First Distribution Invariants and EKR Theorems*

N. MANICKAM*

Department of Mathematics and Computer Science,
DePauw University, Greencastle, Indiana 46135

AND

N. M. SINGHI*

School of Mathematics, Tata Institute of Fundamental Research,
Colaba, Bombay 400005, India

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It is shown by a simple counting argument that, in a projective space $P_{n-1}$, any set of $\left[\frac{d}{n}\right]$ distinct $(d-1)$-subspaces of $P_{n-1}$, $d/n$, contains a $d$-spread. A weight function on $P_{n-1}$ is a real-valued function on the set $I_n$ of points of $P_{n-1}$ such that the sum of the values on all points of $I_n$ is nonnegative. The weight of any subset of $I_n$ is the sum of the weights of all the points in it. It is shown that the number of $(d-1)$-subspaces in $P_{n-1}$ with nonnegative weights is at least $\left[\frac{d}{n}\right]$. The case of equality is characterized by using the Erdős-Ko-Rado theorem. These results are then applied to study the first distribution invariant of $J_d(n, d)$. Analogues of these results are proved for sets and affine spaces when $n \geq 2d$. In the case of affine spaces the problem is essentially solved. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let $\mathcal{X} = (X, \{\mathcal{A}_i\}_{0 \leq i \leq d})$ be a symmetric association scheme and $R^X = V_0 \perp V_1 \perp \cdots \perp V_d$ be the orthogonal decomposition of the vector space $R^X$ with each $V_i$ being the maximal common eigenspace of the adjacency matrices of $\mathcal{X}$. A vector $w \in R^X$ is said to be a general vector if and only if $\langle w, x \rangle \neq 0$ for all $x$ in $X$. For $i, 1 \leq i \leq d$, we define the $i$-th-distribution invariant $vt_i(\mathcal{X})$ as

$$vt_i(\mathcal{X}) = \min_{w} |\{x | w \in V_i, \langle w, x \rangle > 0 \text{ and } w \text{ general}\}|.$$

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The distribution invariants were first introduced by Thomas Bier [2] while attempting to answer certain problems in topology. He also proposed

PROBLEM 1.1. Find all the distribution invariants for all the known classical association schemes.

For a generalized version of distribution invariants see Bier and Delsarte [3]. The following conjecture is motivated by results of Bier and Manickam [4].

Conjecture 1.2. (a) For the Johnson scheme \( J(n, d) \)

\[
V_1(J(n, d)) = \binom{n - 1}{d - 1} \quad \text{if} \quad n \geq 4d.
\]

(b) For the q-analogue Johnson scheme \( J_q(n, d) \)

\[
V_1(J_q(n, d)) = \left[ \frac{n - 1}{d - 1} \right]_q \quad \text{if} \quad n \geq 4d
\]

where \( \left[ \frac{a}{b} \right]_q = \frac{(q^{a-1} - 1)(q^{a-2} - 1) \cdots (q^{a-b+1} - 1)}{(q^a - 1) \cdots (q - 1)} \) is the Guesison q-ary binomial coefficient.

Throughout this paper \( q \) will denote the order of a given field GF(q). We will write \( \left[ \frac{a}{b} \right]_q \) for \( \left[ \frac{a}{b} - 1 \right]_q \).

Part (a) of Conjecture 1.2 was shown to be true in [4] when \( d \) divides \( n \) or when \( n \geq d^3d \). We also note that part (a) is not true when \( n < 4d \), in particular it was shown in [4] that the conjecture is not true when \( n = 3d + 1 \). Similar examples show that part (a) of Conjecture 1.2 is not true whenever \( 2d < n < 3d + \lfloor d/2 \rfloor \) and \( n \neq 3d \). No such examples are known for part (b), to the authors. In Section 2, it is shown that Conjecture 1.2 follows from a conjecture on a combinatorial problem stated as Conjecture 1.4.

Let \( I_n \) be any of the following:

(i) the set of all points of a projective space \( P_{n-1} \) of dimension \( n - 1 \),

(ii) the set of all points of an affine space \( A_{n-1} \) of dimension \( n - 1 \),

(iii) elements of an \( n \)-set.

