Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of MATHEMATICAL

www.elsevier.com/locate/jmaa

Global robust controllability of the triangular integro-differential Volterra systems

V.I. Korobov [a,b], S.S. Pavlichkov [c,*,1], W.H. Schmidt [d]

[a] *Department of Differential Equations and Control, Kharkov National University, Svobody sqr. 4, 61077 Kharkov, Ukraine*

[b] *Institute of Mathematics, Szczecin University, Wielkopolska str. 15, 70451 Szczecin, Poland*

[c] *Department of Mathematical Analysis, Kharkov National University, Svobody sqr. 4, 61077 Kharkov, Ukraine*

[d] *Institute of Mathematics and Informatics, Greifswald University, Friedrich-Ludwig-Jahn-str. 15-a, 17487 Greifswald, Germany*

Received 22 June 2004

Available online 17 March 2005

Submitted by H. Frankowska

Abstract

A solution of the global controllability problem for a class of nonlinear control systems of the Volterra integro-differential equations is presented. It is proven that there exists a family of continuous controls that solve the global controllability problem for this class. The constructed controls depend continuously on the initial and the terminal states. It makes possible to prove the global controllability of the uniformly bounded perturbations of these systems under the global Lipschitz condition for the unperturbed system with respect to the states and the controls.

Keywords: Nonlinear control; Triangular form; Global controllability; Volterra integro-differential control systems

* Corresponding author.

E-mail addresses: vkorobov@univer.kharkov.ua, korobow@sus.univ.szczecin.pl (V.I. Korobov), s_s_pavlichkov@yahoo.com (S.S. Pavlichkov), wschmidt@uni-greifswald.de (W.H. Schmidt).

1 Supported in part by DAAD, Germany.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.11.057

1. Introduction

The Volterra equations appear in modeling various physical and engineering systems (as examples, we mention the aero-elasticity problem [2] or actuarial mathematics [9]). Despite a lot of works concerned with the Volterra systems without a control input (see, for instance, [2,10,13,29–32,37]), the controllability problem for the Volterra systems has been investigated by few authors. The first paper concerned with the controllability problem for the Volterra systems was [1], where an approach based on the reduction of the controllability conditions to the Kakutani fixed point theorem was proposed. The next works in this direction were [3,4]. In these papers, a sufficient condition of the global controllability was proven for linear integro-differential Volterra systems and for their nonlinear bounded perturbations. The obtained results are a generalization of the well-known controllability criterion for the linear systems of ordinary differential equations (ODE). However, for the nonlinear Volterra systems, the controllability problem requires further investigation. Thus, finding new classes of the nonlinear Volterra systems that are globally controllable is of interest.

On the other hand, beginning with [16], the "triangular," or "feedback" (or "pure feedback") form is well known in the case of the nonlinear control systems of ODE (see [6, 7,11,14,15,19,21,22,25,27,28,35,36]). First, the triangular form is physically natural. To explain this, let us consider two systems: (I) $\dot{x} = f(x, y)$ with states x and controls y, and (II) $\dot{z} = g(z, u)$ with states z and controls u; then, by putting $z = y$, we obtain the "cascade" of (I) and (II), i.e., the system of the triangular form $\dot{x} = f(x, y)$, $\dot{y} = g(y, u)$, where $(x, y)^T$ is the state and u is the control. Such chain structures, where the output of a system affects the input of another system, appear in mechanical systems very often (for example, see [5,12,24], etc.). Second, there are effective backstepping design procedures which allow to construct stabilizing feedback laws for the triangular form [11,15,21,22]. Third, the triangular systems are closely related to the general feedback linearization problem [6,14,27], which arises both in general nonlinear control theory [6,12,14,16,27], and in engineering problems [8,19,23,33].

Therefore, it is natural to begin the investigation of the nonlinear control systems of the Volterra equations with the triangular systems as in the case of ODE. The controllability problem for the Volterra systems of the triangular form was considered in [18] by the current authors. However, the robustness properties of the constructed controls were not discussed in this paper. In general, it is not clear how the open-loop control that steers a given initial state into a given terminal one would be changed if the terminal or the initial state were changed continuously or if we deviated from the prescribed route during the driving. Nevertheless, it turns out that the construction proposed in [18] can be modified essentially; in particular, the problem of robustness can be solved for the Volterra systems of the triangular form. In the current work, we consider triangular systems of the Volterra integro-differential equations under more general conditions in comparison with [18]. For this class, we construct a family of continuous open-loop controls parametrized by the initial and the terminal states such that each element of this family steers the corresponding initial state into the corresponding terminal one and depends continuously on them with respect to the metric of $C([t_0, T]; \mathbf{R}^1)$. This remedies the above-mentioned deficiencies of the open-loop controls and allows us to prove the global controllability for the uniformly bounded perturbations of our class.

Following most works devoted to the triangular systems over last 15 years, we use “adding a differentiator” and some kind of backstepping to construct the desired controls (see, for instance, [11,15,21,22,36]). However, our technique differs greatly from that of the above-mentioned works: whereas the backstepping technique is used habitually for constructing *closed-loop controls* for systems of *ODE*, we construct *a family of open-loop controls* for our systems of *integro-differential equations*.

2. Preliminaries

In this paper, we consider a control system of the Volterra integro-differential equations

$$\dot{x}(t) = f\big(t, x(t), u(t)\big) + \int_{t_0}^{t} g\big(t, s, x(s), u(s)\big)\, ds, \quad t \in I = [t_0, T], \tag{1}$$

where $x = (x_1, \dots, x_n)^T \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^1$ is the control, and functions f and g have the following “triangular” form:

$$\begin{aligned} f(t, x, u) &= \big(f_1(t, x_1, x_2), f_2(t, x_1, x_2, x_3), \dots, f_n(t, x_1, \dots, x_n, u)\big)^T, \\ g(t, s, x, u) &= \big(g_1(t, s, x_1, x_2), g_2(t, s, x_1, x_2, x_3), \dots, g_n(t, s, x_1, \dots, x_n, u)\big)^T \end{aligned} \tag{2}$$

and satisfy the conditions:

(i) $f \in C(I \times \mathbf{R}^n \times \mathbf{R}^1; \mathbf{R}^n)$, $g \in C(I^2 \times \mathbf{R}^n \times \mathbf{R}^1; \mathbf{R}^n)$, $\frac{\partial f}{\partial x} \in C(I \times \mathbf{R}^n \times \mathbf{R}^1; \mathbf{R}^{n\times n})$, $\frac{\partial f}{\partial u} \in C(I \times \mathbf{R}^n \times \mathbf{R}^1; \mathbf{R}^n)$, $\frac{\partial g}{\partial x} \in C(I^2 \times \mathbf{R}^n \times \mathbf{R}^1; \mathbf{R}^{n\times n})$, $\frac{\partial g}{\partial u} \in C(I^2 \times \mathbf{R}^n \times \mathbf{R}^1; \mathbf{R}^n)$.

(ii) There exists $a > 0$ such that for each $t \in I$ and each $(x, u) \in \mathbf{R}^n \times \mathbf{R}^1$ we have

$$\left|\frac{\partial f_i}{\partial x_{i+1}}(t, x_1, \dots, x_{i+1})\right| \geqslant a > 0, \quad i = 1, \dots, n-1;$$
$$\left|\frac{\partial f_n}{\partial u}(t, x, u)\right| \geqslant a > 0.$$

(iii) For each $i = 1, \dots, n$ and each compact set $K \subset \mathbf{R}^i$ there exists $l_K > 0$ such that for all $(t, s) \in I^2$, $(x_1, \dots, x_i)^T \in K$, $y \in \mathbf{R}^1$, $z \in \mathbf{R}^1$ we have

$$\big|g_i(t, s, x_1, \dots, x_i, y) - g_i(t, s, x_1, \dots, x_i, z)\big| \leqslant l_K |y - z|.$$

Along with system (1), we consider its perturbation of the form

$$\begin{aligned} \dot{x}(t) = {}& f\big(t, x(t), u(t)\big) + h\big(t, x(t), u(t)\big) + \int_{t_0}^{t} g\big(t, s, x(s), u(s)\big)\, ds \\ &+ \int_{t_0}^{t} r\big(t, s, x(s), u(s)\big)\, ds, \quad t \in I, \end{aligned} \tag{3}$$

where functions h and r satisfy the conditions:

(iv) $h \in C(I \times \mathbf{R}^n \times \mathbf{R}^1; \mathbf{R}^n)$, $r \in C(I^2 \times \mathbf{R}^n \times \mathbf{R}^1; \mathbf{R}^n)$, and for each compact set $Q \subset \mathbf{R}^n \times \mathbf{R}^1$ there exists $L_Q > 0$ such that

$$\left|h(t, x^1, u^1) - h(t, x^2, u^2)\right| \leqslant L_Q\left(|x^1 - x^2| + |u^1 - u^2|\right),$$

$$\left|r(t, s, x^1, u^1) - r(t, s, x^2, u^2)\right| \leqslant L_Q\left(|x^1 - x^2| + |u^1 - u^2|\right)$$

for all $(t, s) \in I^2$, $(x^1, u^1) \in Q$, $(x^2, u^2) \in Q$.

(v) There exists $H > 0$ such that $|h(t, x, u)| \leqslant H$, and $|r(t, s, x, u)| \leqslant H$ for all $(t, s, x, u) \in I^2 \times \mathbf{R}^n \times \mathbf{R}^1$.

For each $x^0 \in \mathbf{R}^n$ and each $u(\cdot) \in C(I; \mathbf{R}^1)$ by $t \mapsto x(t, x^0, u(\cdot))$ we denote the trajectory of (1), that is defined by this control $u(\cdot)$ and by the initial condition $x(t_0, x^0, u(\cdot)) = x^0$ on some maximal subinterval $J \subset I$. Throughout the paper, the abbreviation "w.r.t." means "with respect to."

As in the case of ODE, we say that a system of the Volterra integro-differential equations is globally controllable in time $I = [t_0, T]$, iff for each initial state x^0 and each terminal state x^T there exists a control $u(\cdot)$ that "steers x^0 into x^T w.r.t. the system," i.e., the trajectory $x(\cdot)$ of the system with this control $u(\cdot)$ such that $x(t_0) = x^0$ satisfies $x(T) = x^T$.

3. Main results

Theorem 3.1. *Assume that for system* (1) *functions f and g have triangular form* (2) *and satisfy conditions* (i)–(iii). *Then there exists a family of controls $\{u_{(x^0, x^T)}(\cdot)\}_{(x^0, x^T) \in \mathbf{R}^n \times \mathbf{R}^n}$ such that the map $(x^0, x^T) \mapsto u_{(x^0, x^T)}(\cdot)$ is of class $C(\mathbf{R}^n \times \mathbf{R}^n; C(I; \mathbf{R}^1))$, and for each $(x^0, x^T) \in \mathbf{R}^n \times \mathbf{R}^n$ the trajectory $t \mapsto x(t, x^0, u_{(x^0, x^T)}(\cdot))$ is defined for all $t \in I$ and satisfies the condition $x(T, x^0, u_{(x^0, x^T)}(\cdot)) = x^T$.*

As a corollary, we obtain the following result.

