alpha \) of \( n - m \). For each \( k \leq n - m \) the subsequence of \( (\Omega^c) \) consisting of all the numbers \( k' < k \) which occur before \( k \) in that sequence, coincides with \( \alpha_k \). Since

\[
n - m = \Omega^c(i, j - 1), \quad n - m - 1 = \Omega^c(i - 1, j - 1)
\]
and $a_{n-m} = a_n - i$, applying the inductive assumption to the partition $\omega$ we get the equality

$$\lim_{r \to 0} \left( \theta_n(r) \cdot \prod_{j=i \ldots j-1} \psi_{a_{j-1}+i-m}(u', u(i,j)) \right) = 0.$$  \hspace{1cm} (6.5)

We shall derive (6.4) from (6.5). By the definition of the partition $\omega$ and of the product $\tilde{m}(r)$ we have

$$\tilde{m}(r) = \prod_{k=n-m+1 \ldots n-i} \left( \prod_{p=1 \ldots a_k} \psi_{k-p}(u_k, u_{\mathcal{A}_k(p)}) \right) \cdot \prod_{p=1 \ldots a_{n-1} - 1} \left( \prod_{k=n-i+1 \ldots n-2} \psi_{n-1-p}(u_{n-1}, u_{\mathcal{A}_{n-1}(p)}) \right) \cdot \prod_{p=1 \ldots a_{n-j+i-1}} \psi_{n-1-p}(u_{n-1}, u_{\mathcal{A}_{n-1}(p)}).$$

If $k = n-i+1, \ldots, n-2$ then $\mathcal{A}_k \supset \mathcal{A}'_n$ and $A'(i-1, i-1), \ldots, A'(i-1, j-1), A'(i, j-1) \in \mathcal{A}'_k \setminus \mathcal{A}'_n$. Hence for $p = 1, \ldots, a_k$ we obtain that

$$k - p \geq a_k' + 1 \geq 0 + j - i + 3 = n - a_n + j - i + 2 = a - i + 2.$$

For $p = 1, \ldots, a_{n-1} - 1$ we have

$$n - 1 - p \geq n - a_{n-1} - 1 = n - a_n + j - i + 1 = a - i + 1.$$

Furthermore, if $p = 1, \ldots, a_n - j + i - 2$ then

$$n - p \geq n - a_n + j - i + 2 = a - i + 2.$$

Therefore each factor in the last three lines of the above product commutes with $\psi(r)$. Moreover, if the pair $(k, p)$ satisfies one of the three conditions

- $k = n-i+1, \ldots, n-2; \quad p = 1, \ldots, a_k$
- $k = n-1; \quad p = 1, \ldots, a_{n-1} - 1$
- $k = n; \quad p = 1, \ldots, a_n - j + i - 2$
and $u_{\mathcal{A}(p)} = u(i', j')$ for some $i', j'$ such that $j - i = j' - i'$, then the element $\psi_{k-p-1}(u_{\mathcal{A}(p+1)}, u_{\mathcal{A}(p)})$ also commutes with $\psi(r)$. Thus to get (6.4) it suffices to prove that

$$
\lim_{r \to 0} \left( \theta(r) \cdot \prod_{k=n-m+1}^{\infty} \left( \prod_{p=1}^{\infty} \psi_{k-p}(u_k, u_{\mathcal{A}(p)}) \right) \cdot \psi(r) \right) = 0. \tag{6.6}
$$

Let $k \in \{n - m + 1, \ldots, n - i\}$. By the definition of the sequence $\mathcal{A}_k$ we have

$$
\mathcal{A}_k(a_n - j) = A'(i - 1, j - 2); \quad \mathcal{A}_k(a_n - j + 1) = A'(i - 1, j - 1);
$$
$$
\mathcal{A}_k(a_n - j + j - 2) = A'(i, j), \quad j = i, \ldots, j - 1.
$$

Denote by $\mathcal{B}_k$ the sequence obtained from $\mathcal{A}_k$ by a cyclic permutation of the numbers $A'(i - 1, j - 1), A'(i, i) \cdots A'(i, j - 1)$:

$$
\mathcal{B}_k(p) = \begin{cases} 
\mathcal{A}_k(p + 1) & \text{if } p = a_n - j + 1, \ldots, a_n - i; \\
\mathcal{A}_k(a_n - j + 1) & \text{if } p = a_n - i + 1; \\
\mathcal{A}_k(p) & \text{otherwise.}
\end{cases}
$$

Then by the second and the third equalities in (5.1) we get

$$
\lim_{p \to \infty} \left( \prod_{p=1}^{\infty} \psi_{k-p}(u_k, u_{\mathcal{A}(p)}) \right) \cdot \prod_{j=i}^{j=i-j-1} \psi_{a-n-j-1}(u', u(i, j))
$$
$$
= \prod_{j=i-j-1}^{j=i-j-1} \psi_{a-n-j-1}(u', u(i, j)) \cdot \prod_{p=1}^{\infty} \psi_{k-p}(u_k, u_{\mathcal{A}(p)}).
$$

Therefore

$$
\theta_{\mathcal{A}}(r) \cdot \prod_{k=n-m+1}^{\infty} \left( \prod_{p=1}^{\infty} \psi_{k-p}(u_k, u_{\mathcal{A}(p)}) \right) \cdot \psi(r)
$$
$$
= \theta_{\mathcal{B}}(r) \cdot \prod_{j=i-j-1}^{j=i-j-1} \psi_{a-n-j-1}(u', u(i, j)) \cdot \theta(r), \tag{6.7}
$$

where

$$
\theta(r) = \prod_{k=n-m+1}^{\infty} \left( \prod_{p=1}^{\infty} \psi_{k-p}(u_k, u_{\mathcal{A}(p)}) \right).
$$
Suppose that $R(p) = A'(i', j')$. Note that we have
\[ k = A'(i + k + m - n, j - 1), \quad u_k = u(i + k + m - n, j - 1). \]

Denote by $\theta(r)$ the product obtained from $\theta(r)$ by inserting before each factor $\psi_k - p(u_k, u_{\rho_k(p)})$ such that
\[ j' - i' = (j - 1) - (i + k + m - n), \quad (6.8) \]
the expression
\[ \psi_{k - p - 1}(u_{\rho_k(p)}, u_{\rho_k(p + 1)}) \cdot (u_{\rho_k(p + 1)} - u_{\rho_k(p)})/2. \]

The product $\theta(r)$ has a limit at $r = 0$. Indeed, let the pair $(i', j')$ satisfy the condition (6.8). Since $k \geq n - m + 1$, we have $j' - i' \leq j - i - 2$. But
\[ R_2(a_n - j) = A'(i - 1, j - 2), \quad R_1(a_n - i) = A'(i, j - 1), \]
\[ R_2(a_n - i + 1) = A'(i - 1, j - 1) \]
and $p \neq a_n - j, a_n - i, a_n - i + 1$. Therefore either
\[ R_2(p) = \mathcal{A}(p) \quad \text{and} \quad R_2(p + 1) = \mathcal{A}(p + 1) \]
or
\[ R_1(p) = \mathcal{A}(p + 1) \quad \text{and} \quad R_1(p + 1) = \mathcal{A}(p + 2). \]

Thus $u_{\rho_k(p + 1)} = u(i', j' + 1)$ and the product
\[ \psi_{k - p - 1}(u_{\rho_k(p + 1)}, u_{\rho_k(p)}) \psi_{k - p}(u_k, u_{\rho_k(p)}) \psi_{k - p - 1}(u_k, u_{\rho_k(p + 1)}) \]
has a limit at $r = 0$ by Lemma 5.1. We shall prove that
\[ \theta_{\alpha}(r) \cdot \prod_{j = i \ldots j - 1} \psi_{u - j - 1 + i - m}(u', u(i, j)) \cdot \theta(r) \]
\[ = \theta_{\alpha}(r) \cdot \prod_{j = i \ldots j - 1} \psi_{u - j - 1 + i - m}(u', u(i, j)) \cdot \theta(r). \]

Then, gathering the equalities (6.5), (6.7), (6.10) together, we shall get (6.6).

Let the indices $k \in \{n - m + 1, \ldots, n - i\}$ and $p \leq a_k$ such that the pair $(i', j')$ satisfies the condition (6.8), be fixed. To prove the equality (6.10) it suffices to show that the product of the factors which occur before
\[ \psi_{k \ldots j}(u_k, u_{\sigma(j,p)}) \] in the left hand side of (6.10), is equal to the same product multiplied on the right by (6.9). There are the following three possibilities:

\[ p \leq a_n - j - 1, \quad p \geq a_n - i + 2, \quad a_n - j + 1 \leq p \leq a_n - i - 1. \]

We will treat each of these possibilities separately.

(i) Suppose that \( p \leq a_n - j - 1 \), then

\[ \mathcal{A}(p) = \mathcal{A}(p) \quad \text{and} \quad \mathcal{A}(p + 1) = \mathcal{A}(p + 1). \]

Therefore we have already proved that the product of the factors which occur before

\[ \psi_{k \ldots j}(u_k, u_{\sigma(i,p)}) = \psi_{k \ldots j}(u_k, u_{\sigma(j,p)}) \]

in the left hand side of (6.7), is equal to the same product multiplied on the right by (6.9). But for \( j = i, \ldots, j - 1 \)

\[ k - p - 3 \geq k - a_n + j - 2 = k + a - n - 2 \geq a - j - 1 - (n - i - k + 1). \]

Therefore the expression (6.9) commutes with the product

\[ \prod_{j=i, \ldots, j-1} \psi_{a - j - 1 - (n - i - k + 1)}(u(j), u(i, j)). \]

Thus the equality (6.7) implies that the product of the factors which occur before \( \psi_{k \ldots j}(u_k, u_{\sigma(j,p)}) \) in the left hand side of (6.10), is equal to the same product multiplied on the right by (6.9).

(ii) Now suppose that \( p \geq a_n - i + 2 \), then again

\[ \mathcal{A}(p) = \mathcal{A}(p) \quad \text{and} \quad \mathcal{A}(p + 1) = \mathcal{A}(p + 1). \]

Therefore we have already proved that the product of the factors which occur before

\[ \psi_{k \ldots j}(u_k, u_{\sigma(i,p)}) = \psi_{k \ldots j}(u_k, u_{\sigma(j,p)}) \]

in the left hand side of (6.7), is equal to the same product multiplied on the right by (6.9). But for \( j = i, \ldots, j - 1 \)

\[ k - p + 1 \leq k - a_n + i - 1 = k + a - n + i - j - 1 \leq a - j - 1 - (n - i - k + 1). \]
Therefore the expression (6.9) commutes with the product
\[
\prod_{j=i \cdots j-1} \psi_{a-j-1-(n-i-k+1)}(u', u(i, j)).
\]
Thus the equality (6.7) implies that the product of the factors which occur before \(\psi_{k-p}(u_k, u_{\rho_k(p)})\) in the left hand side of (6.10), is again equal to the same product multiplied on the right by (6.9).