A function \( a: I_n \to \mathbb{R} \) is called a weight function on \( I_n \) if \( \sum_{v \in I_n} a(v) \geq 0 \). We will write \( a_v \) for \( a(v) \) and \( a = (a_v)_{v \in I_n} \). The weight \( a(W) \) of any subspace (subset) \( W \) is the sum of the weights of all points in it.

EXAMPLE 1.3. Let \( \mu \) be a fixed point in \( I_n \). Define a weight function \( a_\mu: I_n \to \mathbb{R} \) by \( a_\mu(v) = |I_n| - 1 \) if \( v = \mu \) and \( -1 \) otherwise.

Clearly \( a_\mu \geq 0 \) for a \( (d - 1) \)-subspace (d-subset) \( W \) if and only if \( \mu \in W \).
Thus the number of \((d-1)\)-subspaces \((d-\text{subsets})\) \(W\) with nonnegative weights is \(N(I_n, d)\), where

\[
N(I_n, d) = \left[\binom{n}{d-1}\right] \text{ if } I_n \text{ is the set of points of } A_{n-1} \text{ or } P_{n-1} \text{ and } \\
\text{is equal to } \left(\binom{n}{d-1}\right) \text{ if } I_n \text{ is an } n \text{-set.}
\]

For a weight function \(a\) defined on \(I_n\), we let \(\phi(a, d) = |\{W| a(W) \geq 0\}|\), where \(W\)'s are \((d-1)\)-subspaces or \(d\)-subsets according to the case under consideration. Note that \(\phi(a_n, d) = N(I_n, d)\). We can now state conjecture 1.4.

**Conjecture 1.4.** For all \(n \geq 4d\) and for all weight functions \(a: I_n \to \mathbb{R}\),

\[
\phi(a, d) \geq N(I_n, d).
\]

Again from the example in [4] one can easily see that Conjecture 1.4 is not true for all pairs \((n, d)\) in the case of sets. In particular it is not true for \(n = 3d + 1\). Though no such example is known to the authors in the case of projective spaces. We believe that the conjecture may be true when \(n \geq 4d\).

In this paper we will show that Conjecture 1.4 is true for all pairs \((n, d)\) in the case of the affine spaces. Thus condition \(n \geq 4d\) is not needed in this case. We also show that conjecture is true if \(d|n\) in the case of projective spaces and sets. These results will be proved in Section 3 (Theorems 3.1, 3.2, and 3.3). We first prove the result in the case of projective spaces and then show how the proof can be modified in the other cases. The main tool is Theorem 3.5 (and its analogues) which shows that any set of \(d\)-subspaces of a vector space of sufficiently large size contains a \(d\)-spread. Theorem 3.5 is proved by a simple counting argument.

We also characterize in these cases the extremal case, i.e., when \(\phi(a, d) = N(I_n, d)\). The main tool is the Erdős–Ko–Rado theorem (see [7]) for sets and vector spaces. Infact proving that conjecture 1.4 is true is in some sense dual to the Erdős–Ko–Rado theorem (see Section 4).

In Section 4 we also prove that in the case of projective space or sets if Conjecture 1.4 is true for \((n, d)\), then it is also true for \((kn, d)\) for any positive integer \(k\).

Finally we remark that all our proofs are essentially based on one simple fact that for \(P_n, A_n\) or sets, all \(d-1\) subspaces (or \(d\)-subsets) behave in the same manner. Infact almost all results and conjectures of this paper can be easily generalized to any matroid of rank \(n\) whose automorphisms group acts transitively in flats of rank \(d\). Examples of such matroids can be constructed from geometric groups of sharp sets of permutations (see [5]).
2. Structure of the First Eigenspace

Let us assume that $V$ is a vector space of dimension $n$ over GF($q$) and that the vertices of $J_q(n, d)$ are the set of all $d$-subspaces of $V$. In this section we will establish an algebraic expression for the eigenspace $V_1$.