Theorem 3.2. *Assume that functions f and g have triangular form* (2), *satisfy* (i)–(iii), *and satisfy the global Lipschitz condition w.r.t. x and u, i.e., there exists $L > 0$ such that for each $(t, s) \in I^2$, each $(x^1, u^1) \in \mathbf{R}^n \times \mathbf{R}^1$, and each $(x^2, u^2) \in \mathbf{R}^n \times \mathbf{R}^1$ we have*

$$\left|f(t, x^1, u^1) - f(t, x^2, u^2)\right| \leqslant L\left(|x^1 - x^2| + |u^1 - u^2|\right),$$

$$\left|g(t, s, x^1, u^1) - g(t, s, x^2, u^2)\right| \leqslant L\left(|x^1 - x^2| + |u^1 - u^2|\right).$$

Suppose that h and r satisfy (iv), (v). *Then* (3) *is globally controllable in time I by means of controls of class $C(I; \mathbf{R}^1)$.*

In particular, the following statement, which is a generalization of the main result of [18], is a mere partial case of the statement of Theorem 3.1.

Theorem 3.3. *Assume that f and g have triangular form* (2) *and satisfy* (i)–(iii)*. Then system* (1) *is globally controllable in time I by means of controls of class $C(I;\mathbf{R}^1)$.*

Remark 3.1. We assume that u and x_i are scalar only to simplify the notation and to make the argument clearer. For the case $u \in \mathbf{R}^m$, $x = (x_1, \dots, x_k)^T \in \mathbf{R}^n$, $x_i \in \mathbf{R}^m$, $n = km$, we can replace condition (ii) with the following one:

(II) For each $i = 1, \dots, k$, and each $(t, x_1, \dots, x_i) \in I \times \mathbf{R}^{im}$, $f_i(t, x_1, \dots, x_i, \cdot)$ is a diffeomorphism of $\mathbf{R}^m$ onto $\mathbf{R}^m$ such that the inverse diffeomorphism $U_i(t, x_1, \dots, x_i, \cdot)$ satisfies the following condition: for every compact set $K \subset \mathbf{R}^{im}$ there exists $\Lambda_K > 0$ such that $|U_i(t, x_1, \dots, x_i, v) - U_i(t, x_1, \dots, x_i, w)| \leqslant \Lambda_K |v - w|$ for all $(t, x_1, \dots, x_i)$ in $I \times K$, $v \in \mathbf{R}^m$, $w \in \mathbf{R}^m$.

Then, using the same argument as below, we can prove Theorems 3.1–3.3 for each system (1) such that (i), (II), and (iii) hold regardless of whether u and x_i are scalars or vectors.

Example 3.1. Consider the following system:

$$\begin{cases} \dot{x}_1 = 2x_2 + \sin x_2 + \int_0^t e^{ts} x_2(s)\,ds + h_1(t, x_1, x_2, u) \\ \qquad + \int_0^t r_1(t, s, x_1(s), x_2(s), u(s))\,ds, \\ \dot{x}_2 = u + h_2(t, x_1, x_2, u) + \int_0^t r_2(t, s, x_1(s), x_2(s), u(s))\,ds, \end{cases} \quad t \in [0, 1], \tag{4}$$

with states $(x_1, x_2)^T \in \mathbf{R}^2$ and controls $u \in \mathbf{R}^1$, where h_i and r_i are arbitrary functions such that (iv), (v) hold. If we had $h_i(t, x, u) = r_i(t, s, x, u) = 0$ for all $(t, s) \in [0, 1]^2$, $x \in \mathbf{R}^2$, $u \in \mathbf{R}^1$, then (4) would be a mere triangular system satisfying the global Lipschitz condition w.r.t. (x, u), and we could refer to the results of [18]. However, if the perturbation does not vanish, the results of [18] are no longer applicable. Nevertheless, by our Theorem 3.2, system (4) being a bounded perturbation of a triangular system, it is globally controllable in time $[0, 1]$.

Example 3.2. Consider the system

$$\begin{cases} \dot{x}_1 = x_2^3 + x_2 + \int_0^t e^{2ts} x_1(s) x_2(s)\,ds, \\ \dot{x}_2 = u^3 + u + \int^t e^{ts} x_2^2(s) u(s)\,ds, \end{cases} \quad t \in [0, 1], \tag{5}$$

with states $(x_1, x_2)^T$ and controls u. System (5) satisfies conditions (i)–(iii). Therefore, by Theorem 3.1, there exists a family $\{u_{(x^0, x^T)}(\cdot)\}_{(x^0, x^T) \in \mathbf{R}^2 \times \mathbf{R}^2}$ of controls such that $(x^0, x^T) \mapsto u_{(x^0, x^T)}(\cdot)$ is of class $C(\mathbf{R}^2 \times \mathbf{R}^2; C([0, 1]; \mathbf{R}^1))$, and $u_{(x^0, x^T)}(\cdot)$ steers x^0 into x^T w.r.t. (5) whatever $x^0 = (x_1^0, x_2^0)^T \in \mathbf{R}^2$ and $x^T = (x_1^T, x_2^T)^T \in \mathbf{R}^2$. In particular, (5) is globally controllable (Theorem 3.3), whereas the results of [18] cannot be applied to (5) because the global Lipschitz condition w.r.t. (x, u) does not hold for the right-hand side of (5).

Example 3.3. Let us show how Theorem 3.2 can work in the case of ODE. Consider the system

$$\begin{cases} \dot{x}_1 = f_1(x_2), \\ \dot{x}_2 = u, \end{cases} \quad t \in I = [t_0, T], \tag{6}$$

where $(x_1, x_2)^T \in \mathbf{R}^2$ is the state, $u \in \mathbf{R}^1$ is the control, $f_1(\cdot)$ is given by $f_1(x_2) = (x_2 - 2\sin x_2)(1 - \psi(x_2))$, and $\psi(\cdot)$ is an arbitrary function of class $C^1(\mathbf{R}; \mathbf{R})$ such that $0 \leqslant \psi(x_2) \leqslant 1$, if $x_2 \in \mathbf{R}$; $\psi(x_2) = 0$, if $x_2 \in \mathbf{R} \setminus [-3, 3]$; and $\psi(x_2) = 1$, if $x_2 \in [-2, 2]$. System (6) is a uniformly bounded perturbation of the canonical system $\dot{x}_1 = x_2$, $\dot{x}_2 = u$. Applying Theorem 3.2, we obtain that (6) is globally controllable in time I. Let us point out that (6) is a triangular system of ODE but it is easy to prove that (6) is not globally feedback equivalent to the canonical linear system $\dot{z}_1 = z_2$, $\dot{z}_2 = v$. In particular, for system (6) the usual regularity condition $\frac{\partial f_1}{\partial x_2} \neq 0$ does not hold; thus, we obtain the triangular form in the so-called singular case (see [6,27]). This observation leads us to a more wide class of the triangular systems of ODE in comparison with those investigated previously [6,16,27], which is globally controllable, but the set of its regular points is no longer open and dense in the state space. This question is studied in [17,26].

The paper is organized as follows. In Section 4, we prove that Theorem 3.2 follows from Theorem 3.1, and then we reduce Theorem 3.1 to Theorem 4.1, Theorem 4.1 being the main point of our approach. In Section 5, we prove Theorem 4.1.

4. The reduction of the main results to a backstepping procedure

Let us first prove that Theorem 3.2 follows from Theorem 3.1. Denote by

$$\{u_{(x^0, x^T)}(\cdot)\}_{(x^0, x^T) \in \mathbf{R}^n \times \mathbf{R}^n}$$

the family obtained from Theorem 3.1. Take any $x^0 \in \mathbf{R}^n$. Let $y(x^T, \cdot)$ be the trajectory of (3), defined by the control $u_{(x^0, x^T)}(\cdot)$ and by the initial condition $y(x^T, t_0) = x^0$, whenever $x^T \in \mathbf{R}^n$. Put $x(x^T, t) := x(t, x^0, u_{(x^0, x^T)}(\cdot))$ for all $t \in I$ and $x^T \in \mathbf{R}^n$. Then, using standard arguments based on the Gronwall–Bellman lemma, we get the existence of $D > 0$ such that

$$\left| x(x^T, t) - y(x^T, t) \right| \leqslant D, \quad \text{whenever } t \in I, \text{ and } x^T \in \mathbf{R}^n. \tag{7}$$

Since $x(x^T, T) = x^T$, we get $|y(x^T, T) - x^T| \leqslant D$ for all $x^T \in \mathbf{R}^n$. From Theorem 3.1, it follows that the map $x^T \mapsto y(x^T, T)$ is of class $C(\mathbf{R}^n; \mathbf{R}^n)$. Using the statement from [20, p. 277], which is based on the Brouwer fixed point theorem, we obtain that for each $y^T \in \mathbf{R}^n$ there exists $x^T \in \mathbf{R}^n$ such that $y(x^T, T) = y^T$, i.e., the control $u_{(x^0, x^T)}(\cdot)$ steers x^0 into y^T in time I w.r.t. (3). Finally, since $x^0 \in \mathbf{R}^n$ was an arbitrary initial state, this completes the proof of Theorem 3.2.

Next, we reduce Theorem 3.1 to a theorem which roughly speaking states that the controllability of a triangular system implies its controllability with any prescribed boundary conditions for the controls.

For each fixed $k = 1, \dots, n$, consider the following control system:

$$\dot{y}(t) = \varphi\big(t, y(t), v(t)\big) + \int_{t_0}^{t} \psi\big(t, s, y(s), v(s)\big)\, ds, \quad t \in I, \tag{8}$$

where $y = (x_1, \dots, x_k)^T \in \mathbf{R}^k$ is the state, $v \in \mathbf{R}^1$ is the control, and

$$\varphi(t, y, v) = \big(f_1(t, x_1, x_2), f_2(t, x_1, x_2, x_3), \dots, f_k(t, x_1, \dots, x_k, v)\big)^T,$$
$$\psi(t, s, y, v) = \big(g_1(t, s, x_1, x_2), g_2(t, s, x_1, x_2, x_3), \dots, g_k(t, s, x_1, \dots, x_k, v)\big)^T. \tag{9}$$

In other words, (8) is the k-dimensional subsystem of (1), that consists of the first k equations of (1), where x_{k+1} is treated as the control. For each $y^0 \in \mathbf{R}^k$ and each $v(\cdot) \in C(I; \mathbf{R}^1)$, let $t \mapsto y(t, y^0, v(\cdot))$ be the trajectory of (8), defined by the control $v(\cdot)$ and by the initial condition $y(t_0, y^0, v(\cdot)) = y^0$ on some maximal subinterval $J \subset I$.