(iii) Finally, suppose that \(a_n-j+1 \leq p \leq a_n-i-1\), then
\[
\mathcal{R}(p) = \mathcal{A}(p+1) \quad \text{and} \quad \mathcal{R}(p+1) = \mathcal{A}(p+2).
\]
Therefore we have already proved that the product of the factors which occur before
\[
\prod_{q=1 \cdots a_i} \psi_{k-p}(u_k, u_{\rho_{i(q)}})
\]
in the left hand side of (6.7), equals the same product multiplied on the right by
\[
\psi_{k-p-2}(u_{\rho_{k}(p+2)}, u_{\rho_{k}(p+1)}) \cdot (u_{\rho_{k}(p+2)} - u_{\rho_{k}(p+1)})/2
\]
Furthermore, we have \(i' = i, i \leq j, j' \leq j-2\) and
\[
u_{\rho_{k}(p)} = u(i, j'), \quad u_{\rho_{k}(p+1)} = u(i, j'+1),
\]
\[
k - p - 2 = k - (a_n - i - j + j' + 1) - 2 = a - j' - 2 - (n - i - k + 1).
\]
Therefore by the second and the third relations in (5.1) and by (5.10) we get
\[
\psi_{k-p-2}(u_{\rho_{k}(p+1)}, u_{\rho_{k}(p)}) \cdot \prod_{j=i \cdots j-1} \psi_{a-j-1-(n-i-k+1)}(u', u(i, j))
\]

\[
= \prod_{j=i \cdots j-1} \psi_{a-j-1-(n-i-k+1)}(u', u(i, j)) \cdot \psi_{k-p-1}(u_{\rho_{k}(p+1)}, u_{\rho_{k}(p)})
\]

\[
= \prod_{j=i \cdots j-1} \psi_{a-j-1-(n-i-k+1)}(u', u(i, j))
\]

\[
\times \psi_{k-p-1}(u_{\rho_{k}(p+1)}, u_{\rho_{k}(p)})^2 \cdot (u_{\rho_{k}(p+1)} - u_{\rho_{k}(p)})/2
\]

\[
= \psi_{k-p-2}(u_{\rho_{k}(p+1)}, u_{\rho_{k}(p)}) \cdot \prod_{j=i \cdots j-1} \psi_{a-j-1-(n-i-k+1)}(u', u(i, j))
\]

\[
\times \psi_{k-p-1}(u_{\rho_{k}(p+1)}, u_{\rho_{k}(p)}) \cdot (u_{\rho_{k}(p+1)} - u_{\rho_{k}(p)})/2.
\]
where
\[ u'(i, j) = \begin{cases} 
  u(i, j'+1) & \text{if } j = j'; \\
  u(i, j') & \text{if } j = j' + 1; \\
  u(i, j) & \text{otherwise.} 
\end{cases} \]

Now the equality (6.7) implies that the product of the factors which occur before
\[ \prod_{q=1}^{k} \psi_{k-q}(u_k, u_{\rho(q)}) \]
in the left hand side of (6.10), is equal to the same product multiplied on the right by (6.9). Since each of the factors
\[ \psi_{k-q}(u_k, u_{\rho(q)}); \quad q = 1, \ldots, p - 1 \]
commutes with (6.9), we obtain that the product of the factors which occur before \( \psi_{k-j}(u_k, u_{\rho(j)}) \) in the left hand side of (6.10), is equal to the same product multiplied on the right by (6.9). Thus the equality (6.10) is verified.

Let an index \( k = 1, \ldots, n - 1 \) be fixed. We will need the following property of the element \( \theta_j(r) \) defined by (5.9).

**Proposition 6.2.** Suppose that \( k = A'(i - 1, j) \) and \( k + 1 = A'(i, j) \). Then
\[ \lim_{r \to 0} \psi_{A'}(u(i - 1, j), u(i, j)) \cdot \theta_j(r) = 0. \]

**Proof.** We will use several facts established while proving Theorem 5.6 and Proposition 6.1. As well as in the proof of Proposition 6.1 denote
\[ u(i - 1, j) = u, \quad u(i - 1, j - 1) = u', \quad u(i, j) = u'' \]
for short. Thanks to the definition (5.9) and the equality (5.11) it suffices to prove that the limit
\[ \lim_{r \to 0} \left( \psi_{A'}(u, u'') \cdot \prod_{k'=2}^{k+1} \left( \prod_{p=1}^{a_{k'}} \psi_{k'-p}(u_{k'}, u_{\rho_{j}(p)}) \right) \right) \]
is equal to zero. Thus we will assume that \( k = n - 1 \), otherwise one should replace \( A' \) by the tableau obtained by taking the numbers \( k + 2, \ldots, n \) out of \( A' \).
Consider the tableau $\Omega^*$ obtained by taking the numbers $n-1$ and $n$ out of $\Omega$. Evidently, it is the column tableau for a certain strict partition $\omega$ of $n-2$. Then

$$\theta_{\omega}(r) = \prod_{k'=2 \ldots n-2} \left( \prod_{p=1 \ldots a_{k'}} \psi_{k'-p}(u_{k'}, u_{A_{\omega}(p)}) \right),$$

(6.12)

$$\theta_{\omega}(r) = \theta_{\omega}(r) \cdot \prod_{k'=n-1, n} \left( \prod_{p=1 \ldots a_{k'}} \psi_{k'-p}(u_{k'}, u_{A_{\omega}(p)}) \right).$$

(6.13)

If $k' = 2, \ldots, n-2$ and $p = 1, \ldots, a_{k'}$, then $k' - p \leq n - 3$. Therefore the expression $\psi_{n-2}(u(i-1, j), u(i, j))$ commutes with $\theta_{\omega}(r)$. As well as in the proof of Proposition 6.1 denote $a = j + n - a_{\omega}$; then

$$a_{n-1} = a_n - (j - i + 1) = n + a - i - 1.$$  

(6.14)

We have $u_{n-1} = u(i - 1, j), u_n = u(i, j)$ and $A_{\omega}(a_{n-1}) = \Omega'(i-1, j-1).

$$A_{\omega}(p) = \begin{cases} A_{\omega}-1(p), & p \leq a_{n-1}; \\ n-1, & p = a_{n-1} + 1; \\ \Omega'(i, i + p - a_{n-1} - 1), & p > a_{n-1} + 1. \end{cases}$$

By the second and the third relations in (5.1) we get the equality

$$\psi_{n-1}(u, u^*) \cdot \prod_{k'=n-1, n} \left( \prod_{p=1 \ldots a_{k'}} \psi_{k'-p}(u_{k'}, u_{A_{\omega}(p)}) \right) = \theta(r)$$

$$\times \psi_{\omega-n-1}(u, u^*) \cdot \psi_{\omega-1}(u, u') \cdot \psi_{\omega-2}(u^*, u') \cdot \psi_{\omega-3}(u^*, u)$$

$$\times \prod_{j=i-1} \psi_{\omega-j-1}(u^*, u(i, j))$$

(6.15)

where now $\theta(r)$ is the product

$$\prod_{p=1 \ldots a_{n-1} - 1} \psi_{\omega-p}(u_n, u_{A_{\omega}(p)}) \cdot \prod_{p=1 \ldots a_{\omega} - 1} \psi_{\omega-p}(u_{n-1}, u_{A_{\omega}(p)}).$$

Let $A_{\omega}(p) = \Omega'(i', j')$. Let $\bar{\theta}(r)$ be the product obtained from $\theta(r)$ by inserting before each factor $\psi_{\omega-n-1}(u_n, u_{A_{\omega}(p)})$ such that $j' - i' = j - i$, the expression

$$\psi_{\omega-p-2}(u_{A_{\omega}(p) + 1}, u_{A_{\omega}(p)}) \cdot (u_{A_{\omega}(p) + 1} - u_{A_{\omega}(p))}.)$$
and by also inserting before each factor $\Psi_{n-p}(u_{n-1}, u_{\alpha_{n-1}(p)})$ such that $j' - i' = j - i + 1$, the expression

$$
\psi_{n-p+1}(u_{\alpha_{n-1}(p+1)}, u_{\alpha_{n-1}(p)}) \cdot \frac{u_{\alpha_{n-1}(p+1)} - u_{\alpha_{n-1}(p)}}{2}.
$$

Let $j' - i'$ be equal to $j - i$ or $j - i + 1$. Then $\alpha_{n-1}(p+1) = \alpha'(i', j' + 1)$, and the pair $(u_{\alpha_{n-1}(p+1)}, u_{\alpha_{n-1}(p)})$ satisfies the condition (4.11). Therefore the product $\theta(r)$ has a limit at $r=0$ by Lemma 5.1. Since $\Omega' = \alpha_{n-1}$, applying Proposition 4.3 to the partition $\omega$ and making use of (5.10), we obtain that

$$
\theta_{\omega}(r) \cdot \Psi_{n-p-2}(u_{\alpha_{n-1}(p+1)}, u_{\alpha_{n-1}(p)}) \cdot \frac{u_{\alpha_{n-1}(p+1)} - u_{\alpha_{n-1}(p)}}{2}
$$

is equal to $\theta_{\omega}(r)$. This equality, the second and the third relations in (5.1) together with (5.10) imply that $\theta_{\omega}(r) \cdot \bar{\theta}(r) = \theta_{\omega}(r) \cdot \theta(r)$. So the product $\theta_{\omega}(r) \cdot \theta(r)$ has a limit at $r=0$.