Let $M_{id}$ denote the $[n_i] \times [n_d]$ matrix whose rows and columns are indexed by $i$-dimensional and $d$-dimensional subspaces of $V$, respectively, and $(A, B)$th entry is 1 if $A \subseteq B$ and 0 otherwise. Let $U_i$ denote the row space of $M_{id}$. It is easy to prove:

**Lemma 2.1.** $U_{i-1} \subseteq U_i$ for $1 \leq i \leq d$ and $\dim U_i = [n_i]$. 

**Lemma 2.2.** Let $U, W$ be $i$ and $d$ subspace of $V$, respectively. If $U \subseteq W$, then there exists $[d-i][n-d]q$ $d$-subspaces of $V$ containing $U$ and having $(d-1)$ as the dimension of the intersection space with $W$. If $\dim(U \cap W) = i-1$, then there exists $[d-i+1]d$-subspaces of $V$ containing $U$ and having $(d-1)$ as the dimension of the intersection space with $W$.

**Proof.** Straightforward.

**Lemma 2.3.** $U_i = V_i \perp U_{i-1}$ for $1 \leq i \leq d$.

Let $W_i$ be the orthogonal complement of $U_{i-1}$ in $U_i$. Any vector $v \in U_i$ can be written as $v = v_1 + v_2$ where $v_1 \in W_i$ and $v_2 \in U_{i-1}$. Let $\Pi_i : U_i \rightarrow W_i$ be the projection map. To prove the lemma, we need to show $W_i = V_i$.

**Claim.** $Aw = \lambda_i w$, for all $w \in W_i$, where $\lambda_i$ is the eigenvalue of $A$, the adjacency matrix of the graph $J_q(n, d)$, on $V_i$. $\lambda_i = [d-i][n-d]q - [n_i][d-i+1]$ (see [1]).

Let $X_i$ denote the collection of $j$-subspaces of $V$ for $1 \leq j \leq d$. Let $u \in X_i$. To prove the claim. We need to show $A(\Pi u) = \lambda_i(\Pi u)$, where $u$ denotes the row of $u$ in $M_{id}$. By the definition of $u$, 

$$Au = A \sum_{u \in X} x \quad (x \in X_d)$$

$$= \sum_{u \in x} Ax$$

$$= \left( \sum_{u \in x} \sum_{y, y \cap x \in X_{d-1}} y \right) (y \in X_d) \quad \text{(by the definition of adjacency)}$$

$$= \left( \sum_{u \cap y \in X_{d-1}} y \right)[d-i+1] + \left( \sum_{u \in y} \right)[d-i][n-d]q$$
Since \( C_{\lambda, v} \in W_i \) and \( A(\lambda) = A(v) \), we obtain \( v_i(w_i) = A(\lambda) = A(v) \) and hence the claim. Therefore \( W_i \subseteq V_i \).

Also since \( W_d \perp W_{d-1} \perp \cdots \perp W_0 = U_d = V_d \perp V_{d-1} \perp \cdots \perp V_0 \) follows that \( W_i = V_i \). This proves the lemma.

Let \( I_n \) be the set of all projective points of \( V \). It can be easily seen that the map \( f: \mathbb{R}^{1n} \to U_1 \) defined by \( f((a_\lambda)_{\lambda \in I_n}) = \sum_{\lambda \in I_n} a_\lambda m_\lambda \), where \( m_1, m_2, \ldots, m_{[\frac{d}{2}]} \) are the row vectors of \( M_{I_d} \), is an isomorphism. From Lemma 2.3 \( U_1 = V_1 \perp U_0 = V_1 \perp V_0 \) and \( V_0 = \langle (1, 1, \ldots, 1) \rangle \subseteq \mathbb{R}^{\frac{d}{2}} \).

Now from the fact that

\[
\langle m_i, (1, 1, \ldots, 1) \rangle = \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}
\]

for all \( 1 \leq i \leq \left[ \frac{n}{2} \right] \),

it follows that

\[
V_1 = \left\{ f((a_\lambda)_{\lambda \in I_n}) \left| \sum_{\lambda \in I_n} a_\lambda = 0 \right. \right\}.
\]

Since \( f \) is an isomorphism, we conclude that

\[
V_1 \simeq \left\{ (a_\lambda)_{\lambda \in I_n} \left| \sum_{\lambda \in I_n} a_\lambda = 0 \right. \right\}.
\]

Remark 2.4. We will identify \( V_1 \) with \( \{(a_\lambda)_{\lambda \in I_n} \left| \sum_{\lambda \in I_n} a_\lambda = 0 \right. \} \). Thus a vector \( a = (a_\lambda)_{\lambda \in I_n} \) is a general vector if and only if it satisfies

(i) \( \sum_{\lambda \in I_n} a_\lambda = 0 \) and

(ii) \( \sum_{\xi \in \omega} a_\xi \neq 0 \) for every \( d \)-subspace \( W \) of \( V \).