Theorem 4.1. *Assume that f and g are given by* (2) *and satisfy* (i)–(iii). *Suppose that for some fixed $k = 1, \dots, n$ and for system* (8) *with φ and ψ defined by* (9), *there is a family of controls $\{v_{(\zeta,\xi)}(\cdot)\}_{(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k}$ such that*:

(a) *The map given by $(\zeta, \xi) \mapsto v_{(\zeta,\xi)}(\cdot)$ is of class $C(\mathbf{R}^k \times \mathbf{R}^k; C(I; \mathbf{R}^1))$.*
(b) *For each $(\zeta, \xi) \in \mathbf{R}^k \times \mathbf{R}^k$, the trajectory $t \mapsto y(t, \zeta, v_{(\zeta,\xi)}(\cdot))$ is defined for all $t \in I$ and $y(T, \zeta, v_{(\zeta,\xi)}(\cdot)) = \xi$.*

Then there exists a family of controls $\{\hat{v}_{(\zeta,\alpha,\xi,\beta)}(\cdot)\}_{(\zeta,\alpha,\xi,\beta)\in\mathbf{R}^k\times\mathbf{R}^1\times\mathbf{R}^k\times\mathbf{R}^1}$ such that the following three conditions hold:

(c) *For each $(\zeta, \alpha, \xi, \beta) \in \mathbf{R}^k \times \mathbf{R}^1 \times \mathbf{R}^k \times \mathbf{R}^1$, the control $\hat{v}_{(\zeta,\alpha,\xi,\beta)}(\cdot)$ is of class $C^1(I; \mathbf{R}^1)$ and $\hat{v}_{(\zeta,\alpha,\xi,\beta)}(t_0) = \alpha$, $\hat{v}_{(\zeta,\alpha,\xi,\beta)}(T) = \beta$.*
(d) *The map $(\zeta, \alpha, \xi, \beta) \mapsto \hat{v}_{(\zeta,\alpha,\xi,\beta)}(\cdot)$ is of class $C(\mathbf{R}^k \times \mathbf{R}^1 \times \mathbf{R}^k \times \mathbf{R}^1; C^1(I; \mathbf{R}^1))$.*
(e) *For each $(\zeta, \alpha, \xi, \beta) \in \mathbf{R}^k \times \mathbf{R}^1 \times \mathbf{R}^k \times \mathbf{R}^1$, the trajectory $t \mapsto y(t, \zeta, \hat{v}_{(\zeta,\alpha,\xi,\beta)}(\cdot))$ is defined for all $t \in I$ and $y(T, \zeta, \hat{v}_{(\zeta,\alpha,\xi,\beta)}(\cdot)) = \xi$.*

Having proved Theorem 4.1, we can easily obtain Theorem 3.1 by induction over k. Indeed, for $k = 1$, we may define $v_{(\zeta,\xi)}(\cdot)$ as the solution of the Volterra integral equation (w.r.t. unknown function $v(t)$)

$$\frac{d}{dt} x_1(\zeta, \xi, t) = f_1\big(t, x_1(\zeta, \xi, t), v(t)\big) + \int_{t_0}^{t} g_1\big(t, s, x_1(\zeta, \xi, s), v(s)\big)\, ds, \quad t \in I,$$

where $x_1(\zeta, \xi, t) = \zeta \frac{T-t}{T-t_0} + \xi \frac{t-t_0}{T-t_0}$ for all $t \in I$, $(\zeta, \xi) \in \mathbf{R}^1 \times \mathbf{R}^1$. Then, from (i)–(iii) we obtain that $v_{(\zeta,\xi)}(t)$ is well defined for all $t \in I$ and satisfies (a)–(b). Assume that for some $k = 1, \dots, n-1$ there exists $\{v_{(\zeta,\xi)}(\cdot)\}_{(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k}$ such that (a)–(b) hold, and let $\{\hat{v}_{(\zeta,\alpha,\xi,\beta)}(\cdot)\}_{(\zeta,\alpha,\xi,\beta)\in\mathbf{R}^k\times\mathbf{R}^1\times\mathbf{R}^k\times\mathbf{R}^1}$ be the family that satisfies conditions (c)–(e) of Theorem 4.1. For each $\chi = ((\zeta, \alpha), (\xi, \beta)) \in \mathbf{R}^k \times \mathbf{R}^1 \times \mathbf{R}^k \times \mathbf{R}^1$, by definition, put

$y(\chi,t) := y(t,\zeta,\hat{v}_{(\zeta,\alpha,\xi,\beta)}(\cdot))$, $t \in I$, and let $u_\chi(\cdot)$ be the solution of the Volterra integral equation

$$\frac{d}{dt}\hat{v}_\chi(t) = f_{k+1}\bigl(t, y(\chi,t), \hat{v}_\chi(t), u(t)\bigr) + \int_{t_0}^{t} f_{k+1}\bigl(t,s,y(\chi,s),\hat{v}_\chi(s),u(s)\bigr)\,ds \tag{10}$$

w.r.t. unknown function $u(t)$. Then (again, due to (i)–(iii)), $u_\chi(\cdot)$ is defined for all $t \in I$ and the family of controls $\{u_\chi(\cdot)\}_{\chi\in\mathbf{R}^{k+1}\times\mathbf{R}^{k+1}}$ satisfies conditions (a)–(b) for the extended $(k+1)$-dimensional control system

$$\begin{cases} \dot{y}(t) = \varphi(t, y(t), x_{k+1}(t)) + \int_{t_0}^{t} \psi(t,s,y(s),x_{k+1}(s))\,ds, \\ \dot{x}_{k+1}(t) = f_{k+1}(t,y(t),x_{k+1}(t),u(t)) + \int_{t_0}^{t} g_{k+1}(t,s,y(s),x_{k+1}(s),u(s))\,ds. \end{cases}$$

Thus, for $k+1=n$, we obtain the family of controls satisfying Theorem 3.1.

5. Proof of Theorem 4.1

To prove Theorem 4.1, we follow the same way as in the proof of Theorem 4 in [18]. However, in contrast with [18], we have to deal with families of controls and trajectories. This affects the formulations and the proofs of all the lemmas.

For each $y^0 \in \mathbf{R}^k$ and each $r > 0$, we put $B_r(y^0) := \{y \in \mathbf{R}^k \mid |y - y^0| < r\}$, and, for $A \subset \mathbf{R}^k$, by $\overline{A}$ we denote the closure of A.

5.1. The controllability of families of linear systems

Consider a family of control systems of the following form:

$$\dot{z}(t) = A(\xi,t)z(t) + B(\xi,t)w(t) + \int_{t_0}^{t}\bigl[C(\xi,t,s)z(s) + D(\xi,t,s)w(s)\bigr]\,ds, \tag{11}$$

where $z = (z_1,\ldots,z_k)^T \in \mathbf{R}^k$ is the state, $w \in \mathbf{R}^1$ is the control, $\xi \in \mathbf{R}^N$ is the parameter of the family, matrixes $A(\cdot,\cdot)$, $B(\cdot,\cdot)$, $C(\cdot,\cdot,\cdot)$, and $D(\cdot,\cdot,\cdot)$ have the form

$$A(\xi,t) = \begin{pmatrix} a_{11}(\xi,t) & a_{12}(\xi,t) & 0 & \ldots & 0 \\ a_{21}(\xi,t) & a_{22}(\xi,t) & a_{23}(\xi,t) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1\,1}(\xi,t) & a_{k-1\,2}(\xi,t) & a_{k-1\,3}(\xi,t) & \ldots & a_{k-1\,k}(\xi,t) \\ a_{k1}(\xi,t) & a_{k2}(\xi,t) & a_{k3}(\xi,t) & \ldots & a_{kk}(\xi,t) \end{pmatrix}, \tag{12}$$

$$C(\xi,t,s) = \begin{pmatrix} c_{11}(\xi,t,s) & c_{12}(\xi,t,s) & 0 & \ldots & 0 \\ c_{21}(\xi,t,s) & c_{22}(\xi,t,s) & c_{23}(\xi,t,s) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{k-1\,1}(\xi,t,s) & c_{k-1\,2}(\xi,t,s) & \ldots & \ldots & c_{k-1\,k}(\xi,t,s) \\ c_{k1}(\xi,t,s) & c_{k2}(\xi,t,s) & c_{k3}(\xi,t,s) & \ldots & c_{kk}(\xi,t,s) \end{pmatrix}, \tag{13}$$

$$B(\xi,t)=\begin{pmatrix}0\\ \vdots\\ 0\\ a_{k\,k+1}(\xi,t)\end{pmatrix},\qquad D(\xi,t,s)=\begin{pmatrix}0\\ \vdots\\ 0\\ c_{k\,k+1}(\xi,t,s)\end{pmatrix},\tag{14}$$

$$a_{ij}(\cdot,\cdot)\in C(\mathbf{R}^N\times I;\mathbf{R}),\qquad c_{ij}(\cdot,\cdot,\cdot)\in C(\mathbf{R}^N\times I^2;\mathbf{R}).\tag{15}$$

Given $z^0\in\mathbf{R}^k$, $w(\cdot)\in C(I;\mathbf{R}^1)$, and $\xi\in\mathbf{R}^N$, let $t\mapsto z(t,z^0,w(\cdot),\xi)$ be the trajectory of system (11) determined by ξ, that is defined by the control $w(\cdot)$ and by the initial condition $z(t_0,z^0,w(\cdot),\xi)=z^0$.

The goal of this subsection is to prove the following lemma.

Lemma 5.1. *Assume that family* (11) *satisfies* (12)–(15)*, and for each $i=1,\dots,k$ and each $(\xi,t)\in\mathbf{R}^N\times I$ we have $a_{i\,i+1}(\xi,t)\neq 0$. Then for each $z^T\in\mathbf{R}^k$, each $\beta\in\mathbf{R}^1$ and each $\mu\in\mathbf{N}$ there exists a family of controls $\{w(\xi,\cdot)\}_{\xi\in\mathbf{R}^N}$ such that the following conditions hold*:

(a) *For each $\xi\in\mathbf{R}^N$ the control $w(\xi,\cdot)$ is of class $C^\mu(I;\mathbf{R}^1)$ and satisfies the boundary conditions $w(\xi,T)=\beta$; $w(\xi,t_0)=0$.*
(b) *The map $\xi\mapsto w(\xi,\cdot)$ is of class $C(\mathbf{R}^N;C^\mu(I;\mathbf{R}^1))$.*
(c) *For each $\xi\in\mathbf{R}^N$ we have $z(T,0,w(\xi,\cdot),\xi)=z^T$.*

The principal part of the proof is the following lemma.