We will complete the proof of Proposition 6.2 by considering separately each of the two possibilities:

$$
\begin{align*}
\text{(i)} & \quad j = i \quad \text{and} \quad j' \geq i + 1, \\
\text{(ii)} & \quad j' = i + 1. \quad \text{Here and in the proof of Proposition 6.1 we denote by $a$ the same number. Therefore the limit at $r=0$ of the product in the third line of (6.15) coincides with that of the product $\psi(r)$ defined by (6.3). Denote}
\end{align*}
$$

$$
g(r) = \left( \frac{2}{(u + u')^3} - \frac{2}{(u - u')^3} \right).
$$

Then due to Lemma 5.1 and Lemma 5.4 the product in the second line of (6.15) has a limit at $r=0$ equal to that of $\psi_{n-1}(u, u') \cdot g(r)$. Thus the limit at $r=0$ of the product $\psi_{n-1}(u, u') \cdot \bar{\theta}_{\omega}(r)$ is equal to that of

$$
\begin{align*}
\theta_{\omega}(r) \cdot \bar{\theta}(r) \cdot \psi_{n-1}(u, u') \cdot g(r) \cdot \psi(r) \\
= \psi_{n-1}(u, u') \cdot \bar{\theta}(r) \cdot g(r) \prod_{p=1}^{n-1} \psi_{n-1-p}(u_{n-1, u_{\alpha_{n-1}(p)})} \\
\times \prod_{p=1}^{n-1} \psi_{n-p}(u_{n}, u_{\alpha_{n-1}(p)}) \psi(r) \\
= \psi_{n-1}(u, u') \cdot g(r) \cdot \psi(r)
\end{align*}
$$

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where \( \tilde{r}(r) \) is defined by (6.1); to get the above equalities we made use of (6.12) to (6.15). But the right hand side of the above equalities has the zero limit at \( r = 0 \) by (6.4).

The following two properties of the element \( \psi_i(r) \) constitute the main result of this section. The part (a) of the next theorem is matched with Proposition 5.7.

**Theorem 6.3.** Suppose that \( k = A^1(i, j) \) and \( k + 1 = A^2(i, j) \). Then

(a) \( \lim_{r \to 0} \tilde{r}\psi_i(u(i, j), u(i, j)) = 0; \)

(b) \( \lim_{r \to 0} \tilde{r}\psi_i(r) \cdot \tilde{r}\psi_{i}^{-1}(u(i, j), u(i, j)) = 0. \)

**Proof.** All factors at the right hand side of the equality (5.8) have limits at \( r = 0 \). By (5.8) the part (a) of Theorem 6.3 follows from Proposition 6.2. We shall derive (b) from (a). Define an involutive antiautomorphism of the algebra \( M_n \) by

\[
(s_1, ..., s_{n-1}) \mapsto (s_{n-1}, ..., s_1); \quad (c_1, ..., c_n) \mapsto (c_n, ..., c_1).
\]

Evidently, then

\[
\psi_i(u(i, j), u(i, j)) \mapsto \psi_{i}^{-1}(u(i, j), u(i, j)).
\]

We shall prove that the element \( \psi_i(r) \) is a fixed point of this antiautomorphism. By applying it to the equality in (a) we will then obtain (b).

Observe that for the tableau \( A = A^1 \) we have \( \beta_{k^1} = (1, 2, ..., k^1 - 1) \). Therefore by the definition (5.6) we get

\[
\tilde{r}\psi_i(r) = \prod_{k^1 = 2}^{n} \left( \prod_{p = 1}^{k - 1} \tilde{r}\psi_{k^1 - p}(u_{k^1}, u_{p}) \right).
\]

We have to prove that also

\[
\tilde{r}\psi_i(r) = \prod_{k^1 = 2}^{n} \left( \prod_{p = 1}^{k - 1} \tilde{r}\psi_{n - k^1 + p}(u_{k^1}, u_{p}) \right); \tag{6.16}
\]

to do that we will use the reduced decomposition

\[
w_0 = \prod_{k^1 = 2}^{n} (s_{n-1} \cdots s_{n-k^1 + 1}).
\]
For any indices $k' \geq 2$ and $p \leq k' - 1$ consider the elements of $S_n$

$$s = s_{n-k'+p} s_{n-k'+1} \cdot \prod_{k'=2}^{k'+1} (s_{n-k'+1} \cdots s_{n-k'} + 1),$$

$$w = \prod_{k'=2}^{k'+1} (s_{n-k'+1} \cdots s_{n-k'} + 1).$$

then $w_0 = w s_{n-k'} + p s$ and $\text{lenth}(w_0) = \text{lenth}(s) + \text{lenth}(w) + 1$. Moreover, then

$$\pi_s (\Phi_{n-k'+p})(1) = \varphi_{n-k'} + p (u_k', u_p)$$

as

$$s \cdot \chi(x_{n-k'+p}) = \chi(x_{n-k'+1}) = w_0 \cdot \chi(x_k') = u_k',$$

$$s \cdot \chi(x_{n-k'+p+1}) = w_0 \cdot \chi(x_{n-k'+p}) = w_0 \cdot \chi(x_p) = u_p.$$ Therefore due to Corollary 3.4 we obtain that

$$\pi_s (\Phi_{n-k'}) = \prod_{k'=2}^{k'+1} \left( \prod_{k'=2}^{k'+1} \varphi_{n-k'+p} (u_k', u_p) \right).$$

This formula along with the first equality in (5.8) provides (6.16). The proof of Theorem 6.3 has been now completed.

7. REPRESENTATION $V_\lambda$ OF THE ALGEBRA $S_n$

Let $\lambda$ run through the set $S_n^*$ of the standard shifted tableaux of form $\lambda$. Theorem 5.6 allows us to define the elements of the algebra $M_n$

$$\psi_\lambda = \lim_{r \to 0} \psi_\lambda(r).$$

These elements have the following remarkable property. Put

$$u_\lambda(i, j) = \lim_{r \to 0} u(i, j) = \sqrt{(j-i)(j-i+1)}.$$ Let $k = A(i, j)$ and $k+1 = A(i', j')$. Note that if $s_k w_\lambda \succ w_\lambda$, then the tableau $s_k \cdot w_\lambda$ is standard due to Lemma 2.2(b).
Proposition 7.1.  

(a) If \( s_k w_A \succ w_A \), then \( \psi_s(u_0(i, j), u_0(i', j')) \cdot \psi_A = \psi_{s_k \cdot A} \).

(b) If \( s_k \cdot A \not\in \mathcal{F}_j \), then \( \psi_s(u_0(i, j), u_0(i', j')) \cdot \psi_A = 0 \).

Proof. Suppose that \( s_k w_A \succ w_A \) Since
\[
  w_A \cdot \chi(x_k) = u(i, j) \quad \text{and} \quad w_A \cdot \chi(x_{k+1}) = u(i', j')
\]
by the equality (4.1), thanks to Proposition 3.3 and to (4.2) we get
\[
  \pi_j(\Phi_{s_k w_A})(1) = \pi_{w_A}(\Phi_k)(1) \cdot \pi_j(\Phi_{u_0})(1) = \varphi_j(u(i, j), u(i', j')) \cdot \pi_j(\Phi_{u_0})(1).
\]

The element \( \psi_s(r) \) differs from \( \pi_j(\Phi_{s_k w_A})(1) \) only by a scalar factor. The element \( \psi_{s_k \cdot A}(r) \) differs from \( \pi_j(\Phi_{s_k w_A})(1) \) also only by a scalar factor. Therefore
\[
  \psi_{s_k \cdot A}(r) = h(r) \cdot \psi_s(u(i, j), u(i', j')) \cdot \psi_A(r)
\]
for some scalar factor \( h(r) \). As an argument similar to that starting the proof of Theorem 5.6 shows,
\[
  \psi_{s_k \cdot A}(r) = s_k w_A + \sum_{s \in S_n} z_s s, \quad z_s \in \mathbb{Z}_n,
\]
\[
  \psi_A(u(i, j), u(i', j')) \cdot \psi_A(r) = s_k w_A + \sum_{s \in S_n} z'_s s, \quad z'_s \in \mathbb{Z}_n
\]
where \( z_s, z'_s \neq 0 \) only if \( \text{length}(s) < \text{length}(s_k w_A) \). Thus \( h(r) = 1 \). Taking the limit at \( r = 0 \) in the equality (7.1), we get the part (a) of Proposition 7.1.

Now suppose that \( s_k \cdot A \not\in \mathcal{F}_j \). Since the tableau \( A \) is standard, only the two cases are possible: either \( i' = i + 1 \) and \( j' = j + 1 \), or \( i' = i + 1 \) and \( j' = j \). In the former case the part (b) of Proposition 7.1 follows directly from Proposition 5.7.

Assume that \( i' = i + 1 \) and \( j' = j \). Then \( s_k w_A \not\preceq w_A \) by Lemma 2.2, and \( s_k w_A \succ s_j \) by Lemma 2.1. Put \( \tilde{K} = A'(i, j) \), then \( \tilde{K} + 1 = A'(i + 1, j) \) and by the definition of the element \( s_A \) we get \( s_As_k = s_Ks_A \). In particular,
\[
  \text{length}(s_k s_A) = \text{length}(s_A s_k) = \text{length}(s_A) + 1.
\]
Due to the equality (4.1) we have
\[
  w_A \cdot \chi(x_k) = w_A \cdot \chi(x_k) = u(i, j),
\]
\[
  w_A \cdot \chi(x_{k+1}) = w_A \cdot \chi(x_{k+1}) = u(i + 1, j).
\]
Therefore by (4.2), Proposition 3.3 and Lemma 2.1 we get the equalities

\[
\begin{align*}
\pi_{u_3w}(\Phi_{u_3})(1) \cdot \psi_{x}(u(i, j), u(i + 1, j)) &= \pi_{w_{u_3}}(\Phi_{w})(1) = \pi_{w_{u_3}}(\Phi_{u_3})(1) \\
&= \pi_{u_3w}(\Phi_{u_3})(1) = \pi_{u_3w}(\Phi_{u_3})(1) \\
&= \pi_{u_3w}(\Phi_{u_3})(1).
\end{align*}
\]

(7.2)

Write \( u_3 = \gamma(x_v) \); then by (4.3), (5.6) and by the first formula of Proposition 4.1 we get the equality

\[
\begin{align*}
\pi_{u_3w}(\Phi_{u_3})(1) \cdot \psi_{x}(r) &= \psi_{x}(r) \cdot \prod_{q=1}^{n} (u^{2}(q) - u^{2}_{k}). (7.3)
\end{align*}
\]

It follows from (7.2) and (7.3) that

\[
\begin{align*}
\pi_{u_3w}(\Phi_{u_3})(1) \cdot \psi_{x}(u(i, j), u(i + 1, j)) \cdot \psi_{x}(r) &= \psi_{x}(u(i, j), u(i + 1, j)) \cdot \psi_{x}(r) \cdot \prod_{q=1}^{n} (u^{2}(q) - u^{2}_{k}). (7.4)
\end{align*}
\]

By Theorem 6.3(a) the right hand side of the equality (7.4) has the zero limit at \( r = 0 \). We will prove that the factor \( \pi_{u_3w}(\Phi_{u_3})(1) \) at the left hand side of (7.4) is continuous and invertible for any \( r \) such that \( 0 < r < 1 \). Then, taking the limit at \( r = 0 \) in (7.4), we will get the part b) of Proposition 7.1.