Clearly by condition (i) \( w \) is a weight function. The second condition says that the weight of each \( d \)-subspace is non-zero. It is now clear that

First distribution invariant = \( \min_\sigma \{ |\{ B | a(B) > 0 \}| \} \), where the minimum
is taken over all the weight functions \( a \) with \( a(B) \neq 0 \) for any \( d \)-subspace \( B \). The weight function \( a_\mu \) in Example 1 clearly corresponds to a general vector. The following remark is now clear.

\textbf{Remark 2.5.} Let \( I_n \) be the set of all points of projective \( P_{n-1} \) or \( n \)-set. Let \( \min \phi(a, d) \) denote the minimum value over all weight functions \( \phi \) on \( I_n \). Then \( \min \phi(a, d) = N(I_n, d) \) implies \( Vt_1(J_q(n, d)) = N(I_n, d) \).

3. FIRST DISTRIBUTION INVARIANTS AND \( d \)-SPREADS

The aim of this section is to prove the following theorems.

\textbf{Theorem 3.1.} Let \( I_n \) be the set of all points in a projective space \( P_{n-1} \). Let \( d \) be a positive integer such that \( d \mid n \). Let \( a: I_n \to \mathbb{R} \) be any weight function then

\begin{itemize}
  \item[(i)] \( \phi(a, d) \geq \binom{n-1}{d-1} \),
  \item[(ii)] equality holds in (i) if and only if there exists \( \mu \in I_n \) such that for any \( (d-1) \)-subspace \( W \) of \( P_{n-1} \), \( a(W) > 0 \) if and only if \( \mu \in W \).
\end{itemize}

\textbf{Theorem 3.2.} Let \( I_n \) be the set of all points in an affine space \( A_{n-1} \). Let \( d \) be positive integer. Let \( a: I_n \to \mathbb{R} \) be any weight function, then

\begin{itemize}
  \item[(i)] \( \phi(a, d) \geq \binom{n-1}{d-1} \),
  \item[(ii)] when \( n \geq 2d \) equality holds in (i) if and only if there exists \( \mu \in I_n \) such that for any \( (d-1) \)-subspace \( W \) of \( A_{n-1} \), \( a(W) \geq 0 \) if and only if \( \mu \in W \).
\end{itemize}

\textbf{Theorem 3.3.} Let \( I_n \) be the set of order \( n \). Let \( d \) be a positive integer such that \( d \mid n \). Let \( a: I_n \to \mathbb{R} \) be any weight functions, then

\begin{itemize}
  \item[(i)] \( \phi(a, d) \geq \binom{n-1}{d-1} \)
  \item[(ii)] equality holds in (i) if and only if \( \mu \in I_n \) such that for any \( d \)-subset \( W \) of \( I_n \), \( a(W) > 0 \) if and only if \( \mu \in W \).
\end{itemize}

The following corollaries are immediate.

\textbf{Corollary 3.4.} Let \( d \) be positive integer such that \( d \mid n \). Then

\begin{itemize}
  \item[(i)] \( Vt_1J_q(n, d) = \binom{n-1}{d-1} \);
  \item[(ii)] if \( a \) is a general vector then the number of vertices \( x \) of \( J_q(n, d) \) with \( \langle a, x \rangle \geq 0 \) is precisely \( \binom{n-1}{d-1} \) if and only if there exists \( \mu \) in \( V \) such that \( \langle a, x \rangle \geq 0 \) if and only if \( \mu \in X \).
\end{itemize}
**Corollary 3.5.** Let $d$ be a positive integer such that $d | n$. Then

(i) $V_t \mathcal{J}(n, d) = \binom{n}{d+1};$

(ii) if $a$ is a general vector then the number of vertices $x$ of $\mathcal{J}(n, d)$ with $\langle a, x \rangle \geq 0$ is precisely $\binom{n}{d+1}$ if and only if there exists a $\mu$ in the vertex-set such that $\langle a, x \rangle \geq 0$ if and only if $\mu \in X$.