Lemma 5.2. *Suppose that family* (11) *satisfies* (12)–(15)*, and for each $z^T\in\mathbf{R}^k$ there exists a family of controls $\{w(\xi,\cdot)\}_{\xi\in\mathbf{R}^N}$ such that the following conditions hold*:

(a) *The map $\xi\mapsto w(\xi,\cdot)$ is of class $C(\mathbf{R}^N;C(I;\mathbf{R}^1))$.*
(b) *For each $\xi\in\mathbf{R}^N$ we have $z(T,0,w(\xi,\cdot),\xi)=z^T$.*

Then for each $z^T\in\mathbf{R}^k$, each $\beta\in\mathbf{R}^1$ and each $\mu\in\mathbf{N}$ there exists a family of controls $\{\hat{w}(\xi,\cdot)\}_{\xi\in\mathbf{R}^N}$ such that the following conditions hold:

(c) *$\hat{w}(\xi,\cdot)\in C^\mu(I;\mathbf{R}^1)$, $\hat{w}(\xi,T)=\beta$, and $\hat{w}(\xi,t_0)=0$ for all $\xi\in\mathbf{R}^N$.*
(d) *The map $\xi\mapsto\hat{w}(\xi,\cdot)$ is of class $C(\mathbf{R}^N;C^\mu(I;\mathbf{R}^1))$.*
(e) *$z(T,0,\hat{w}(\xi,\cdot),\xi)=z^T$ for all $\xi\in\mathbf{R}^N$.*

Arguing as above, we see that the reduction of Lemma 5.1 to Lemma 5.2 is similar to that of Theorem 3.1 to Theorem 4.1. Therefore, to complete the proof of Lemma 5.1 we need only to prove Lemma 5.2.

Proof of Lemma 5.2. Take any $\beta\in\mathbf{R}^1$, $z^T\in\mathbf{R}^k$, and $\mu\in\mathbf{N}$. Let $z^1,\dots,z^{k+1}$ be in $\mathbf{R}^k$ such that the interior $\operatorname{int}S$ of the simplex $S=\operatorname{conv}\{z^1,\dots,z^{k+1}\}$ is not empty, and $z^T\in\operatorname{int}S$. Then there exist $(k+1)$ families of controls $\{v_i(\xi,\cdot)\}_{\xi\in\mathbf{R}^N}$, $i=1,\dots,k+1$, such that:

(A_1) For each $i = 1, \ldots, k+1$, the map $\xi \mapsto v_i(\xi, \cdot)$ is of class $C(\mathbf{R}^N; C(I; \mathbf{R}^1))$.
(A_2) $z^i = z(T, 0, v_i(\xi, \cdot), \xi)$, whenever $i = 1, \ldots, k+1$ and $\xi \in \mathbf{R}^N$.

Choose $\varepsilon > 0$ such that, for each collection $\{\hat{z}^i\}_{i=1}^{k+1} \subset \mathbf{R}^k$, the condition $\hat{z}^i \in B_\varepsilon(z^i)$, $i = 1, \ldots, k+1$, implies $z^T \in \operatorname{int} \operatorname{conv}\{\hat{z}^1, \ldots, \hat{z}^{k+1}\}$. Our goal is to construct $(k+1)$ families of smooth controls which satisfy the required boundary conditions, continuously depend on ξ and steer $0 \in \mathbf{R}^k$ into $B_\varepsilon(z^i)$, $i = 1, \ldots, k+1$, for all $\xi \in \mathbf{R}^N$. Put

$$R(\xi) := |\beta| + \max_{i=1,\ldots,k+1} \left\| v_i(\xi, \cdot) \right\|_{C(I;\mathbf{R}^1)} + 1, \tag{16}$$

$$M(\xi) := \left\| A(\xi, \cdot) \right\|_{C(I;\mathbf{R}^{k\times k})} + \left\| B(\xi, \cdot) \right\|_{C(I;\mathbf{R}^{k})} + \left\| C(\xi, \cdot, \cdot) \right\|_{C(I^2;\mathbf{R}^{k\times k})} + \left\| D(\xi, \cdot, \cdot) \right\|_{C(I^2;\mathbf{R}^{k})} + 1, \tag{17}$$

$$\delta(\xi) := \min\left\{ \frac{\varepsilon}{2(4R(\xi) + T - t_0)(M(\xi) + M(\xi)(T - t_0))e^{((T-t_0)^2 + (T-t_0))M(\xi)}}, \frac{T - t_0}{3} \right\}, \quad \text{whenever } \xi \in \mathbf{R}^N. \tag{18}$$

Using the well-known theorem on the partitions of unity (see, for instance, [34]), we get the existence of $k+1$ families of controls $\{w_i(\xi, \cdot)\}_{\xi \in \mathbf{R}^N}$, $i = 1, \ldots, k+1$, such that each map $\xi \mapsto w_i(\xi, \cdot)$ is of class $C(\mathbf{R}^N; C^\mu(I; \mathbf{R}^1))$ and

$$\left\| w_i(\xi, \cdot) - v_i(\xi, \cdot) \right\|_{C(I;\mathbf{R}^1)} \leqslant \min\left\{\delta(\xi), 1\right\}, \quad \xi \in \mathbf{R}^N,\ i \in \{1, \ldots, k+1\}. \tag{19}$$

Given $i \in \{1, \ldots, k+1\}$ and $\xi \in \mathbf{R}^N$, define the control $\hat{w}_i(\xi, \cdot)$ by

$$\hat{w}_i(\xi, t) = \begin{cases} w_i(\xi, t) r\left(\frac{t - t_0}{\delta(\xi)}\right) & \text{if t} \in [t_0, t_0 + \delta(\xi)[, \\ w_i(\xi, t) & \text{if t} \in [t_0 + \delta(\xi), T - \delta(\xi)], \\ \left(1 - r\left(\frac{t - T + \delta(\xi)}{\delta(\xi)}\right)\right) w_i(\xi, t) & \\ \quad + r\left(\frac{t - T + \delta(\xi)}{\delta(\xi)}\right)\beta & \text{if } t \in]T - \delta(\xi), T], \end{cases} \tag{20}$$

where $r(\cdot) \in C^\infty(\mathbf{R}; \mathbf{R})$ is some fixed function such that $r(s) = 0$, if $s \leqslant 0$; $0 \leqslant r(s) \leqslant 1$, if $0 \leqslant s \leqslant 1$; and $r(s) = 1$, if $s \geqslant 1$. From (A_1) and from (15)–(18) it follows that functions $M(\xi)$, $R(\xi)$ and $\delta(\xi)$ are of class $C(\mathbf{R}^N;]0, +\infty[)$; therefore from (20) we obtain that the families $\{\hat{w}_i(\xi, \cdot)\}_{\xi \in \mathbf{R}^N}$ satisfy the conditions:

(A_3) $\hat{w}_i(\xi, \cdot) \in C^\mu(I; \mathbf{R}^1)$, $\hat{w}_i(\xi, t_0) = 0$, $\hat{w}_i(\xi, T) = \beta$ for all $\xi \in \mathbf{R}^N$, $i = 1, \ldots, k+1$.
(A_4) The map $\xi \mapsto \hat{w}_i(\xi, \cdot)$ is of class $C(\mathbf{R}^N; C^\mu(I; \mathbf{R}^1))$, whenever $i \in \{1, \ldots, k+1\}$.
(A_5) $\|\hat{w}_i(\xi, \cdot)\|_{C(I;\mathbf{R}^1)} \leqslant R(\xi)$, for all $i \in \{1, \ldots, k+1\}$, and $\xi \in \mathbf{R}^N$.

For the sake of simplicity, by $\hat{z}^i(\xi, \cdot)$ and $z^i(\xi, \cdot)$ we denote the trajectories $t \mapsto z(t, 0, \hat{w}_i(\xi, \cdot), \xi)$ and $t \mapsto z(t, 0, v_i(\xi, \cdot), \xi)$, respectively, for all $i = 1, \ldots, k+1$ and $\xi \in \mathbf{R}^k$. Arguing as in [18] (see the proof of Lemma 2), from the Gronwall–Bellman lemma, we obtain that $\hat{z}^i(\xi, T) \in B_\varepsilon(z^i)$ for all $i = 1, \ldots, k+1$, $\xi \in \mathbf{R}^N$. Hence, by the definition of $\varepsilon > 0$, for each $\xi \in \mathbf{R}^N$ there exists (a unique) collection $\{\lambda_i^*(\xi)\}_{i=1}^{k+1} \subset \mathbf{R}$ such that

$$\sum_{i=1}^{k+1} \lambda_i^*(\xi) = 1; \quad \lambda_i^*(\xi) \geqslant 0,\ i = 1, \ldots, k+1;$$
$$\sum_{i=1}^{k+1} \lambda_i^*(\xi) \hat{z}^i(\xi, T) = z^T. \tag{21}$$

Put $\hat{w}(\xi, t) := \sum_{i=1}^{k+1} \lambda_i^*(\xi) \hat{w}_i(\xi, t)$, for all $\xi \in \mathbf{R}^N$ and $t \in I$. Since each system (11) is linear, we get

$$z\big(t, 0, \hat{w}(\xi, \cdot), \xi\big) = \sum_{i=1}^{k+1} \lambda_i^*(\xi) z\big(t, 0, \hat{w}_i(\xi, \cdot), \xi\big) = \sum_{i=1}^{k+1} \lambda_i^*(\xi) \hat{z}^i(\xi, t).$$

Taking into account (21), we obtain $z(T, 0, \hat{w}(\xi, \cdot), \xi) = z^T$ for all $\xi \in \mathbf{R}^N$. In addition, from (20), (21) and (A_3) it follows that each control $\hat{w}(\xi, \cdot)$ is of class $C^\mu(I; \mathbf{R}^1)$ and satisfies the boundary conditions: $\hat{w}(\xi, T) = \beta,\ \hat{w}(\xi, t_0) = 0,\ \xi \in \mathbf{R}^N$. Thus, family $\{\hat{w}(\xi, \cdot)\}_{\xi \in \mathbf{R}^N}$ satisfies conditions (c) and (e) of Lemma 5.2.