Define the character \( \tilde{\gamma} \) by the equality \( \tilde{\gamma} = \gamma \). Let the sequence \((\tilde{u}_{1}, \ldots, \tilde{u}_{n})\) be obtained from \((u_{1}, \ldots, u_{n})\) by transposition of the terms \( u_{i} \) and \( u_{i+1} \); then \( \tilde{\gamma}(x_v) = \tilde{u}_{k} \). Applying the second formula of Proposition 4.1 to the character \( \tilde{\gamma} \), we get

\[
\begin{align*}
\pi_{u_3w}(\Phi_{u_3})(1) &= \pi_{u_3w}(\Phi_{u_3})(1) \\
&= \prod_{k=1}^{n} \left( \prod_{q=1}^{n} (\tilde{u}^{2}(q) - \tilde{u}^{2}_{k}) \right). (7.5)
\end{align*}
\]

In Section 5 we proved that the pairs \((u_{k}, u_{j})\) does not satisfy the condition (4.11) for any \( r \) such that \( 0 < r < 1 \). By the definition of the sequences \( \mathcal{B}_{k} \) and by the assumption \( i' = i + 1, j' = j \) we have

\[
\begin{align*}
k \in \mathcal{B}_{k} \iff k + 1 \in \mathcal{B}_{k}, \quad k' \neq k, k + 1; \quad \mathcal{B}_{k} = \mathcal{B}_{k+1}.
\end{align*}
\]
Therefore the set of all the pairs \((\bar{u}_k, q), u_k\) coincides with the set of all the pairs \((\bar{u}_k, \bar{q}), \bar{u}_k\). Thus each factor at the right hand side of (7.5) is continuous and invertible for any \(r\) such that \(0 \leq r < 1\).

Denote by \(V_2\) the left ideal in the algebra \(M_n\) generated by the element

\[\psi_{\delta} = \psi_{\delta} = \lim_{r \to 0} \psi_{\delta}(r).\]  

(7.6)

Due to (4.1) one can introduce the character \(\chi_0\) of \(\mathbb{C}[x_1, \ldots, x_n] \subset S_n\) such that

\[w_{\delta} \cdot \chi_0(x_k) = u_0(i, j), \quad A(i, j) = k.\]  

(7.7)

Due to Proposition 3.5 we have defined a homomorphism \(r: S_n \to M_n\) identical on \(M_n\). Consider \(V_2\) as a subspace in the representation space \(M_{n, n}\) of the algebra \(S_n\). Define the set

\[\mathcal{C} = \{c_{k_1} \cdots c_{k_p} \mid 1 \leq k_1 < \cdots < k_p \leq n\} \subset \mathbb{Z}_n.\]

For each \(c \in \mathcal{C}\) put \(u_c(k) = 1\) if the element \(c\) is contained in the subgroup of \(C_{n, n}\) generated by \(c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n\) and put \(u_c(k) = -1\) otherwise.

**Theorem 7.2.** (a) The elements \(c\psi_{\delta}\) where \(c \in \mathcal{C}\) and \(A \in \mathcal{F}_n\), form a basis of \(V_2\).

(b) If \(k = A(i, j)\) then \(\pi_{\delta}(x_k)(c\psi_{\delta}) = u_c(k)u_0(i, j) \cdot c\psi_{\delta}\).

(c) The actions of \(x_k\) and \(t(x_k)\) in \(V_2\) coincide.

**Proof.** We have already proved that the element \(\psi_{\delta}(r)\) has the form

\[w_{\delta} = \sum_{s \in S_n} z_s s, \quad z_s \in \mathbb{Z}_n;\]

where \(z_s \neq 0\) only if \(\text{length}(s) < \text{length}(w_{\delta})\). Therefore the element \(\psi_{\delta}\) has the same form. Thus all the elements \(c\psi_{\delta}\) where \(c \in \mathcal{C}\) and \(A \in \mathcal{F}_n\), are linearly independent. Denote by \(V\) the subspace in \(M_n\) spanned by these elements.

The element \(\psi_{\delta}(r)\) differs from \(\pi_{\delta}(\Phi_{\delta})(1)\) only by a scalar factor. Therefore due to (3.6), (4.1), (7.7) and to Corollary 3.4, we have

\[\pi_{\delta}(x_k)(\psi_{\delta}) = w_{\delta} \cdot \chi_0(x_k).\]  

(7.8)

Furthermore, by the definition of the algebra \(S_n\) we have \(x_k c = u_c(k) \cdot c x_k\).

Now by (3.6), (7.7) and (7.8) we get the part (b) of Theorem 7.2. In particular, the action of the subalgebra \(\mathbb{C}[x_1, \ldots, x_n]\) in \(M_{n, n}\) preserves the subspace \(V\).
The elements $s_k$ and $c_k$ of the algebra $S_n$ act in $M_{2n} = M_n$ via the left multiplication. In particular, for each $A \in \mathcal{S}_k$ the action of $c_1, \ldots, c_n$ preserves the subspace

$$\bigoplus_{c \in \mathcal{C}} \mathbb{C} \cdot c \psi_A$$

in $V$, and does not preserve any of the subspaces

$$\bigoplus_{c \in \mathcal{C}} \mathbb{C} \cdot c \psi_A$$

where $\mathcal{C}'$ is a non-empty proper subset of $\mathcal{C}$. We shall prove that the action of $s_1, \ldots, s_{n-1}$ preserves the subspace $V$, and does not preserve any of the subspaces

$$\bigoplus_{c \in \mathcal{C} \setminus \mathcal{C}' \cup \{A\}} \mathbb{C} \cdot c \psi_A$$ (7.9)

where $\mathcal{C}$ is a non-empty proper subset of $\mathcal{S}_k$. Then $V = M_n \psi_V = V$, and the part (a) of Theorem 7.2 will be proved. Let a tableau $A \in \mathcal{S}_k$ and an index $k \in \{1, \ldots, n-1\}$ be fixed. Let $k = A(i, j)$ and $k+1 = A(i', j')$, as above. Put

$$u = u_0(i, j) = \sqrt{(j-i)(j-i + 1)}, \quad u' = u_0(i', j') = \sqrt{(j'-i')(j'-i' + 1)},$$

$$y = 1 - \frac{1}{(u-u')^2} - \frac{1}{(u+u')^2}. \quad (7.10)$$

Suppose that the tableau $s_k \cdot A$ is standard and that $i < i'$. Then $s_k \cdot w_A \succ w_A$ by Lemma 2.2(b), and by Proposition 7.1(a) we have $\psi_k(u, u') \cdot \psi_A = \psi_{s_k \cdot A}$. By rewriting the latter equality, we get

$$s_k \cdot \psi_A = \left(\frac{-1}{u-u'} + \frac{c_k \cdot c_{k+1}}{u+u'}\right) \psi_A + \psi_{s_k \cdot A}, \quad s_k \cdot A \in \mathcal{S}_k, \quad i < i'. \quad (7.11)$$

Now suppose that the tableau $s_k \cdot A$ is standard and that $i > i'$. Then $w_A \succ s_k \cdot w_A$. Applying Proposition 7.1(a) to the tableau $s_k \cdot A$ instead of $A$, we obtain the equality $\psi_k(u', u') \cdot \psi_{s_k \cdot A} = \psi_A$. Multiplying this equality on the left by $\psi_k(u, u') \cdot \psi_A = \psi_{s_k \cdot A}$, we get $\psi_k(u, u') \cdot \psi_A = \psi_{s_k \cdot A}$. By rewriting the latter equality, we get

$$s_k \cdot \psi_A = \left(\frac{-1}{u-u'} + \frac{c_k \cdot c_{k+1}}{u+u'}\right) \psi_A + y \psi_{s_k \cdot A}, \quad s_k \cdot A \in \mathcal{S}_k, \quad i > i'. \quad (7.12)$$
Finally, suppose that the tableau $s_k \cdot A$ is not standard. Then due to Proposition 7.1(b) we have the equality $\psi_k(u, u') \psi_A = 0$. By rewriting this equality, we get

$$s_k \cdot \psi_A = \left( \frac{-1}{u - u'} + \frac{c_k c_{k+1}}{u + u'} \right) \psi_A; \quad s_k \cdot A \notin \mathcal{S}.$$  \hfill (7.13)

Since $C_{\mathcal{S}}$ is a normal subgroup of $D_n$, the equalities (7.11) to (7.13) imply that the left multiplication in $M_n$ by the element $s_k$ also preserves the subspace $V$. Moreover, let $\mathcal{S}$ be a non-empty proper subset of $\mathcal{S}$. Then by Lemma 2.2(b) and Proposition 2.3 there exist a tableau $A$ and an index $k$ such that $A \in \mathcal{S}$ and $s_k \cdot A \in \mathcal{S} \setminus \mathcal{S}$. By the equalities (7.11) and (7.12) the action of the element $s_k$ then does not preserve the subspace (7.9) in $V$. So we get the part (a) of the Theorem 7.2.

Since $A(1, 1) = 1$ for any tableau $A \in \mathcal{S}$, the parts (a) and (b) imply that the restriction $\pi_\mathcal{S}(x_i) \mid V_\mathcal{S} = 0$. Thus the actions of $x_i$ and $t(x_i)$ in $V_\mathcal{S}$ coincide. Now the part (c) of Theorem 7.2 follows from Proposition 3.5.

By definition the algebra $M_n$ is the quotient $\mathbb{C}[D_n] / \langle \zeta = 1 \rangle$. Thus we get a representation of the group $D_n$ in the space $V_\mathcal{S}$. Theorem 7.2(a) together with the formulas (7.10) to (7.13) describes the action of the generators $s_k$ and $c_k$ of $D_n$ in $V_\mathcal{S}$ explicitly. However, this representation of $D_n$ turns out to be reducible. In the next section we shall point out the irreducible components of this representation.