We continue to assume that $V$ is a vector space of dimension $n$ over $GF(q)$. A projective point is a subspace of dimension one in $V$. A $d$-spread $\xi$ of $V$ is a set of $d$-subspaces of $V$ such that each projective point is contained in one and only one $d$-subspace in $\xi$ (see [6]).

**Lemma 3.4.** If $s$ denotes the total number of $d$-spreads and if $r$ is the total number of $d$-spreads containing a fixed $d$-subspace, then

$$ s/r = \binom{n}{d} \frac{(q^d - 1)/(q^n - 1)}{\binom{n}{d+1}} = \binom{n - 1}{d - 1}. $$

**Proof.** Observe that

$$ s/r = \frac{\text{total number of } d\text{-subspaces}}{\text{total number of } d\text{-subspaces in a spread}} $$

$$ = \binom{n}{d} / \{(q^n - 1)/(q^d - 1)\} $$

$$ = \binom{n}{d} \{(q^d - 1)\}/\{(q^n - 1)\}. $$

**Theorem 3.5.** If $d | n$ and if $D$ is a collection of $d$-subspaces of an $n$-dimensional vector space $V$ over $GF(q)$ with $|D| > \binom{n}{d} - \genfrac{[}{]}{0pt}{}{n}{d+1}$, then $D$ contains a $d$-spread of $V$.

**Proof.** Let us assume that $D$ is a collection of $d$-subspaces of $V$ which does not contain any $d$-spread of $V$. Now we count in two ways the number of pairs $(\xi, A)$, where $\xi$ is a $d$-spread and $A$ is an element of $D$ which is contained in $\xi$. Let $r$ and $s$ be as in Lemma 3.4. Clearly,

$$ \text{total number of pairs } (\xi, A) = |D| \cdot (r). $$

On the other hand, since we assumed that there does not exist a $d$-spread consisting of $d$-subspaces only from $D$, any given $d$-spread $\xi$ contains at the most $\{(q^n - 1)/(q^d - 1)\} - 1$ elements from $D$. Therefore

$$ \text{the total number of pairs } (\xi, A) \leq s \left[ \{(q^n - 1)/(q^d - 1)\} - 1 \right]. $$

$$ 1 $$

$$ 2 $$
Using (1) and (2) we see that

$$|D| \leq (s/r)\{((q^n-1)/(q^d-1))-1\}.$$ 

From Lemma 3.4, we know that

$$s/r = \binom{n}{d} / \{((q^n-1)/(q^d-1))\}.$$ 

Therefore, $|D| \leq \binom{n}{d} / (q^n-1/q^d-1)(q^n-1/q^d-1-1) = \binom{n}{d} - \binom{n}{d-1}$. This proves the theorem.

**Proof of Theorem 3.1.** We will take $I_n$ to be the set of all projective points of $V$. Let $(a_p)_{p \in I_n}$ be a weight function. For any $d$-subspace $W$ of $V$ the weight $a(W)$ of $W$ is defined to be the sum of an the weights of projective points contained in $W$. Let $D = \{\text{negatively weighted } d\text{-subspaces of } V\}$. Then we must have $|D| \leq \binom{n}{d} - \binom{n}{d-1}$. Otherwise, if $|D| > \binom{n}{d} - \binom{n}{d-1}$, then by the above theorem, $D$ contains a $d$-spread $\xi$. Since each $d$-subspace in this spread is negatively weighted, $0 > \sum_{W \in \xi} a_W = \sum_{W \in D} a_W$ contradicting (i). Hence the number of $d$-subspaces of $V$ with nonnegative weight must be $\geq \binom{n}{d} - |D| = \binom{n}{d-1}$. This proves the first part of the theorem.

When $|D| = \binom{n}{d} - \binom{n}{d-1}$ we must have equality in Eq. (2) which clearly implies that each $d$-spread contains precisely one $d$-subspace not from $D$, i.e., with non-negative weight. This implies that for any two $d$-subspaces $W_1$ and $W_2$ of nonnegative weights $\dim(W_1 \cap W_2) \geq 1$, since otherwise they would be contained in a spread. Thus the set of all $d$-subspaces with non-negative weights is an intersecting family in the sense of the Erdős–Ko–Rado (see [7]). Now the last part of the theorem follows from the Erdős–Ko–Rado theorem for vector spaces (case $t=1$). This completes the proof of Theorem 3.1.