From (15), and (A_4), it follows that each map $\xi \mapsto \hat{z}^i(\xi, T)$ is of class $C(\mathbf{R}^N; \mathbf{R}^k)$. On the other hand, every collection $\{\hat{z}^i\}_{i=1}^{k+1} \subset \mathbf{R}^k$ such that $\hat{z}^i \in B_\varepsilon(z^i),\ i = 1, \ldots, k+1$, determines a unique collection $\{\lambda_i(\hat{z}^1, \ldots, \hat{z}^{k+1})\}_{i=1}^{k+1}$ that satisfies the conditions

$$\lambda_i(\hat{z}^1, \ldots, \hat{z}^{k+1}) \geqslant 0; \qquad \sum_{i=1}^{k+1} \lambda_i(\hat{z}^1, \ldots, \hat{z}^{k+1}) = 1;$$
$$\sum_{i=1}^{k+1} \lambda_i(\hat{z}^1, \ldots, \hat{z}^{k+1}) \hat{z}^i = z^T; \tag{22}$$

and all the mappings $(\hat{z}^1, \ldots, \hat{z}^{k+1}) \mapsto \lambda_i(\hat{z}^1, \ldots, \hat{z}^{k+1}),\ i = 1, \ldots, k+1$, are of class $C(\mathbf{R}^{k \times (k+1)}; \mathbf{R})$. Indeed, (22) is equivalent to the system of linear algebraic equations $\sum_{i=2}^{k+1} \lambda_i(\hat{z}^i - \hat{z}^1) = z^T - \hat{z}^1$ w.r.t. unknown variables $\lambda_i,\ i = 2, \ldots, k+1$. By the definition of $\varepsilon > 0$, the set $\{(\hat{z}^i - \hat{z}^1)\}_{i=2}^{k+1}$ is a basis of $\mathbf{R}^k$. Thus, the solution of this nonsingular linear system is uniquely determined and depends continuously on the coefficients of the system. Therefore, the maps $\xi \mapsto \lambda_i^*(\xi),\ i = 1, \ldots, k+1$, defined by (21) are of class $C(\mathbf{R}^N; \mathbf{R})$; finally, it follows from (A_4) and from the definition of $\hat{w}(\xi, \cdot)$ that $\xi \mapsto \hat{w}(\xi, \cdot)$ is of class $C(\mathbf{R}^N; C^\mu(I; \mathbf{R}^1))$, i.e., condition (d) of the statement of Lemma 5.2 holds as well. This completes the proofs of Lemmas 5.2 and 5.1. □

5.2. *Proof of Theorem 4.1*

Let $\{v_{(\zeta,\xi)}(\cdot)\}_{(\zeta,\xi) \in \mathbf{R}^k \times \mathbf{R}^k}$ be a family of controls such that conditions (a)–(b) of Theorem 4.1 hold. Consider the following family of k-dimensional linear control systems:

$$\dot{z}(t) = \frac{\partial \varphi}{\partial y}\big(t, y\big(t, \zeta, v_{(\zeta,\xi)}(\cdot)\big), v_{(\zeta,\xi)}(t)\big) z(t) + \frac{\partial \varphi}{\partial v}\big(t, y\big(t, \zeta, v_{(\zeta,\xi)}(\cdot)\big), v_{(\zeta,\xi)}(t)\big) w(t)$$
$$+ \int_{t_0}^{t} \bigg[\frac{\partial \psi}{\partial y}\big(t, s, y\big(s, \zeta, v_{(\zeta,\xi)}(\cdot)\big), v_{(\zeta,\xi)}(s)\big) z(s)$$

$$+\frac{\partial\psi}{\partial v}\big(t,s,y\big(s,\zeta,v_{(\zeta,\xi)}(\cdot)\big),v_{(\zeta,\xi)}(s)\big)w(s)\bigg]\,ds,\quad t\in I, \tag{23}$$

where $(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k$ is the parameter of the family, $z=(z_1,\dots,z_k)^T\in\mathbf{R}^k$ is the state, $w\in\mathbf{R}^1$ is the control. From conditions (a)–(b) of Theorem 4.1 and from (i), (ii) we obtain that family (23) satisfies the conditions of Lemma 5.1. Then, from Lemma 5.1 it follows that there exist k families $\{w_i(\zeta,\xi,\cdot)\}_{(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k}$, $i=1,\dots,k$, of controls of class $C^1(I;\mathbf{R}^1)$ such that for each $i=1,\dots,k$ the map $(\zeta,\xi)\mapsto w_i(\zeta,\xi,\cdot)$ is of class $C(\mathbf{R}^k\times\mathbf{R}^k;C^1(I;\mathbf{R}^1))$, and for each $(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k$ the control $w_i(\zeta,\xi,\cdot)$ steers $0\in\mathbf{R}^k$ into $e_i=(0,\dots,0,1,0,\dots,0)^T\in\mathbf{R}^k$ in time I with respect to (23) and satisfies the boundary conditions $w_i(\zeta,\xi,t_0)=w_i(\zeta,\xi,T)=0$. For each $\lambda=(\lambda_1,\dots,\lambda_k)^T\in\mathbf{R}^k$, define the family of controls $\{v_\lambda(\zeta,\xi,\cdot)\}_{(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k}$ by $v_\lambda(\zeta,\xi,t)=v_{(\zeta,\xi)}(t)+\sum_{j=1}^k\lambda_j w_j(\zeta,\xi,t)$, for all $t\in I$, $(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k$.

For each $(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k$ and each $\lambda\in\mathbf{R}^k$ such that $t\mapsto y(t,\zeta,v_\lambda(\zeta,\xi,\cdot))$ is defined for all $t\in I$, put $y_\lambda(\zeta,\xi,t):=y(t,\zeta,v_\lambda(\zeta,\xi,\cdot))$, $t\in I$. For each $\mu=(\mu_1,\dots,\mu_k)^T\in\mathbf{R}^k$, by $z_{\mu,\lambda}(\zeta,\xi,\cdot)$ denote the trajectory of the system

$$\begin{aligned}\dot z(t)={}&\frac{\partial\varphi}{\partial y}\big(t,y_\lambda(\zeta,\xi,t),v_\lambda(\zeta,\xi,t)\big)z(t)+\frac{\partial\varphi}{\partial v}\big(t,y_\lambda(\zeta,\xi,t),v_\lambda(\zeta,\xi,t)\big)w(t)\\&+\int_{t_0}^{t}\bigg[\frac{\partial\psi}{\partial y}\big(t,s,y_\lambda(\zeta,\xi,s),v_\lambda(\zeta,\xi,s)\big)z(s)\\&+\frac{\partial\psi}{\partial v}\big(t,s,y_\lambda(\zeta,\xi,s),v_\lambda(\zeta,\xi,s)\big)w(s)\bigg]\,ds,\quad t\in I,\end{aligned}\tag{24}$$

defined by the control $w(\cdot):=w_\mu(\zeta,\xi,\cdot)=\sum_{j=1}^k\mu_j w_j(\zeta,\xi,\cdot)$ and by the initial condition $z_{\mu,\lambda}(\zeta,\xi,t_0)=0\in\mathbf{R}^k$. Define the families $\{F(\zeta,\xi,\cdot)\}_{(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k}$ and $\{G(\zeta,\xi,\cdot,\cdot)\}_{(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k}$ of maps from $\mathbf{R}^k$ and $\mathbf{R}^k\times\mathbf{R}^k$, respectively, to $\mathbf{R}^k$ as follows: for each (ζ,ξ,μ,λ) in $\mathbf{R}^k\times\mathbf{R}^k\times\mathbf{R}^k\times\mathbf{R}^k$ such that $t\mapsto y(t,\zeta,v_\lambda(\zeta,\xi,\cdot))$ is defined for all $t\in I$, put $F(\zeta,\xi,\lambda):=y_\lambda(\zeta,\xi,T)$, and $G(\zeta,\xi,\mu,\lambda):=z_{\mu,\lambda}(\zeta,\xi,T)$.

Lemma 5.3.

(a) *There exists a function $\varepsilon(\cdot,\cdot)$ of class $C(\mathbf{R}^k\times\mathbf{R}^k;\,]0,+\infty[)$ such that, for each (ζ,ξ,λ) in $\Omega=\{(\zeta,\xi,\lambda)\in\mathbf{R}^k\times\mathbf{R}^k\times\mathbf{R}^k\mid\lambda\in B_{\varepsilon(\zeta,\xi)}(0)\}$, the trajectory $t\mapsto y(t,\zeta,v_\lambda(\zeta,\xi,\cdot))$ is defined for all $t\in I$, and, therefore, $F(\zeta,\xi,\lambda)$ and $G(\zeta,\xi,\cdot,\lambda)$ are well defined.*

(b) *For each $(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k$ the map $\lambda\mapsto F(\zeta,\xi,\lambda)$ is differentiable at every $\lambda\in B_{\varepsilon(\zeta,\xi)}(0)$, and $\frac{\partial F}{\partial\lambda}(\zeta,\xi,\lambda)\mu=G(\zeta,\xi,\mu,\lambda)$, whenever $\lambda\in B_{\varepsilon(\zeta,\xi)}(0)$, $\mu\in\mathbf{R}^k$.*

(c) *The maps $(\zeta,\xi,\lambda)\mapsto y_\lambda(\zeta,\xi,\cdot)$, $(\zeta,\xi,\lambda)\mapsto F(\zeta,\xi,\lambda)$ and $(\zeta,\xi,\lambda)\mapsto\frac{\partial F}{\partial\lambda}(\zeta,\xi,\lambda)$ are of classes $C(\Omega;C(I;\mathbf{R}^k))$, $C(\Omega;\mathbf{R}^k)$ and $C(\Omega;\mathbf{R}^{k\times k})$, respectively.*

Lemma 5.3 is a version of the standard statement on the differentiability of the input–output map of a control system. It can be proved, for instance, by using the Gronwall–

Bellman lemma (one can find this argument in [18], see the proof of Theorem 4 (Steps 2 and 3)). We omit the proof of Lemma 5.3 due to space limits.

We will use also the following lemma, which is a direct corollary of the well-known Lagrange theorem.

Lemma 5.4. *Assume that $B \subset \mathbf{R}^k$ is a convex open set, and for each $\lambda_0 \in B$ the Jakoby matrix $\frac{\partial F}{\partial \lambda}(\lambda_0)$ of a map $F(\cdot) \in C^1(B; \mathbf{R}^k)$ is positive definite. Then, $\lambda \mapsto F(\lambda)$ is a diffeomorphism of B onto $F(B)$.*

By the definition of $\{w_i(\zeta,\xi,\cdot)\}_{(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k}$, for each $(\zeta,\xi) \in \mathbf{R}^k \times \mathbf{R}^k$ we have $G(\zeta,\xi, e_i,\lambda)|_{\lambda=0} = e_i$, i.e. (by Lemma 5.3), $\frac{\partial F}{\partial \lambda}(\zeta,\xi,0) = E$, where $E \in \mathbf{R}^{k\times k}$ is the identity matrix. Fix some $\sigma > 0$ such that each matrix $A \in \mathbf{R}^{k\times k}$ that satisfies the inequality $\|A - E\| < 2\sigma$ is positive definite. For each $r > 0$ we put $\Xi_r := \{(\zeta,\xi) \in \mathbf{R}^k \times \mathbf{R}^k \mid |\zeta| + |\xi| \leqslant r\}$.