8. YOUNG’S ORTHOGONAL FORM

For each $k = 1, \ldots, n - 1$ put $z_k = (1 + c_k c_{k+1})/\sqrt{2}$. The elements $z_1, \ldots, z_{n-1} \in Z_n$ are invertible and satisfy the following relations:

$$z_k z_{k'} = z_{k'} z_k, \quad k' - k > 1;$$
$$z_k c_k z_{k-1}^{-1} = c_{k+1}, \quad z_k c_k + 1 z_{k-1}^{-1} = -c_k;$$
$$z_k c_k z_{k-1}^{-1} = c_k, \quad k' \neq k, k + 1;$$

$$\left(c_k - c_{k+1}\right)/\sqrt{2} z_{k-1}^{-1} = c_k = z_k \left(c_k - c_{k+1}\right)/\sqrt{2};$$
$$z_k z_{k+1} = z_{k+1} z_k z_{k+1}.$$

\hfill (8.1)
Lemma 8.1. One can define a family of the elements $z_d \in Z_n$ where $A \in \mathcal{A}$, by the two conditions: (a) $z_d = 1$; (b) if $s_k w_d \succ w_d$ and $y$ is determined by $A$ and $k$ via (7.10), then

$$z_{d_{k+1}} = \frac{1}{\sqrt{y'}} \left\{ \begin{array}{ll} z_d z_k, & k+1 \in \langle A \rangle; \\ z_d, & \text{otherwise.} \end{array} \right.$$ 

Proof. Let a tableau $A \in \mathcal{A}$ be fixed. Due to Lemma 2.2(b) and to Proposition 2.3, there exist indices $k_1, \ldots, k_p$ such that $A = s_{k_p} \cdots s_{k_1}, A'$ and

$$s_{k_p} s_{k_{p-1}} \cdots s_{k_1} w_d > s_{k_{p-1}} \cdots s_{k_1} w_{d'}, \quad q = 1, \ldots, p.$$ 

Hence, making use of the condition (a) and applying (b) consecutively to

$$A = A', A_1, \ldots, A_{p-1} \quad \text{and} \quad k = k_1, k_2, \ldots, k_p$$ 

we get an element $z_d \in Z_n$. Let us prove that this element does not depend on the choice of the indices $k_1, \ldots, k_p$. Since $\text{length}(s_{k_1} \cdots s_{k_p}) = p$, it suffices to consider the following two situations.

(i) Suppose that

$$s_{k_p} s_{k_{p-1}} \cdots s_{k_1} w_{d'} > s_{k_{p-1}} \cdots s_{k_1} w_{d'}, \quad q = 1, \ldots, p. \quad (8.2)$$

Let the number $y$ be defined by the equalities (7.10). Define the numbers $y'$ and $y''$ by substituting in (7.10), instead of the pairs $(i, j)$, $(i', j')$, respectively the pairs $(i, j)$, $(i', j')$, $(i'', j'')$. Then, making use of the chain (8.2), we get the expression

$$z_{d_{k+1}} = \frac{1}{\sqrt{y' y''}} \left\{ \begin{array}{ll} z_d z_{k+1} z_k, & z + 2 \in \langle A \rangle; \\ z_d, & \text{otherwise.} \end{array} \right.$$ 

Making use of the chain (8.3), we get the same expression.

(ii) Suppose that for a certain index $k' \neq k, k+1$ we have $s_{k'} s_{k+1} w_d > s_k w_d > w_d$. Then $s_{k'} s_{k+1} w_d > s_k w_d > w_d$. By the first relation in (8.1), the above two chains provide the same expression for $z_{d_{k+1}}$. \]

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In Section 2 for each tableau \( A \in S \) we defined the element \( g_A \in S_n \).

**Proposition 8.2.** For each index \( k = 1, \ldots, n \) and any tableau \( A \in S \)

\[
Z_A c_k Z_A^{-1} = e_{g_A(k)}^{*} \begin{cases} 
1 & \text{if } k \in \langle A \rangle \\
(-1)^{k - g_A(k)} & \text{otherwise}.
\end{cases}
\]

**Proof.** We will use the induction on the length of \( w_A \). If \( A = A' \) then \( g_A = 1 \) by the definition (2.9), and \( Z_A = 1 \) also by definition; so the statement of Proposition 8.2 becomes trivial in that case.

Now assume that the statement of Proposition 8.2 is valid for a fixed \( A \in S \) and all the indices \( k = 1, \ldots, n \). Suppose that \( w_{x_A} \succ A \) for a certain index \( k' \). Then thanks to the relations in the second and the third lines of (8.1) we obtain that \( z_{x_A} c_k z_{x_A}^{-1} \) equals

\[
\begin{cases} 
Z_A c_{k+1} Z_A^{-1} & \text{if } k' + 1 \in \langle A \rangle \text{ and } k = k', \\
-Z_A c_{k-1} Z_A^{-1} & \text{if } k' + 1 \in \langle A \rangle \text{ and } k = k' + 1, \\
z_A c_k z_A^{-1} & \text{otherwise};
\end{cases}
\]

\[
e_{g_A(k+1)} c_{g_A(k)} c_{g_A(k-1)} (-1)^{k - g_A(k-1)} \begin{cases} 
1 & \text{if } k + 1 \in \langle A \rangle \text{ and } k = k', \\
(-1)^{k - g_A(k)} & \text{if } k \in \langle A \rangle \text{ and } k \neq k', k' + 1,
\end{cases}
\]

\[
e_{g_A(k)} c_{g_A(k-1)} (-1)^{k - g_A(k)} \begin{cases} 
1 & \text{if } k \in \langle A \rangle \text{ and } k = k' + 1, \\
(-1)^{k - g_A(k)} & \text{otherwise};
\end{cases}
\]

\[
e_{g_{x_A}(k)} \begin{cases} 
1 & \text{if } k \in \langle x_A \rangle \langle A \rangle, \\
(-1)^{k - g_{x_A}(k)} & \text{otherwise}.
\end{cases}
\]

Here the first equality follows from the inductive assumption and from that \( k' \notin \langle A \rangle \); the last equality is provided by Lemma 2.7.

In Section 1 we defined a certain representation \( \rho_x \) of the group \( D_n \). By Theorem 1.3, this representation is irreducible if the number \( l \) of the parts of \( x \) is even. If \( l \) is odd, then \( \rho_x \) splits into two non-equivalent irreducible components.

**Theorem 8.3.** The representation of the group \( D_n \) in the space \( V_x \) is isomorphic to the direct sum of \( 2^{l/2} \) copies of the representation \( \rho_x \).
Proof. Let an index $k \in \{1, \ldots, n-1\}$ and a tableau $A \in \mathscr{C}$ be fixed. Determine the numbers $u, u'$ and $y$ by (7.10). By Proposition 1.2(b), for any $c \in \mathcal{C}_{n}$ we have

$$s_{k} \cdot \psi_{A} = s_{k} \cdot c_{A} \cdot s_{k} \cdot \psi_{A} = \eta(t_{k}) c_{A} \cdot s_{k} \cdot \psi_{A} \cdot (-1)^{\deg(c)}$$

$$= \eta(t_{k}) (c_{k} + c_{k+1}) / \sqrt{2} \cdot s_{k} \cdot \psi_{A} \cdot (-1)^{\deg(c)}.$$ 

Therefore by (7.11) to (7.13) we get the following equalities.

$$s_{k} \cdot \psi_{A} = \eta(t_{k}) c_{A} \cdot \frac{u c_{k} - u' c_{k+1}}{u^{2} - u'^{2}} \cdot \frac{1}{\sqrt{2}} \cdot \psi_{A} \cdot (-1)^{\deg(c)} + \eta(t_{k}) (c_{k} + c_{k+1}) / \sqrt{2} \cdot \psi_{A} \cdot (-1)^{\deg(c)}; \quad s_{k} \cdot A \in \mathscr{C}, \quad i < i'; \quad (8.4)$$

$$s_{k} \cdot \psi_{A} = \eta(t_{k}) c_{A} \cdot \frac{u c_{k} - u' c_{k+1}}{u^{2} - u'^{2}} \cdot \frac{1}{\sqrt{2}} \cdot \psi_{A} \cdot (-1)^{\deg(c)} + \eta(t_{k}) c_{A} \cdot \frac{1}{\sqrt{2}} \cdot \psi_{A} \cdot (-1)^{\deg(c)}; \quad s_{k} \cdot A \in \mathscr{C}, \quad i > i'; \quad (8.5)$$

$$s_{k} \cdot \psi_{A} = \eta(t_{k}) c_{A} \cdot \frac{u c_{k} - u' c_{k+1}}{u^{2} - u'^{2}} \cdot \frac{1}{\sqrt{2}} \cdot \psi_{A} \cdot (-1)^{\deg(c)}; \quad s_{k} \cdot A \not\in \mathscr{C}. \quad (8.6)$$

By Theorem 7.2(a) the elements $c \psi_{A}$ where $c \in \mathcal{C}$ and $A \in \mathscr{C}$, form a basis of $V_{j}$. Put $\sigma_{A} = z_{A} \psi_{A}$. Since $z_{A} \in \mathbb{Z}_{n}$ is invertible, the elements $c \sigma_{A}$ also form a basis of $V_{j}$. We will describe the action of the generators $s_{1}, \ldots, s_{n-1}$ in this basis. Let $\langle A' \rangle' = \langle k_{1}, \ldots, k_{n-1} \rangle$; introduce the elements of the algebra $\mathbb{Z}_{n}$

$$d_{p} = c_{k_{p}} \cdot (-1)^{k_{p}}; \quad p = 1, \ldots, n-1.$$ 

Let $p$ be the quantity of the terms in the sequence $\langle A \rangle'$ which do not exceed $k + 1$. Evidently, then

$$k, k+1 \not\in \langle A \rangle \quad \Rightarrow \quad g_{A}(k) = k_{p-1}, \quad g_{A}(k+1) = k_{p};$$

$$k+1 \in \langle A \rangle \quad \Rightarrow \quad g_{A}(k) = k_{p};$$

$$k \in \langle A \rangle \quad \Rightarrow \quad g_{A}(k+1) = k_{p}. \quad (8.7)$$
Note that $\deg(z_A) = 0$. Consider the following five situations.