We now consider the cases of sets and affine spaces. In the case of sets it can be easily seen that if we take $V$ to be an $n$-set, then the corresponding statements of Lemma 3.4 and Theorem 3.5 can be obtained by replacing the words “subspace” by “subsets,” “spreads” by “partition” and “Guession integers” by usual “binomial numbers” (e.g., the equation in Lemma 3.4 will change to $s/r - \binom{n}{d}(d/n) = \binom{a-1}{d-1}$). Proofs are also essentially the same with similar modifications. Proof of Theorem 3.3 can also be obtained in the same way.

We now consider the affine case. We fix a subspace $H$ of $V$, of dimension $n-1$, and take the set $P$ of points of the affine space $A_{n-1}$ to be the set of all projective points of $V$ not contained in $H$. Each subspace $W$ of $V$ corresponds to a unique $(d-1)$-subspace $W$ of $A_{n-1}$ consisting of all projective points of $W$ not contained in $H$. A $(d-1)$-spread $\xi$ in $A_{n-1}$ is a set
of mutually disjoint \((d-1)\)-subspaces of \(A_{n-1}\) such that each element of \(I_n\) is contained in exactly one \((d-1)\)-subspace in \(\xi\).

One can easily prove Lemma 3.4 and Theorem 3.5 with similar modifications (e.g., in Lemma 3.4 \(s/r = (\binom{n}{d-1} - \binom{n}{d-1}) q^d/q^n = \binom{n-1}{d-1}\).)

The proof of the first part of Theorem 3.2 can also be obtained as in the projective case. For the last part of the theorem suppose that the number of \((d-1)\)-subspaces of \(A_{n-1}\) with nonnegative weights is equal to \(\binom{n}{d-1}\). As in the projective case this implies that each \((d-1)\)-spread of \(A_{n-1}\) contains precisely one \((d-1)\)-subspace of \(A_{n-1}\) with nonnegative weight. Remark 3.6 given below shows that in this case for any two \((d-1)\)-subspaces of \(A_{n-1}\) with nonnegative weights the corresponding \(d\)-subspaces of \(V\) must intersect. Since \(n \geq 2d\), this implies, using Erdős-Ko-Rado theorem for vector spaces (see [7]), that there exists a projective point \(\mu\) of \(V\) contained in each of these \(d\)-subspaces of \(V\). It is clear that necessarily \(\mu\) is not contained in \(H\). This proves the last part of the theorem.

Remark 3.6. Let \(W_1\) and \(W_2\) be two \((d-1)\)-subspaces of \(A_{n-1}\) such that the intersection of the corresponding subspaces \(W'_1\) and \(W'_2\) is \(\{0\}\). Then there exists a \(d\)-spread \(\xi\) of \(A_{n-1}\) containing \(W_1\) and \(W_2\). One construction for \(\xi\) is as follows. Let \(S'\) be the subspace of \(V\) generated by \(W'_1\) and \(W'_2\). The dimension of \(S'\) is \(2d\). Now for each \((2d-1)\)-subspace \(U\) of the unique parallel class \(P_S\) of \((2d-1)\)-subspaces of \(A_{n-1}\) containing \(S\), we choose a \(d\)-spread \(\xi_U\) of \(U\), where spread \(\xi_S\) is chosen to be \(\{W'_1, W'_2\}\).

Now it can be easily seen that

\[ \xi = \bigcup \{W | W' \in \xi_U, U \in P_S\} \]

is the required spread of \(A_{n-1}\).

We now give another very simple proof of the first part of Theorem 3.2. The proof is essentially similar to the one given for the set case in [4].

Alternative proof of part (i) of Theorem 3.2

From Example 1 it is clear that \(\phi(a, d) \leq N(I_n, d)\). Let \(P_1, \ldots, P_m\), where \(m = \binom{n-1}{d-1}\), be the parallel classes of \(d\)-subspaces of \(A_{n-1}\). Let \((a_v)_{v \in I_n}\) be a weight function. Now clearly for each \(P_i\), \(1 \leq i \leq \binom{n-1}{d-1}\).

\[ \sum_{v \in P_i} a_v = \sum_{v \in I_n} a_v \geq 0. \]

This implies each \(P_i\) contains at least one \(d\)-subspace with nonnegative weight. This proves the theorem.