Lemma 5.5. *There exist $\varepsilon_1(\cdot,\cdot)$ and $\varepsilon_2(\cdot,\cdot)$ of class $C(\mathbf{R}^k \times \mathbf{R}^k;]0,+\infty[)$ such that, for each $(\zeta,\xi) \in \mathbf{R}^k \times \mathbf{R}^k$, we obtain*

$$\varepsilon_1(\zeta,\xi) < \frac{1}{2}\varepsilon(\zeta,\xi), \tag{25}$$

$$\left\|\frac{\partial F}{\partial \lambda}(\zeta,\xi,\lambda) - E\right\| < \sigma, \quad \text{whenever } \lambda \in \overline{B_{\varepsilon_1(\zeta,\xi)}(0)}, \tag{26}$$

$$\overline{B_{\varepsilon_2(\zeta,\xi)}(\xi)} \subset F\big(\zeta,\xi,B_{\varepsilon_1(\zeta,\xi)}(0)\big). \tag{27}$$

Proof. Let us first prove the existence of $\varepsilon_1(\cdot,\cdot)$. To do this it is sufficient to prove that for each $m \in \mathbf{N}$ there exists $\bar\varepsilon_m \in]0, \frac{1}{2}\min_{(\zeta,\xi)\in\Xi_m} \varepsilon(\zeta,\xi)[$ such that for every $(\zeta,\xi) \in \Xi_m$ and every $\lambda \in \overline{B_{\bar\varepsilon_m}(0)}$ we have $\|\frac{\partial F}{\partial \lambda}(\zeta,\xi,\lambda) - E\| < \sigma$. Without loss of generality, we can assume that $\bar\varepsilon_{m+1} \leqslant \bar\varepsilon_m$, $m \in \mathbf{N}$ (otherwise, consider $\tilde\varepsilon_m = \min_{1\leqslant l\leqslant m} \bar\varepsilon_l$, $m \in \mathbf{N}$, instead of $\bar\varepsilon_m$). Then the function $\varepsilon_1(\cdot,\cdot)$ given by $\varepsilon_1(\zeta,\xi) = \bar\varepsilon_{m+1} + (\bar\varepsilon_{m+2} - \bar\varepsilon_{m+1})(|\zeta| + |\xi| - m)$ for all $m \leqslant |\zeta| + |\xi| < m+1$, $m \geqslant 0$, and $(\zeta,\xi) \in \mathbf{R}^k \times \mathbf{R}^k$ will satisfy (25), (26).

Indeed, if such $\{\bar\varepsilon_m\}_{m=1}^{\infty}$ does not exist, there is $m_0 \in \mathbf{N}$ such that for each $\bar\varepsilon \in]0, \frac{1}{2}\min_{(\zeta,\xi)\in\Xi_{m_0}} \varepsilon(\zeta,\xi)[$ there exists $(\zeta,\xi) \in \Xi_{m_0}$ and $\lambda \in \overline{B_{\bar\varepsilon}(0)}$ satisfying the inequality $\|\frac{\partial F}{\partial \lambda}(\zeta,\xi,\lambda) - E\| \geqslant \sigma$. Hence, we get the existence of sequences $\{(\zeta_q,\xi_q)\}_{q=1}^{\infty} \subset \Xi_{m_0}$ and $\{\lambda_q\}_{q=1}^{\infty} \subset B_{\frac{1}{2}\min_{(\zeta,\xi)\in\Xi_{m_0}} \varepsilon(\zeta,\xi)}(0)$ such that $\lambda_q \to 0$ as $q \to +\infty$, and for all $q \in \mathbf{N}$ we have $\|\frac{\partial F}{\partial \lambda}(\zeta_q,\xi_q,\lambda_q) - E\| \geqslant \sigma$. Choose a subsequence $\{(\zeta_{q_p},\xi_{q_p})\}_{p=1}^{\infty}$ of $\{(\zeta_q,\xi_q)\}_{q=1}^{\infty}$ such that $(\zeta_{q_p},\xi_{q_p}) \to (\zeta^*,\xi^*)$ as $p \to \infty$ for some $(\zeta^*,\xi^*) \in \Xi_{m_0}$. From Lemma 5.3 and from the inequality $\|\frac{\partial F}{\partial \lambda}(\zeta_{q_p},\xi_{q_p},\lambda_{q_p}) - E\| \geqslant \sigma$ we get $\|\frac{\partial F}{\partial \lambda}(\zeta^*,\xi^*,0) - E\| \geqslant \sigma$. Since $\frac{\partial F}{\partial \lambda}(\zeta^*,\xi^*,0) = E$, this contradicts the definition of $F(\cdot,\cdot,\cdot)$ and proves the existence of $\varepsilon_1(\cdot,\cdot) \in C(\mathbf{R}^k \times \mathbf{R}^k;]0,+\infty[)$ such that (25) and (26) hold.

Let us prove the existence of $\varepsilon_2(\cdot,\cdot) \in C(\mathbf{R}^k \times \mathbf{R}^k;]0;+\infty[)$ such that (27) holds for all $(\zeta,\xi) \in \mathbf{R}^k \times \mathbf{R}^k$. For each $m \in \mathbf{N}$ denote $\hat\varepsilon_m = \min_{(\zeta,\xi)\in\Xi_m} \varepsilon_1(\zeta,\xi)$. It is sufficient to prove the existence of $\{\tilde\varepsilon_m\}_{m=1}^{\infty} \subset]0,+\infty[$ such that for all $m \in \mathbf{N}$ and $(\zeta,\xi) \in \Xi_m$ we have $\overline{B_{\tilde\varepsilon_m}(\xi)} \subset F(\zeta,\xi,B_{\hat\varepsilon_m}(0))$.

Assume the converse, then there exist $m_0 \in \mathbf{N}$ and sequences $\{(\zeta_q, \xi_q)\}_{q=1}^{\infty} \subset \Xi_{m_0}$ and $\{\eta_q\}_{q=1}^{\infty} \subset \mathbf{R}^k$ satisfying $|\eta_q - \xi_q| \to 0$ as $q \to \infty$, and $\eta_q \notin F(\zeta_q, \xi_q, B_{\hat{\varepsilon}_{m_0}}(0))$ for all $q \in \mathbf{N}$. Since $\{(\zeta_q, \xi_q)\}_{q=1}^{\infty}$ is a bounded sequence, there is a subsequence $\{(\zeta_{q_p}, \xi_{q_p})\}_{p=1}^{\infty}$ and a point $(\bar{\zeta}, \bar{\xi}) \in \Xi_{m_0}$ such that $(\zeta_{q_p}, \xi_{q_p}) \to (\bar{\zeta}, \bar{\xi})$ as $p \to \infty$. By definition, put $\eta^p = \eta_{q_p}$, $\zeta^p = \zeta_{q_p}$, $\xi^p = \xi_{q_p}$, whenever $p \in \mathbf{N}$. From the definition of $\hat{\varepsilon}_{m_0}$, from (26), and from Lemma 5.4, it follows that $F(\bar{\zeta}, \bar{\xi}, \cdot)$ is a diffeomorphism of $B_{\hat{\varepsilon}_{m_0}}(0)$ onto $F(\bar{\zeta}, \bar{\xi}, B_{\hat{\varepsilon}_{m_0}}(0))$. Hence there exists $\bar{\varepsilon} > 0$ such that $\overline{B_{\bar{\varepsilon}}(\bar{\xi})} \subset F(\bar{\zeta}, \bar{\xi}, B_{\hat{\varepsilon}_{m_0}}(0))$. The continuous function $F(\cdot, \cdot, \cdot)$ is uniformly continuous on the compact set $\Xi_{m_0} \times \overline{B_{\hat{\varepsilon}_{m_0}}(0)}$; and $\eta^p \to \bar{\xi}$, $\zeta^p \to \bar{\zeta}$, and $\xi^p \to \bar{\xi}$ as $p \to \infty$. Therefore, there exists $p_0 \in \mathbf{N}$ such that for each $p \geqslant p_0$, $p \in \mathbf{N}$, and each $\lambda \in \overline{B_{\hat{\varepsilon}_{m_0}}(0)}$ we have $|F(\zeta^p, \xi^p, \lambda) - F(\bar{\zeta}, \bar{\xi}, \lambda)| < \frac{\bar{\varepsilon}}{2}$, and $|\eta^p - \bar{\xi}| < \frac{\bar{\varepsilon}}{2}$. By $F^{-1}(\bar{\zeta}, \bar{\xi}, \cdot)$ we denote the map of $\overline{B_{\bar{\varepsilon}}(\bar{\xi})}$ to $B_{\hat{\varepsilon}_{m_0}}(0)$ that is inverse to the diffeomorphism $\lambda \mapsto F(\bar{\zeta}, \bar{\xi}, \lambda)$ of $B_{\hat{\varepsilon}_{m_0}}(0)$ to $F(\bar{\zeta}, \bar{\xi}, B_{\hat{\varepsilon}_{m_0}}(0))$. For each $p \geqslant p_0$, $p \in \mathbf{N}$, consider the map of $\overline{B_{\bar{\varepsilon}}(\bar{\xi})}$ to $\mathbf{R}^k$ given by $\eta \mapsto \eta - F(\zeta^p, \xi^p, F^{-1}(\bar{\zeta}, \bar{\xi}, \eta)) + \eta^p$, this continuous function maps the closed ball $\overline{B_{\bar{\varepsilon}/2}(\eta^p)} \subset \overline{B_{\bar{\varepsilon}}(\bar{\xi})}$ into itself. Then, from the Brouwer fixed point theorem, we get the existence of $\eta_p^* \in \overline{B_{\bar{\varepsilon}/2}(\eta^p)}$ such that $\eta^p = F(\zeta^p, \xi^p, F^{-1}(\bar{\zeta}, \bar{\xi}, \eta_p^*))$. Finally, define $\lambda_p^* = F^{-1}(\bar{\zeta}, \bar{\xi}, \eta_p^*)$, then, we obtain that for each $p \geqslant p_0$, $p \in \mathbf{N}$ there exists $\lambda_p^* \in B_{\hat{\varepsilon}_{m_0}}(0)$ such that $\eta^p = F(\zeta^p, \xi^p, \lambda_p^*)$. This contradicts the definition of $\{\eta^p\}_{p=1}^{\infty}$, $\{\zeta^p\}_{p=1}^{\infty}$, $\{\xi^p\}_{p=1}^{\infty}$. The proof of Lemma 5.5 is complete. □

To simplify the notation, let $\chi = (\zeta, \alpha, \xi, \beta) \in \mathbf{R}^{2k+2}$ mean $\zeta \in \mathbf{R}^k$, $\alpha \in \mathbf{R}^1$, $\xi \in \mathbf{R}^k$, and $\beta \in \mathbf{R}^1$.