(i) Let $s_k w_A > w_A$ and $k + 1 \notin \langle A \rangle$. Then $s_k \cdot A \in S_s^e$, $k \notin \langle A \rangle$, $S_{s_k \cdot A} = g_A$ and $z_{s_k \cdot A} = -z_A/\sqrt{y}$ by Lemma 8.1(b). Therefore by (8.4), (8.5), (8.7) and by Proposition 8.2 we get

$$s_k \cdot c \left[ \begin{array}{c} \sigma_A \\ \sigma_{s_k \cdot A} \end{array} \right] = \eta(t_k) c z_A (c_k P - c_{k+1} Q) z_A^{-1} \cdot (-1)^{\deg(c)} \left[ \begin{array}{c} \sigma_A \\ \sigma_{s_k \cdot A} \end{array} \right]$$

$$= \eta(t_k) c (d_{p-1} P + d_p Q) \cdot (-1)^{\deg(c) + k} \left[ \begin{array}{c} \sigma_A \\ \sigma_{s_k \cdot A} \end{array} \right].$$

Here we use the matrices

$$P = \left[ \begin{array}{cc} \sqrt[2]{u^2 - u'^2} & \sqrt[2]{y^2} \\ \sqrt[2]{y^2} & -u' \sqrt[2]{u^2 - u'^2} \end{array} \right]$$

and

$$Q = \left[ \begin{array}{cc} u' \sqrt[2]{u^2 - u'^2} & \sqrt[2]{y^2} \\ \sqrt[2]{y^2} & -u \sqrt[2]{u^2 - u'^2} \end{array} \right].$$

(ii) Let $s_k \cdot A \notin S_s^e$ and $k, k + 1 \notin \langle A \rangle$. Then by (8.6), (8.7) and by Proposition 8.2

$$s_k \cdot c \sigma_A = \eta(t_k) c z_A \left( c_k \frac{u \sqrt[2]{u^2 - u'^2}}{u^2 - u'^2} - c_{k+1} \frac{u' \sqrt[2]{u^2 - u'^2}}{u^2 - u'^2} \right) z_A^{-1} \sigma_A \cdot (-1)^{\deg(c)}$$

$$= \eta(t_k) c (d_{p-1} \frac{u \sqrt[2]{u^2 - u'^2}}{u^2 - u'^2} + d_p \frac{u' \sqrt[2]{u^2 - u'^2}}{u^2 - u'^2}) \sigma_A \cdot (-1)^{\deg(c) + k}.$$ 

(iii) Let $s_k w_A > w_A$ and $k + 1 \in \langle A \rangle$. Then $s_k \cdot A \in S_s^e$, $k \notin \langle A \rangle$ and we have $z_{s_k \cdot A} = -z_A/\sqrt{y}$ by Lemma 8.1(b). Moreover, then $u' = 0$. Therefore by (8.4), (8.5), (8.7), Proposition 8.2 and by the relations in the fourth line of (8.1) we get

$$s_k \cdot c \left[ \begin{array}{c} \sigma_A \\ \sigma_{s_k \cdot A} \end{array} \right] = \eta(t_k) c z_A c_k z_A^{-1} R \cdot (-1)^{\deg(c)} \left[ \begin{array}{c} \sigma_A \\ \sigma_{s_k \cdot A} \end{array} \right]$$

$$= \eta(t_k) c d_p R \cdot (-1)^{\deg(c) + k} \left[ \begin{array}{c} \sigma_A \\ \sigma_{s_k \cdot A} \end{array} \right].$$
here we use the matrix

$$\begin{bmatrix}
\sqrt{2}/u & \sqrt{y} \\
\sqrt{y} & -\sqrt{2}/u
\end{bmatrix}.$$ 

(iv) Let $s_k \cdot A \notin \mathcal{S}_k$ and $k + 1 \in \langle A \rangle$; then $k \notin \langle A \rangle$. Then $u = \sqrt{2}$ and $u' = 0$. Therefore by (8.6), (8.7) and by Proposition 8.2 we get

$$s_k \cdot c_A = \eta(t_k) cz_A c_k z_A^{-1} \cdot c_A \cdot (-1)^{\deg(c)}$$

$$= \eta(t_k) c d_p \cdot c_A \cdot (-1)^{\deg(c)} + k.$$

(v) Let $s_k \cdot A \notin \mathcal{S}_k$ and $k + 1 \notin \langle A \rangle$. Moreover, then $u = 0$ and $u' = \sqrt{2}$. Therefore by (8.6), (8.7) and by Proposition 8.2 we get

$$s_k \cdot c_A = \eta(t_k) cz_A c_k z_A^{-1} \cdot c_A \cdot (-1)^{\deg(c)}$$

$$= \eta(t_k) c d_p \cdot c_A \cdot (-1)^{\deg(c)} + k + 1.$$

Now we can point out the irreducible components of the representation of the group $D_n$ in $V_*$. Denote by $X_j$ the vector space with the basis $\{\sigma_A \mid A \in \mathcal{S}_j\}$. Put

$$X_{A,k} = \begin{cases} 
C \cdot \sigma_A \oplus C \cdot \sigma_{s_k \cdot A} & \text{if } s_k \cdot A \in \mathcal{S}_j \text{ and } s_k \cdot \sigma_{s_k \cdot A} \notin \mathcal{S}_j; \\
C \cdot \sigma_A & \text{if } s_k \cdot A \notin \mathcal{S}_j.
\end{cases}$$

Since the elements $c_A$ form a basis of $V_j$, we have the decomposition $V_j = Z_n \oplus X_j$. From now on we will treat separately the cases when $n$ is even and when it is odd.

Suppose that $n$ is even, then there is only one irreducible representation $\nu$ of the algebra $Z_n$; let $U$ be the representation space of $\nu$. The vector space $Z_n$ can be identified with $U \otimes U$ so that the left and the right multiplication in $Z_n$ by $c \in U$ take the forms $\nu(c) \otimes \text{id}$ and $\text{id} \otimes \nu(c^{-1})$, respectively. Moreover, the linear operator $c \mapsto c \cdot (-1)^{\deg(c)}$ in $Z_n$ takes the form $\nu(c_0) \otimes \nu(c_0)$. Here we make use of the element $c_0$ introduced in Section 1. The linear operator in $Z_n$

$$c \mapsto \eta(t_k) c d_p \cdot (-1)^{\deg(c)} + 1$$

then takes the form $\nu(c_0 \eta(t_k)) \otimes \nu(c_0 d_p)$.

Thus we have $V_j = U \otimes U \oplus X_j$, and an element $c \in Cl_n$ acts in $V_j$ as $\nu(c) \otimes \text{id} \otimes \text{id}$. The action of the generator $s_k$ in $V_j$ preserves the subspace

$$U \otimes U \oplus X_{A,k} \subseteq U \otimes U \oplus X_j.$$

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In this subspace
\[ s_k \mapsto v(c_0 \eta(t_k)) \otimes \alpha, \]
where \( \alpha \) is the linear operator in \( U \otimes X_{A,k} \) defined by
\[
\begin{align*}
    s_k w_{A, k+1} & \mapsto w_{A, k+1} \otimes P + v(c_0 d_{p-1}) \otimes Q \cdot (-1)^{k+1}; \\
    s_k \cdot A \not\in A, k+1 & \in \langle A \rangle \\
    s_k \cdot A \not\in A, k \not\in \langle A \rangle \\
    s_k \cdot A \not\in A, k \in \langle A \rangle 
\end{align*}
\]
\[
\alpha = \left( \frac{u}{u'} \right)^{\frac{1}{2}} \left( \frac{u'}{u} \right)^{\frac{1}{2}} \cdot (-1)^{k+1}; \quad (8.8)
\]
Here \( P, Q \) and \( R \) are the above defined \( 2 \times 2 \)-matrices.

Due to Proposition 1.2(a) and to the relations (1.3), the map \( t_k \mapsto i \cdot v(c_0 \eta(t_k)) \) defines a representation of the group \( T_n \) in the space \( U \otimes X_{A,k} \), such that \( \xi \mapsto \eta \), as follows: the action of the generator \( t_k \) preserves the subspace
\[
U \otimes X_{A,k} \subset U \otimes X_{A,k}, \quad (8.9)
\]
and in this subspace
\[
t_k \mapsto -i \cdot \alpha. \quad (8.10)
\]

Now suppose that \( n \) is odd. Then there are two non-equivalent irreducible representations \( v \) and \( v' \) of the algebra \( Z_n \); they have a common representation space \( U \). The vector space \( Z_n \) can be identified with \( C^2 \otimes U \otimes U \) so that the left and the right multiplication in \( Z_n \) by \( c \in C \) take the forms \( J \otimes v(c) \otimes \text{id} \) and \( J \otimes \text{id} \otimes v(c^{-1}) \), respectively. Moreover, we can assume that the linear operator \( c \mapsto c \cdot (-1)^{\deg(c)} \) in \( Z_n \) takes the form \( I \otimes \text{id} \otimes \text{id} \). Then the linear operator in \( Z_n \)
\[
c \mapsto \eta(t_k) \cdot c \cdot d_p \cdot (-1)^{\deg(c)+1}
\]
takes the form $I \otimes v(\eta(t_k)) \otimes v(d_p)$. Thus we have $V_j = \mathbb{C}^2 \otimes U \otimes U \otimes X_j$, and an element $c \in C_n$ acts in $V_j$ as $J \otimes v(c) \otimes \text{id} \otimes \text{id}$. The action of the generator $s_k$ in $V_j$ preserves the subspace

$$\mathbb{C}^2 \otimes U \otimes U \otimes X_{A,k} \subset \mathbb{C}^2 \otimes U \otimes U \otimes X_j.$$

In this subspace

$$s_k \mapsto I \otimes v(\eta(t_k)) \otimes \alpha,$$

where now $\alpha$ is the linear operator in $U \otimes X_{A,k}$ defined by

$$s_k w \mapsto w, k \neq \langle A \rangle \Rightarrow \alpha = (v(d_{p-1}) \otimes P + v(d_p) \otimes Q) \cdot (-1)^{k+1};$$

$$s_k \cdot A \notin G^k; \quad k, k + 1 \neq \langle A \rangle \Rightarrow \alpha = \left( v(d_{p-1}) \otimes \frac{u \sqrt{3}}{u^2 - u'^2} + v(d_p) \otimes \frac{u' \sqrt{3}}{u^2 - u'^2} \right) \cdot (-1)^{k+1};$$

$$s_k w \mapsto w, k \neq \langle A \rangle \Rightarrow \alpha = v(d_p) \otimes P \cdot (-1)^{k+1};$$

$$s_k \cdot A \notin G^k, k \neq \langle A \rangle \Rightarrow \alpha = v(d_p) \otimes 1 \cdot (-1)^{k+1};$$

$$s_k \cdot A \notin G^k, k \neq \langle A \rangle \Rightarrow \alpha = v(d_p) \otimes 1 \cdot (-1)^k.$$

By Proposition 1.2(a), the map $t_k \mapsto v(\eta(t_k))$ defines a representation of the group $T_n$ in the space $U \otimes X_j$, such that $\zeta \mapsto -\text{id}$. Therefore one can define a representation of the group $T_n$ in the space $U \otimes X_j$, such that $\zeta \mapsto -\text{id}$, as follows: the action of the generator $t_k$ preserves the subspace

$$U \otimes X_{A,k} \subset U \otimes X_j,$$

In this subspace

$$t_k \mapsto \alpha.$$

Now we will again treat simultaneously the cases when $n$ is even and when it is odd. If $n$ is even, denote by $Z$ the subalgebra in $\mathbb{Z}_n$ generated by the elements $c_0, d_1, \ldots, c_0, d_{n-1}$. Then there is only one irreducible representation $v_{l,1}$ of the algebra $Z$; denote by $U_{l,1}$ the representation space of $v_{l,1}$.