4. Certain Translitrivities

In this section we obtain a lower bound for \(\phi(a, d)\) and hence for the first distribution invariant of \(J_q(n, d)\). We also prove that if \(\phi(a, d) \geq \binom{k-1}{d-1}\) for all weight functions \(a: I_n \to \mathbb{R}\), then \(\phi(a, d) \geq \binom{k-1}{d-1}\) for all weight functions \(a: I_{kn} \to \mathbb{R}\) where \(k\) is any positive integer.
Lemma 4.1. Let \( a = (a_v)_{v \in I_n} \) be a weight function on \( I_n \), the set of all projective points of \( V \). Then for any \( s \) with \( 1 \leq s \leq n \) there exists subspace \( W \) of \( V \) of dimension \( s \) such that \( a(W) \geq 0 \).

Proof. For any non-zero vector \( u \) in \( V \), there exist \( \left[ \frac{n-r}{s} \right] \) \( s \)-dimensional subspaces containing the projective point \( \langle u \rangle \).

Therefore, \( \sum_W a(W) = \left[ \frac{n-r}{s} \right] \left( \sum_v a_v \right) \geq 0 \), where the first sum on the left-hand side varies over the set of all \( s \)-subspaces of \( V \). This implies that there exists at least one \( s \)-subspace \( W \) with \( a(W) \geq 0 \).

Lemma 4.2. \( \phi(a, d) \geq \left[ \frac{n-r}{d-1} \right] \) if \( n = pd + r \) with \( 0 \leq r < d \), where \( a \) is a weight function on \( I_n \), the set of points of a projective space \( P_{n-1} \).

Proof. Let \( a = (a_v)_{v \in I_n} \) be any weight function on \( I_n \). Let \( s = n - r \). By Lemma 4.1 there exists a subspace \( W \) of \( V \) of dimension \( s \) satisfying \( a(W) \geq 0 \). Since \( d \) divides \( s \), by Theorem 3.1, there exist at least \( \left[ \frac{s}{d-1} \right] \) \( d \)-subspaces of \( W \) having positive weights. These are also \( d \)-subspaces of \( V \) having the same weight, hence \( \phi(a, d) \geq \left[ \frac{s}{d-1} \right] = \left[ \frac{n-r}{d-1} \right] \).

Corollary 4.3. \( \Xi \left( J_\ast \left( n, \frac{d}{d-1} \right) \right) \geq \left[ \frac{n-r}{d-1} \right] \).

Theorem 4.4. Conjecture 1.4 is true for \( I_{kn} \) for all positive integers \( k \) if it is true for \( I_n \).

Proof. We shall give the proof for \( P_{kn-1} \). For sets the result follows similarly. Let \( a = (a_v)_{v \in I_{kn}} \) be a weight function.

Let \( \mathcal{N} = \{ W \mid W \text{ is a } d\text{-subspace of } V \text{ and } a(W) < 0 \} \), where \( V \) is a vector space of dimension \( kn \) over \( GF(q) \).

Claim. \( |N| \leq \left[ \frac{k}{d} \right] - \left[ \frac{k}{d-1} \right] \).

Count the number of pairs \((\xi, A)\) where \( \xi \) is an \( n \)-spread of \( V \), \( A \) is a \( d \)-subspace in \( \mathcal{N} \), and \( A \) is contained in one of the \( n \)-dimensional subspaces in \( \xi \). For any fixed \( d \)-subspace \( A \), the number of pairs of \((\xi, A)\) is exactly equal to

\[
\left[ \frac{kn-d}{n-d} \right] \{ \text{number of } n \text{-spreads containing a fixed } n \text{-subspace} \}.
\]

Therefore, by varying \( A \) over the elements of \( \mathcal{N} \) we conclude that in total we have
such pairs.