Lemma 5.6. *For each $\Delta(\cdot, \cdot, \cdot, \cdot) \in C(\mathbf{R}^{2k+2};]0, +\infty[)$ there exists a family*

$$\{v_\Delta(\chi, \cdot)\}_{\chi \in \mathbf{R}^{2k+2}}$$

of controls of class $C^1(I; \mathbf{R}^1)$ such that

(a) *The map $\chi \mapsto v_\Delta(\chi, \cdot)$ is of class $C(\mathbf{R}^{2k+2}; C^1(I; \mathbf{R}^1))$.*
(b) *For each $\chi = (\zeta, \alpha, \xi, \beta) \in \mathbf{R}^{2k+2}$ we have*

$$v_\Delta(\chi, t_0) = \alpha, \qquad v_\Delta(\chi, T) = \beta; \tag{28}$$

$$\left\| v_\Delta(\chi, \cdot) - v_{(\zeta,\xi)}(\cdot) \right\|_{L_1(I;\mathbf{R}^1)} < \Delta(\chi), \qquad \left\| v_\Delta(\chi, \cdot) \right\|_{C(I;\mathbf{R}^1)} < R(\chi), \tag{29}$$

$$\textit{where } R(\chi) = 2\max\left\{ |\alpha|, |\beta|, \max_{\lambda \in \overline{B_{\varepsilon_1(\zeta,\xi)}(0)}} \left\| v_\lambda(\zeta, \xi, \cdot) \right\|_{C(I;\mathbf{R}^1)} \right\} + 1. \tag{30}$$

The proof of Lemma 5.6 is based on the theorem on the partitions of unity, and is similar to the construction of the families $\{\hat{w}_i(\xi, \cdot)\}_{\xi \in \mathbf{R}^N}$ from the proof of Lemma 5.2. We omit the proof of Lemma 5.6 due to space limits.

To each function $\Delta(\cdot, \cdot, \cdot, \cdot)$ of class $C(\mathbf{R}^{2k+2};]0, +\infty[)$ assign the family of controls $\{v_\Delta(\chi, \cdot)\}_{\chi \in \mathbf{R}^{2k+2}}$ obtained from Lemma 5.6 such that conditions (a) and (b) of Lemma 5.6

hold. Then, to each $\lambda = (\lambda_1, \dots, \lambda_k)^T \in \mathbf{R}^k$ assign the family $\{\hat{v}_{\Delta,\lambda}(\chi, \cdot)\}_{\chi \in \mathbf{R}^{2k+2}}$ of controls of class $C^1(I; \mathbf{R}^1)$ given by $\hat{v}_{\Delta,\lambda}(\chi, t) = v_\Delta(\chi, t) + \sum_{j=1}^k \lambda_j w_j(\zeta, \xi, t)$, for all $t \in I$ and $\chi = (\zeta, \alpha, \xi, \beta) \in \mathbf{R}^{2k+2}$. For each $\chi = (\zeta, \alpha, \xi, \beta) \in \mathbf{R}^{2k+2}$ and each $\lambda \in \mathbf{R}^k$ such that $t \mapsto y(t, \zeta, \hat{v}_{\Delta,\lambda}(\chi, \cdot))$ is defined for all $t \in I$, we put by definition: $\hat{y}_{\Delta,\lambda}(\chi, t) = y(t, \zeta, \hat{v}_{\Delta,\lambda}(\chi, \cdot))$, $t \in I$; and then, for each $\mu = (\mu_1, \dots, \mu_k)^T \in \mathbf{R}^k$, by $\hat{z}_{\Delta,\mu,\lambda}(\chi, \cdot)$ we denote the trajectory of the system

$$\dot{z}(t) = \frac{\partial \varphi}{\partial y}\big(t, \hat{y}_{\Delta,\lambda}(\chi, t), \hat{v}_{\Delta,\lambda}(\chi, t)\big) z(t) + \frac{\partial \varphi}{\partial v}\big(t, \hat{y}_{\Delta,\lambda}(\chi, t), \hat{v}_{\Delta,\lambda}(\chi, t)\big) w(t)$$
$$+ \int_{t_0}^{t} \bigg[\frac{\partial \psi}{\partial y}\big(t, s, \hat{y}_{\Delta,\lambda}(\chi, s), \hat{v}_{\Delta,\lambda}(\chi, s)\big) z(s)$$
$$+ \frac{\partial \psi}{\partial v}\big(t, s, \hat{y}_{\Delta,\lambda}(\chi, s), \hat{v}_{\Delta,\lambda}(\chi, s)\big) w(s) \bigg] ds, \quad t \in I, \tag{31}$$

defined by the control $w_\mu(\zeta, \xi, \cdot) := \sum_{j=1}^k \mu_j w_j(\zeta, \xi, \cdot)$ and by the initial condition $\hat{z}_{\Delta,\mu,\lambda}(\chi, t_0) = 0 \in \mathbf{R}^k$. Define the families

$$\big\{\hat{F}_\Delta(\chi, \cdot)\big\}_{\chi = (\zeta,\alpha,\xi,\beta) \in \mathbf{R}^{2k+2}} \quad \text{and} \quad \big\{\hat{G}_\Delta(\chi, \cdot, \cdot)\big\}_{\chi = (\zeta,\alpha,\xi,\beta) \in \mathbf{R}^{2k+2}}$$

of maps from $\mathbf{R}^k$ and $\mathbf{R}^k \times \mathbf{R}^k$, respectively, to $\mathbf{R}^k$ as follows: for each $\chi \in \mathbf{R}^{2k+2}$, each $\mu \in \mathbf{R}^k$, and each $\lambda \in \mathbf{R}^k$ such that $t \mapsto y(t, \zeta, \hat{v}_{\Delta,\lambda}(\chi, \cdot))$ is defined for all $t \in I$, by definition, put $\hat{F}_\Delta(\chi, \lambda) = \hat{y}_{\Delta,\lambda}(\chi, T)$, $\hat{G}_\Delta(\chi, \mu, \lambda) = \hat{z}_{\Delta,\mu,\lambda}(\chi, T)$. In addition, we introduce the following notation: for each $\chi = (\zeta, \alpha, \xi, \beta) \in \mathbf{R}^{2k+2}$ we put $|\chi| = |\zeta| + |\alpha| + |\xi| + |\beta|$, and for each $r > 0$ by Υ_r we denote the set $\Upsilon_r := \{\chi \in \mathbf{R}^{2k+2} \mid |\chi| \leqslant r\}$.

Lemma 5.7. *There exists a function* $\Delta(\cdot, \cdot, \cdot, \cdot) \in C(\mathbf{R}^{2k+2};]0, +\infty[)$ *such that the following statements hold*:

(a) *For each* (χ, λ) *in* $\Omega_1 := \{(\zeta, \alpha, \xi, \beta, \lambda) \in \mathbf{R}^{2k+2} \times \mathbf{R}^k \mid \lambda \in B_{\varepsilon_1(\zeta,\xi)}(0)\}$, *the trajectory* $t \mapsto y(t, \zeta, \hat{v}_{\Delta,\lambda}(\chi, \cdot))$ *is defined for all* $t \in I$, *and, therefore,* $\hat{F}_\Delta(\chi, \lambda)$ *and* $\hat{G}_\Delta(\chi, \cdot, \lambda)$ *are well defined.*

(b) *For each* $\chi = (\zeta, \alpha, \xi, \beta) \in \mathbf{R}^{2k+2}$ *the map* $\lambda \mapsto \hat{F}_\Delta(\chi, \lambda)$ *is differentiable for all* $\lambda \in B_{\varepsilon_1(\zeta,\xi)}(0)$, *and for every* $\mu \in \mathbf{R}^k$, *we have* $\frac{\partial \hat{F}_\Delta}{\partial \lambda}(\chi, \lambda)\mu = \hat{G}_\Delta(\chi, \mu, \lambda)$.

(c) *The maps* $(\chi, \lambda) \mapsto \hat{F}_\Delta(\chi, \lambda)$ *and* $(\chi, \lambda) \mapsto \frac{\partial \hat{F}_\Delta}{\partial \lambda}(\chi, \lambda)$ *are of classes* $C(\Omega_1; \mathbf{R}^k)$ *and* $C(\Omega_1; \mathbf{R}^{k \times k})$, *respectively.*

(d) *For each* $(\chi, \lambda) \in \Omega_1$ *we have*

$$\big|\hat{F}_\Delta(\chi, \lambda) - F(\zeta, \xi, \lambda)\big| < \frac{\varepsilon_2(\zeta, \xi)}{2};$$
$$\left\| \frac{\partial \hat{F}_\Delta}{\partial \lambda}(\chi, \lambda) - \frac{\partial F}{\partial \lambda}(\zeta, \xi, \lambda) \right\| < \sigma. \tag{32}$$

The construction of a $\Delta(\cdot,\cdot,\cdot,\cdot)$ such that conditions (a)–(c) hold is similar to that of $\varepsilon(\cdot,\cdot)$ from Lemma 5.3. To comply with item (d) of Lemma 5.7, we use the same argument as [18, proof of Theorem 4, Step 5]. The input–output map of system (8) (as a map of $L_\infty(I;\mathbf{R}^1)$ to $C(I;\mathbf{R}^k)$) being continuous w.r.t. the norms of spaces $L_1(I;\mathbf{R}^1)$ and $C(I;\mathbf{R}^k)$, respectively, the left-hand sides of the inequalities from (32) are small enough whenever $\|v_\Delta(\chi,\cdot)-v_{(\zeta,\xi)}(\cdot)\|_{L_1(I;\mathbf{R}^1)}$ is small enough and $\chi\in\Omega_1$. Then, the construction of the desired $\Delta(\cdot,\cdot,\cdot,\cdot)$ becomes similar to that of $\varepsilon_1(\cdot,\cdot)$ and $\varepsilon_2(\cdot,\cdot)$ in Lemma 5.5. We must omit the proof of Lemma 5.7 due to space limits.