If $n$ is odd, we will denote by $Z$ the the subalgebra in $\mathbb{Z}_n$ generated by the elements $d_1, \ldots, d_{n-1}$. Then the algebra $Z$ has two irreducible non-equivalent representations $v_{l,1}$ and $v_{l,-1}$; denote by $U_{l,1}$ and $U_{l,-1}$ their representation spaces.
Let $\varepsilon = (\pm 1)^{n-\ell}$. Substitute $v_{l,\varepsilon}$ and $U_{l,\varepsilon}$ for $v$ and $U$ in (8.8) to (8.10) if $n$ is even, or in (8.11) to (8.13) if $n$ is odd. Then we get the same explicit realization of the representation $\tau_{l,\varepsilon}$ of the group $T_n$ as produced in [N1]. In particular, the representation of $T_n$ obtained by this substitution is irreducible, and its space is

$$U_{l,\varepsilon} \otimes X_\varepsilon = U_{l,\varepsilon}. \quad (8.14)$$

If $n-\ell$ is even, the restriction of the representation $\psi$ onto $Z$ splits into the direct sum of $2^{|(l-1)/2|}$ copies of $v_{l,1}$. If $n-\ell$ is odd, this restriction splits into the direct sum of $2^{|(l-1)/2|}$ copies of $v_{l,1} \oplus v_{l,-1}$. By the construction of the representation $\rho_{ij}$ of the group $D_n$ in Section 1, we obtain that the representation of the group $D_n$ in the space $V_j$ splits into the direct sum of $2^{|(l-1)/2|}$ copies of the representation $\rho_{ij}$.}

Remark 8.4. By the formulas (8.8) to (8.10) if $n$ is even, and by (8.11) to (8.13) if $n$ is odd, we obtain a new proof of Theorem 1 from [N1].

9. YOUNG'S SYMMETRIZER

Denote by $Y_\varepsilon$ the Young subgroup in $S_n$ corresponding to the partition $\varepsilon$. This subgroup consists of all the elements which preserve each set of the numbers occurring in the same row of the tableau $\lambda$. Consider the restriction of the representation of $D_n$ in the space $V_\varepsilon$ onto the subgroup $D_\varepsilon = Y_\varepsilon \rtimes C_{m}$. By (7.13), this restriction preserves the subspace $Z_n \psi_\varepsilon = Z_n \psi_{\varepsilon'}$ in $V_\varepsilon$. Moreover, if $s_k \in Y_\varepsilon$ then

$$s_k \cdot \psi_\varepsilon = \left( \frac{1}{u-u'} + \frac{c_k s_{k+1}}{u'+u} \right) \psi_\varepsilon \quad (9.1)$$

where $\lambda(i, j) = k$ and

$$u = \sqrt{(j-i)(j-i+1)}, \quad u' = \sqrt{(j-i+1)(j-i+2)}.$$

Proposition 9.1. The representation of the group $D_\varepsilon$ in the space $Z_n \psi_\varepsilon$ coincides with its isotypical component in $V_\varepsilon$.

Proof. Let $V$ be any subspace in $V_\varepsilon$ preserved by the action of the group $D_\varepsilon$. Suppose there is a surjective $D_\varepsilon$-intertwining operator

$$Z_n \psi_{\varepsilon'} \rightarrow V. \quad (9.2)$$
We have to prove that \( V \subset Z_n \psi \sigma \). We will use several facts established in the proof of Theorem 8.3, and keep to the notation introduced therein. Our arguments will depend on the parity of \( n \). Let an element \( s_k \in Y_j \) be fixed.

Suppose that \( n \) is even, then the space \( V_j \) has the form \( U \otimes U \otimes X_j \). The subspace \( Z_n \psi \sigma \) then takes the form \( U \otimes U \otimes C \cdot \sigma \). Since an element \( c \in Cl_n \) acts in \( V_j \) as \( v(c) \otimes \text{id} \otimes \text{id} \) and \( V \) is preserved by this action, we obtain that \( V = U \otimes X \) for a certain subspace \( X \subset U \otimes X_j \). Moreover, it follows from Proposition 1.2(b) and from the irreducibility of \( v \) that in the space \( V \) for a certain element \( \beta \in \text{End}(X) \)

\[ s_k \mapsto v(c, \eta(t_k)) \otimes \beta. \]

Denote by \( T_j \) the inverse image in \( T_n \) of the subgroup \( Y_j \subset S_n \). By Proposition 1.2 the map \( t_k \mapsto -i \cdot \beta \) defines a representation of the group \( T_j \) in the space \( X \), such that \( \zeta \mapsto -\text{id} \). It follows again from the irreducibility of \( v \) that the operator (9.2) takes the form \( \text{id} \otimes \zeta \), where the operator \( \zeta : U \otimes C \cdot \sigma \sigma \rightarrow X \) is surjective and intertwining with respect to \( T_j \).

Each irreducible component of the representation space \( U \otimes X_j \) of the group \( T_j \), has the form (8.14) where \( U_{\mu, j} \subset U \). The action of the subgroup \( T_k \subset T_n \) in the space (8.14) preserves the subspace \( U_{\mu, j} \otimes C \cdot \sigma \sigma \). It was proved in [S] that this subspace is irreducible and coincides with its isotypical component in (8.14). Thus \( X \subset U \otimes C \cdot \sigma \sigma \), and Proposition 9.1 is proved for an even \( n \).

Now suppose that \( n \) is odd, then \( V_j = C^2 \otimes U \otimes U \otimes X_j \). The subspace \( Z_n \psi \sigma \) takes the form \( C^2 \otimes U \otimes U \otimes C \cdot \sigma \sigma \). An element \( c \in Cl_n \) acts in the space \( V_j \) as \( J \otimes v(c) \otimes \text{id} \otimes \text{id} \), and \( V \) is preserved by this action. The algebra \( Z_n \) is \( Z_2 \)-graded, and the restriction of \( v \) onto the subalgebra consisting of all the elements of the zero degree, is irreducible. Therefore by Proposition 1.2 for a certain subspace \( X \subset U \otimes X_j \) we obtain that \( V = C^2 \otimes U \otimes X \), and that in the space \( V \)

\[ s_k \mapsto I \otimes v(\eta(t_k)) \otimes \beta_0 + K \otimes v(\eta(t_k)) \otimes \beta_1 \]

for some elements \( \beta_0, \beta_1 \in \text{End}(X) \). The map \( t_k \mapsto I \otimes \beta_0 + K \otimes \beta_1 \) then defines a representation of the group \( T_j \) in the space \( C^2 \otimes X \), such that \( \zeta \mapsto -\text{id} \). It follows again from the irreducibility of \( v \) that the operator (9.2) takes the form \( E \otimes \text{id} \otimes \zeta_0 + J \otimes \text{id} \otimes \zeta_1 \). The operator

\[ E \otimes \zeta_0 + J \otimes \zeta_1 : C^2 \otimes U \otimes C \cdot \sigma \sigma \longrightarrow C^2 \otimes X \]
is surjective and intertwining with respect to $T_*$. For each $A \in \mathcal{A}$ the
element $t_k$ acts in the subspace
\[ C^2 \otimes U \otimes X_{t_k} \subset C^2 \otimes U \otimes X_\lambda \]
by $I \otimes \lambda$, where the operator $\lambda$ is defined by (8.11). As an argument similar
to that used for an even $n$ shows, the subspace
\[ C^2 \otimes U \otimes C \cdot \sigma_{\mathcal{A}} \subset C^2 \otimes U \otimes X_\lambda \]
coincides with its isotypical component with respect to $T_*$. Therefore
\[ C^2 \otimes X \subset C^2 \otimes U \otimes C \cdot \sigma_{\mathcal{A}}, \]
and Proposition 9.1 is proved for an odd $n$. 

In Section 7 we defined the vector space $V_\lambda$ to be the left ideal in the
algebra $M_n$ generated by the element $\psi_\lambda = \psi_{\mathcal{A}}$. By Theorem 7.2(a) we have
$\dim V_\lambda = 2^n \cdot f_\lambda$, where $f_\lambda$ is the number of the standard shifted tableaux
of shape $\lambda$. There is an explicit formula for this number:
\[ f_\lambda = \frac{n!}{\lambda_1! \cdots \lambda_l!} \prod_{1 \leq i < j \leq l} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}; \]
see [M, Ex. III.7.8]. Consider the element of the algebra $M_n$
\[ v_\lambda = \frac{f_\lambda}{n!} \psi_{\mathcal{A}} w_{\mathcal{A}}^{-1} = \frac{f_\lambda}{n!} \psi_{\mathcal{A}} w_{\mathcal{A}}^{-1}. \]

Denote by $Z_\lambda$ the subalgebra in $Z_n$ generated by the elements $c_k$ where
$k \in \langle \mathcal{A} \rangle$.

**Theorem 9.2.** For some non-zero element $z_\lambda \in Z_\lambda$ we have $v_\lambda^2 = z_\lambda v_\lambda$.

**Proof.** Consider the operator $\mu$ of the right multiplication in $M_n$ by
$w_{\mathcal{A}}^{-1} \psi_{\mathcal{A}} \cdot f_\lambda/n!$. This operator commutes with the action of the group $D_n$
and preserves the subspace $V_\lambda$. Therefore by Proposition 9.1 we get
$\mu(\mathcal{Z}_n \psi_{\mathcal{A}}) \subset \mathcal{Z}_n \psi_{\mathcal{A}}$. In particular, we get the equality
\[ \psi_{\mathcal{A}} w_{\mathcal{A}}^{-1} \psi_{\mathcal{A}}^* f_\lambda/n! = z_\lambda \psi_{\mathcal{A}} \]
for a certain element $z_\lambda \in \mathcal{Z}_n$. By the definition of $v_\lambda$ we then get $v_\lambda^2 = z_\lambda \cdot v_\lambda$.