On the other hand, suppose we fix an $n$-spread $\xi$ and count the number of pairs $(\xi, A)$. $\xi$ contains \( (q^{kn} - 1)/(q^n - 1) = r \) (say) $n$-dimensional subspaces of $V$. Let $W_1, \ldots, W_r$ be all of them and let

\[ N_i = N \cap \{ d\text{-subspaces of } W_i \}. \]

As \( \sum_{1 \leq i \leq r} a(W_i) = \sum_{v \in V} a_v \geq 0 \), there exists at least one $j$ such that $a(W_j) \geq 0$. Hence $a$ is a weight function on $W_j$. Now from the hypothesis, we have at least \( \lceil n - a \rceil \) positively weighted $d$-subspaces of $W_j$. Hence the cardinality of $N_j$ is at most \( \lceil n - a \rceil - \lceil n - a - 1 \rceil \). Therefore, for any fixed $n$-spread $\xi$, we have at most \( \lceil (q^{kn} - 1)/(q^n - 1) \rceil \lceil n - a \rceil - \lceil n - a - 1 \rceil \) pairs $(\xi, A)$, which is equal to \( \lceil (q^{kn} - 1)/(q^n - 1) \rceil \lceil n - a \rceil - \lceil n - a - 1 \rceil \). Now, by varying $\xi$ over the set of all $n$-spreads of $V$, we conclude that

\[
|N| \leq \left( \text{total number of } n\text{-spreads} \right) \left( \frac{q^{kn} - 1}{q^n - 1} \right) \left[ n \atop d \right] - \left[ n \atop d - 1 \right].
\]

(2)

Let $s = \text{the total number of } n\text{-spreads}$ and $t = \text{the total number of } n\text{-spreads containing a fixed } n\text{-subspace}$. By comparing (1) and (2) we see that

\[
|N| \leq \left\{ \frac{s}{t} \right\} \left[ (q^{kn} - 1)/(q^n - 1) \right] \left[ n \atop d \right] - \left[ n \atop d - 1 \right].
\]

(3)

By Lemma 3.4,

\[
\frac{\text{the total number of } n\text{-spreads}}{\text{the total number of } n\text{-spreads containing a fixed } n\text{-subspace}} = \frac{\text{the total number of } n\text{-subspace}}{\text{the total number of } n\text{-subspaces in a spread}} = \frac{\left[ \frac{kn}{n} \right]}{(q^n - 1)/(q^{kn} - 1)}
\]

(4)
Substituting (4) in (3) we get that

\[ |N| \leq \left[ \frac{kn}{d} \right] \frac{(q^n - 1)/(q^{kn} - 1)}{\binom{n}{d} - \binom{n-1}{d-1}/\binom{kn-d}{n-d}} \]

and hence the claim that \( |N| \leq \left[ \frac{kn}{d} \right] - \left[ \frac{kn}{d-1} \right] \) is settled.

Finally we note some relationships with Erdős–Ko–Rado inequalities. We first note an equivalent statement of Theorem 3.4.

**Theorem 4.5.** If \( d \mid n \) and if \( D \) is a collection of \( d \)-subspaces of an \( n \)-dimensional vector space \( V \) over \( \mathbb{GF}(q) \) such that each \( d \)-spread contains at least one element of \( D \), then \( |D| \geq \left[ \frac{n}{d} - 1 \right] \).

By reversing all the inequalities in the proof of Theorem 3.5 and making similar other necessary changes one can easily prove the following inequality “dual” to Theorem 4.5.

**Theorem 4.6.** If \( d \mid n \) and if \( D \) is a collection of \( d \)-subspaces of an \( n \)-dimensional vector space \( V \) over \( \mathbb{GF}(q) \) such that each \( d \)-spread contains at most one element of \( D \), then \( |D| \leq \left[ \frac{n}{d} - 1 \right] \).

Note that since any intersecting family \( D \) of \( d \)-subspaces of \( V \) clearly satisfies the conditions of Theorem 4.6, the above theorem implies the Erdős–Ko–Rado inequality for vector spaces when \( d \mid n \) and \( t = 1 \). Similar analogies for sets and affine spaces also be easily seen. Note that the condition \( d \mid n \) is not needed in the case of affine spaces. It may be interesting to search for a proper analogue of \( d \)-spreads, when \( d \) does not divide \( n \), for which an analogue of Theorem 4.5 or 4.6 is valid. This will help in proving conjecture 1.2 and 1.4 and possibly will give a new proof for Erdős–Ko–Rado inequality at least for \( t = 1 \) or even more generally.

**REFERENCES**