Now we can complete the proof of Theorem 4.1. Using (26), item (d) of Lemma 5.7, the definition of σ, and Lemma 5.4, we obtain that for each $(\zeta,\alpha,\xi,\beta)\in\mathbf{R}^{2k+2}$ the map $\hat F_\Delta(\zeta,\alpha,\xi,\beta,\cdot)$ is a diffeomorphism of $B_{\varepsilon_1(\zeta,\xi)}(0)$ onto the set $\hat F_\Delta(\zeta,\alpha,\xi,\beta,B_{\varepsilon_1(\zeta,\xi)}(0))$. Furthermore, from (26), it follows that for each $(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k$ the map $F(\zeta,\xi,\cdot)$ is a diffeomorphism of $B_{\varepsilon_1(\zeta,\xi)}(0)$ onto its image $F(\zeta,\xi,B_{\varepsilon_1(\zeta,\xi)}(0))$ as well. By $F^{-1}(\zeta,\xi,\cdot)$ we denote the diffeomorphism of $F(\zeta,\xi,B_{\varepsilon_1(\zeta,\xi)}(0))$ onto $B_{\varepsilon_1(\zeta,\xi)}(0)$ that is inverse to $F(\zeta,\xi,\cdot)$. From statement (c) of Lemma 5.7, (27), and (32), it follows that for each $(\zeta,\alpha,\xi,\beta)\in\mathbf{R}^{2k+2}$ the map $\eta\mapsto\hat F_\Delta(\zeta,\alpha,\xi,\beta,F^{-1}(\zeta,\xi,\eta))$ is well defined and continuous at each $\eta\in\overline{B_{\varepsilon_2(\zeta,\xi)}(\xi)}$, and $|\eta-\hat F_\Delta(\zeta,\alpha,\xi,\beta,F^{-1}(\zeta,\xi,\eta))|<\frac{\varepsilon_2(\zeta,\xi)}{2}$ for all $\eta\in\overline{B_{\varepsilon_2(\zeta,\xi)}(\xi)}$. Therefore, from the statement of [20, p. 277], we get the existence of $\eta^*=\eta^*(\zeta,\alpha,\xi,\beta)\in\overline{B_{\varepsilon_2(\zeta,\xi)}(\xi)}$ such that $\xi=\hat F_\Delta(\zeta,\alpha,\xi,\beta,F^{-1}(\zeta,\xi,\eta^*))$. For each $(\zeta,\alpha,\xi,\beta)\in\mathbf{R}^{2k+2}$, put $\lambda^*(\zeta,\alpha,\xi,\beta)=F^{-1}(\zeta,\xi,\eta^*(\zeta,\alpha,\xi,\beta))$; then,

$$\xi=\hat F_\Delta\big(\zeta,\alpha,\xi,\beta,\lambda^*(\zeta,\alpha,\xi,\beta)\big). \tag{33}$$

In addition, by the construction, $\lambda^*(\zeta,\alpha,\xi,\beta)\in B_{\varepsilon_1(\zeta,\xi)}(0)$, and the map $\hat F_\Delta(\zeta,\alpha,\xi,\beta,\cdot)$ is a diffeomorphism of $B_{\varepsilon_1(\zeta,\xi)}(0)$ onto its image. From this, we get the uniqueness of $\lambda^*(\zeta,\alpha,\xi,\beta)\in B_{\varepsilon_1(\zeta,\xi)}(0)$ such that (33) holds. From statement (c) of Lemma 5.7 and from the implicit function theorem, we obtain that the map $\chi\mapsto\lambda^*(\chi)$ is continuous at each $\chi\in\mathbf{R}^{2k+2}$. For each $(\zeta,\alpha,\xi,\beta)\in\mathbf{R}^{2k+2}$ let $\hat v_{(\zeta,\alpha,\xi,\beta)}(\cdot)$ be the control given by

$$\hat v_{(\zeta,\alpha,\xi,\beta)}(t)=\hat v_{\Delta,\lambda^*(\zeta,\alpha,\xi,\beta)}(\zeta,\alpha,\xi,\beta,t)\quad\text{for all } t\in I. \tag{34}$$

It is clear that the family

$$\big\{\hat v_{(\zeta,\alpha,\xi,\beta)}(\cdot)\big\}_{(\zeta,\alpha,\xi,\beta)\in\mathbf{R}^{2k+2}}$$

given by (34) satisfies the statement of Theorem 4.1. Indeed, condition (e) of Theorem 4.1 follows from (33) and from the definition of $\hat F_\Delta$; condition (d) follows from the continuity of $\lambda^*(\cdot,\cdot,\cdot,\cdot)$, item (a) of Lemma 5.6, and the definition of $\{w_i(\zeta,\xi,\cdot)\}_{(\zeta,\xi)\in\mathbf{R}^k\times\mathbf{R}^k}$. Finally, statement (c) follows from (28) and from the fact that all the $w_i(\zeta,\xi,\cdot)$ satisfy the homogeneous boundary conditions. This completes the proof of Theorem 4.1. □

Acknowledgments

This work was complete while the second author was visiting Institute of Mathematics and Informatics, Ernst-Moritz-Arndt University of Greifswald. Svyatoslav Pavlichkov is grateful for the warm hospitality provided by Prof. Dr. W.H. Schmidt, and Dr. V. Azmjakov.

References

[1] T.S. Angell, The controllability problem for nonlinear Volterra systems, J. Optim. Theory Appl. 41 (1983) 9–35.

[2] I.S. Astapov, S.M. Belotserkovsky, B.O. Kachanov, Yu.A. Kochetkov, O sistemah integro-differencial'nih uravnenij, opisivayushih neustanovivsheesya dvizhenie tel v sploshnoi srede, Differ. Uravn. 18 (1982) 1628–1637.

[3] K. Balachandran, Controllability of nonlinear Volterra integro-differential systems, Kybernetika 25 (1989) 505–518.

[4] K. Balachandran, P. Balasubramaniam, A note on controllability of nonlinear Volterra integro-differential systems, Kybernetika 28 (1992) 284–291.

[5] V.F. Borisov, M.I. Zelikin, Chattering arcs in the time-optimal robots control problem, Prikl. Mat. Mekh. 52 (1988) 939–946.

[6] S. Celikovsky, H. Nijmeijer, Equivalence of nonlinear systems to triangular form: The singular case, Systems Control Lett. 27 (1996) 135–144.

[7] J.-M. Coron, L. Praly, Adding an integrator for the stabilization problem, Systems Control Lett. 17 (1991) 89–104.

[8] B. D'Andrea, J. Levine, C.A.D. for nonlinear systems decoupling, perturbations rejection and feedback linearization with applications to the dynamic control of a robot arm, in: M. Fliess, M. Hazewinkel (Eds.), Algebraic and Geometric Methods in Nonlinear Control Theory, Reidel, Dordrecht, 1986, pp. 545–572.

[9] D.C.M. Dickson, H.R. Waters, Ruin theory, Inst. of Actuaries, Faculty of Actuaries, Dept. Actuarial Math. and Statistics, Heriott-Watt, 1992.

[10] A.I. Egorov, P.I. Kogut, Ob ustoichivosti po sostoyaniyu sistemy integro-differencial'nyh uravnenij nestacionarnoi aerouprugosti, Vychisl. Prikl. Mat. (Kiev) 70 (1990) 112–121.

[11] R.A. Freeman, P.V. Kokotovic, Backstepping design of robust controllers for a class of nonlinear systems, in: Nonlinear Control Systems Design Symposium NOLCOS 92, Proc. IFAC, Bordeaux, 1992, pp. 307–312.

[12] M. Fliess, J. Levine, Ph. Martin, P. Rouchon, Flatness and defect of nonlinear systems: introductory theory and examples, Internat. J. Control 61 (1995) 1327–1361.

[13] Y. Hino, S. Murikami, Stability properties of linear Volterra equations, J. Differential Equations 89 (1991) 121–137.

[14] B. Jakubczyk, W. Respondek, On linearization of control systems, Bull. Acad. Sci. Polonaise Ser. Sci. Math. 28 (1980) 517–522.

[15] P.V. Kokotovic, H.J. Sussmann, A positive real condition for global stabilization of nonlinear systems, Systems Control Lett. 13 (1989) 125–133.

[16] V.I. Korobov, Upravlyaemost', ustoichivost' nekotoryh nelineinyh sistem, Differ. Uravn. 9 (1973) 614–619.

[17] V.I. Korobov, S.S. Pavlichkov, The global controllability of a class of the triangular systems in the singular case, Math. Preprints Server (Elsevier Science) on 29.11.2003, submitted for publication.

[18] V.I. Korobov, S.S. Pavlichkov, W.H. Schmidt, The controllability problem for certain nonlinear integro-differential Volterra systems, Optimization 50 (2001) 155–186.

[19] A.M. Kovalev, Nelineijnie Zadachi Upravleniya i Nabludeniya v Teorii Dinamicheskih Sistem, Naukova Dumka, Kiev, 1980.

[20] A.B. Lee, L. Marcus, Foundations of the Optimal Control Theory, Nauka, Moscow, 1972.

[21] J.-S. Lin, I. Kanellakopoulos, Nonlinearities enhance parameter convergence in strict-feedback systems, IEEE Trans. Automat. Control 43 (1998) 1–5.

[22] W. Lin, C. Quan, Adding one power integrator: A tool for global stabilization of high order lower-triangular systems, Systems Control Lett. 39 (2000) 339–351.

[23] R. Marino, Feedback linearization techniques in robotics and power systems, in: M. Fliess, M. Hazewinkel (Eds.), Algebraic and Geometric Methods in Nonlinear Control Theory, Reidel, Dordrecht, 1986, pp. 523–543.

[24] R.M. Murray, Trajectory generation for a towed cable flight control system, in: Proceedings IFAC World Congress, San Francisco, 1996, pp. 395–400.

[25] K. Nam, A. Arapostathis, A model reference adaptive control scheme for pure-feedback nonlinear systems, IEEE Trans. Automat. Control 33 (1988) 803–811.

[26] S.S. Pavlichkov, The complete controllability of some classes of triangular systems which are not reducible to the canonical form, Ph.D. thesis, Manuscript, V.N. Karazin Kharkov National Univ., Kharkov, 2001.
[27] W. Respondek, Global aspects of linearization, equivalence to polynomial forms and decomposition of nonlinear control systems, in: M. Fliess, M. Hazewinkel (Eds.), Algebraic and Geometric Methods in Nonlinear Control Theory, Reidel, Dordrecht, 1986, pp. 257–284.
[28] A. Saberi, P.V. Kokotovic, H.J. Sussmann, Global stabilization of partially linear composite systems, SIAM J. Control Optim. 28 (1990) 1491–1503.
[29] V.S. Sergeev, O neustoichivosti nulevogo resheniya odnogo klassa sistem integro-differencial'nyh uravnenij, Differ. Uravn. 24 (1988) 1443–1454.
[30] V.S. Sergeev, Ob asimptoticheskoi ustoichivosti i otsenke oblasti prityazheniya v nekotoryh sistemah s posledeistviem, Prikl. Mat. Mekh. 60 (1996) 744–751.
[31] V.S. Sergeev, O neustoichivosti v kriticheskom sluchae pary chisto mnimyh kornei dla odnogo klassa sistem s posledeistviem, Prikl. Mat. Mekh. 62 (1998) 79–86.
[32] R. Shigui, Stability of Volterra integro-differential systems, J. Math. Anal. Appl. 137 (1989) 471–476.
[33] S.N. Singh, T.C. Bossart, Exact feedback linearization and control of space station using CMG, IEEE Trans. Automat. Control 38 (1993) 184–187.
[34] S. Sternberg, Lectures on Differential Geometry, Mir, Moscow, 1970.
[35] J. Tsinias, A theorem on global stabilization of nonlinear systems by linear feedback, Systems Control Lett. 17 (1991) 357–362.
[36] J. Tsinias, Triangular systems: A global extension of the Coron–Praly theorem on the existence of feedback-integrator stabilisers, European J. Control 3 (1997) 37–46.
[37] V. Volterra, Theory of Functionals and of Integral and Integro-Differential Equations, Nauka, Moscow, 1982.