Let $f$ be the coefficient of $1$ in the decomposition of $z_\lambda$ with respect to the
basis $\mathcal{B}$ in $\mathcal{Z}_n$. We shall prove that $f = 1$. To do that let us calculate the
trace of the operator $\mu$ in two different ways (cf. [W]). Thanks to (5.6) we have the expansion
\[
w_{-1}^{\mu} \psi_{\xi} = 1 + \sum_{z \in S_n \setminus \{1\}} z \cdot z, \quad z \in Z_n.
\] (9.4)

By choosing in the space $M_n$ the basis $\mathcal{C} \cdot S_n$ we get $\text{tr}(\mu) = 2^n : f_j$.

On the other hand, we have $\mu(M_n) = V_j$. Due to Theorem 7.2 there exists a subset $\mathcal{S}$ in $S_n$ such that the elements $sc\psi_{\xi}$ where $(s, c) \in \mathcal{S} \times \mathcal{C}$, form a basis in $V_j$. By the equality (9.3) we have $\mu(sc\psi_{\xi}) = scz_j \psi_{\xi}$. Since $\#(s, c) = f_j$, we then get $\text{tr}(\mu) = 2^n : f_j$. By comparing the above two expressions for the trace of $\mu$ we obtain the equality $f = 1$. Now let us prove that $z_j \in Z_j$. Let an element $s_k \in Y_j$ be fixed. Denote
\[
z = \left( \frac{-1}{u - u'} + \frac{c_k c_{k+1}}{u + u'} \right)
\] where $u$ and $u'$ are as in (9.1). Multiplying both sides of the equality (9.3) by $s_k$ on the left and making use of (9.1), we get
\[
z \psi_{\xi} w_{-1}^{\mu} \psi_{\xi} = s_k z_j s_k z \psi_{\xi}.
\] (9.5)

Using the equality (9.3) once more and taking into account (9.4) we obtain from (9.5) that
\[
z z_j = s_k z_j s_k z \psi_{\xi}.
\] (9.6)

Consider the expansion $z_j = z_0 + c_k z_1 + c_{k+1} z_1' + c_k c_{k+1} z_0'$ where $z_0$, $z_1$, and $z_0'$, $z_1'$ are the elements of the subalgebra in $Z_n$ generated by $z_1$, $z_2$, ..., $z_n$, $z_{k+2}$, ..., $z_n$. Then (9.6) is equivalent to
\[
z_0 = 0 \quad \text{and} \quad uz_1 = u' z_1'.
\] (9.7)

Let us observe that $u' \neq 0$ always and that $u = 0$ if the number $k$ occurs on the main diagonal of $A'$. Apply the equalities (9.7) consecutively to
\[
k = \lambda_1 + \ldots + \lambda_{i-1} + 1, \ldots, \lambda_1 + \ldots + \lambda_i - 1
\] for each $i = 1, \ldots, l$. Then we obtain that $z_j$ belongs to the subalgebra in $Z_n$ generated by the elements $c_k$ where $k \in \langle A' \rangle$.

**Corollary 9.3.** If the partition $\lambda$ consists of only one part then $z_j = 1$. 

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Proof. In Section 1 we defined a $\mathbb{Z}_2$-gradation on the algebra $\mathbb{Z}_n$ by $\deg(c_i) = 1$. It follows from (5.6) and from the equality $v^2 = z_j v_j$ that $\deg(z_j) = 0$. If the partition $\lambda$ consists of only one part then we have $\mathbb{Z}_2 = \mathbb{C} \cdot 1 \oplus \mathbb{C} \cdot c_1$ and $z_j = 1$ by the equality $f = 1$ established in the proof of Theorem 9.2.

Conjecture 9.4. The equality $z_j = 1$ holds for any $\lambda$.

The next theorem justifies the claim that the element $v_j \in M_n$ is an analogue of Young’s symmetrizer in $\mathbb{C}[S_n]$. Put $u = \sqrt{(j-i)(j-i+1)}$, $u' = \sqrt{(j-i-1)(j-i)}$.

Theorem 9.5. (a) If $k = A'(i, j-1)$ and $k + 1 = A'(i, j)$ then
\[
\left((k, k + 1) - \frac{1}{u - u'} - \frac{c_k c_{k+1}}{u + u'}\right) \cdot v_j = 0;
\]

(b) if $k = A'(i, j)$ and $k' = A'(i + 1, j)$ then
\[
v_j \cdot \left((k, k') + \frac{1}{u - u'} + \frac{c_k c_{k'}}{u + u'}\right) = 0.
\]

Proof. If the numbers $k$ and $k + 1$ occur in the same row of a tableau $A \in \mathcal{P}$, then $s_k \cdot A \notin \mathcal{P}$. By applying Proposition 7.1(b) to the tableau $A = A'$ we get the part (a) of Theorem 9.5. Let us now prove the part (b).

Denote $A'(i, j) = p$, then $A'(i + 1, j) = p + 1$. By applying Theorem 6.3(b) to the number $p$ instead of $k$ we obtain the equality
\[
\psi_{x'} \cdot \left(\left(n-p, n-p+1\right) + \frac{1}{u - u'} - \frac{c_p c_{n-p+1}}{u + u'}\right) = 0. \tag{9.8}
\]

The argument starting the proof of Theorem 5.6 shows that $\psi_{x'} = \psi_{x'}$ for a certain invertible element $\psi \in M_n$. Furthermore, by the definitions of the number $p$ and of the element $w_{x'} \in S_n$ we have
\[w_{x'}(n-p) = A'(i + 1, j) = k', \quad w_{x'}(n-p+1) = A'(i, j) = k.
\]

Therefore multiplying the equality (9.8) by $\psi^{-1}$ on the left and by $w_{x'}^{-1}$ on the right, we obtain the part (b) of Theorem 9.5.

Remark 9.6. The proof of Theorem 5.6 provides an explicit expression for the element $\psi_{x'} = \psi_{x'}$. Up to a scalar factor, this expression is a product of the elements of the form $\psi_{x'}(u, u')$ or $\psi_{x'}(u, u')$. The latter elements were defined at the beginning of Section 5.
**Example 9.7.** Let \( \lambda = (4, 3, 1) \). Then for the corresponding row tableau \( A' \) we have

\[
\begin{align*}
B_1 &= (1), \\
B_2 &= (12), \\
B_3 &= (123), \\
B_4 &= (12), \\
B_5 &= (1235), \\
B_6 &= (1253), \\
B_7 &= (125364), \\
B_8 &= (12536).
\end{align*}
\]

By Lemma 2.4 we get the reduced decomposition

\[
W_{x'} = S_1 \cdot S_2 S_1 \cdot S_3 S_2 S_1 \cdot S_4 S_2 S_3 S_2 S_1 \cdot S_5 S_3 S_5 S_2 S_1 \cdot S_7 S_6 S_3 S_4 S_1 \cdot S_7 S_6 S_3 S_4 S_1.
\]

Furthermore, for the tableau \( A' \) we have

\[
\begin{align*}
B'_1 &= B_1 = B_2 = B_3 = \emptyset, \\
B'_4 &= (34), \\
B'_5 &= (4), \\
B'_6 &= (47).
\end{align*}
\]

Therefore by the equality (5.7) we obtain that

\[
\psi_s(\sqrt{12}, 0) \psi_s(\sqrt{6}, 0) \psi_s(\sqrt{12}, \sqrt{2}) \psi_s(\sqrt{6}, 0) \psi_s(\sqrt{12}, 0) \cdot \psi_{x'} = \psi_{x'}.
\]

For the corresponding column tableau \( A' \) we have

\[
\begin{align*}
A'_2 &= A'_3 = A'_5 = A'_6 = \emptyset, \\
A'_4 &= (3), \\
A'_7 &= (356), \\
A'_8 &= (6).
\end{align*}
\]

Therefore by the equality (5.8) we obtain that

\[
\psi_{x'} = \lim_{r \to 0} \theta_{x}(r) \cdot \psi_s(\sqrt{6}, 0) \psi_s(\sqrt{12}, 0) \psi_s(\sqrt{12}, \sqrt{2}) \psi_s(\sqrt{6}, 0) \psi_s(\sqrt{6}, 0).
\]

Furthermore, for the tableau \( A' \) we have

\[
\begin{align*}
A_2 &= (1), \\
A_3 &= (12), \\
A_4 &= (12), \\
A_5 &= (1243), \\
A_6 &= (12435), \\
A_7 &= (124), \\
A_8 &= (124735).
\end{align*}
\]

Therefore by the definition (5.9), the equality (5.11) and by Lemma 5.1 we get

\[
\lim_{r \to 0} \theta_{x}(r) = 1/\sqrt{2} \cdot \theta_{x}(\sqrt{2}, 0) \psi_s(\sqrt{6}, 0) \psi_s(\sqrt{2}, 0) \theta_{x}(\sqrt{6}, \sqrt{2})
\]

\[
\times \theta_{x}(\sqrt{2}, 0) \psi_s(\sqrt{6}, \sqrt{2}) \theta_{x}(\sqrt{6}, 0) \psi_s(\sqrt{6}, 0) \psi_s(\sqrt{6}, \sqrt{2})
\]

\[
\times \psi_{x'}(\sqrt{6}, 0) \psi_s(\sqrt{6}, \sqrt{2}) \theta_{x}(\sqrt{12}, \sqrt{6}) \psi_s(\sqrt{6}, 0) \psi_s(\sqrt{6}, \sqrt{2}).
\]
ACKNOWLEDGMENTS

The present work has been launched in Moscow and completed in Kyoto. I am very grateful to I. Cherednik for numerous discussions in both of the places. I am also grateful to G. Olshanski for helpful advice and suggestions. I thank P. Kulish, A. Morris, J. Stembridge, and V. Tarasov for valuable comments. I am especially indebted to T. Miwa for his kind interest in this work, and for the hospitality at RIMS. Support from EPSRC by an Advanced Research Fellowship in York is gratefully acknowledged.

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