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Note

## On 3-connected hamiltonian line graphs

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## ABSTRACT

Thomassen conjectured that every 4-connected line graph is hamiltonian. It has been proved that every 4-connected line graph of a claw-free graph, or an almost claw-free graph, or a quasi-claw-free graph, is hamiltonian. In 1998, Ainouche et al. [2] introduced the class of DCT graphs, which properly contains both the almost claw-free graphs and the quasi-claw-free graphs. Recently, Broersma and Vumar (2009) [5] found another family of graphs, called P3D graphs, which properly contain all quasi-claw-free graphs. In this paper, we investigate the hamiltonicity of 3-connected line graphs of DCT graphs and P3D graphs, and prove that if  $G$  is a DCT graph or a P3D graph with  $\kappa(L(G)) \geq 3$  and if  $L(G)$  does not have an independent vertex 3-cut, then  $L(G)$  is hamiltonian. Consequently, every 4-connected line graph of a DCT graph or a P3D graph is hamiltonian.

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## 1. Introduction

Graphs in this paper are finite and may have multiple edges but no loops. Terms and notations not defined here are referred to [4]. In particular,  $\kappa(G)$  and  $\kappa'(G)$  represent the connectivity and edge connectivity of a graph  $G$ , respectively. As in [4], if  $U \subseteq V(G) \cup E(G)$ , then  $G[U]$  denotes the subgraph of  $G$  induced by  $U$ . A graph is **nontrivial** if it contains at least one edge. For a vertex  $v \in V(G)$ , define  $N_G(v)$  to be the set of vertices that are adjacent to  $v$ ,  $N_G[v] = N_G(v) \cup \{v\}$  and

$$E_G(v) = \{e \in E(G); e \text{ is incident with } v \text{ in } G\}.$$

An edge cut  $X$  of  $G$  is **peripheral** if  $E_G(v) = X$  for some  $v \in V(G)$ . An edge cut  $X$  is **essential** if each component of  $G - X$  has at least one edge. For an integer  $k > 0$ , a graph  $G$  is **essentially  $k$ -edge-connected** if  $G$  does not have an essential edge cut  $X$  with  $|X| < k$ .

The line graph of a graph  $G$ , denoted by  $L(G)$ , has  $E(G)$  as its vertex set, where two vertices in  $L(G)$  are adjacent if and only if the corresponding edges in  $G$  have at least one vertex in common.

If  $H \cong K_{1,3}$  is an induced subgraph of  $G$ , then  $H$  is called the **claw** of  $G$ . The only vertex of degree 3 in  $H$  is the **center** of  $H$ , and the vertices of degree 1 are the **toes** of  $H$ . If the vertices  $\{z, a_1, a_2, a_3\}$  of  $G$  induces a claw with center  $z$ , then we denote this claw by  $G[z, a_1, a_2, a_3]$ . A graph  $G$  is **claw-free** if  $G$  does not have an induced subgraph isomorphic to  $K_{1,3}$ .

For a connected graph  $G$  and any  $x, y \in V(G)$ , the **distance** between  $x$  and  $y$  in  $G$ , denoted by  $dist_G(x, y)$ , is the length of a shortest  $(x, y)$ -path of  $G$ . For vertices  $x, y \in V(G)$  with  $dist_G(x, y) = 2$ , define

$$J_G(x, y) = \{u \in N_G(x) \cap N_G(y) : N_G[u] \subseteq N_G[x] \cup N_G[y]\}, \quad (1)$$

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$$J'_G(x, y) = \{u \in N_G(x) \cap N_G(y) : \text{if } v \in N_G(u) - (N_G[x] \cup N_G[y]), \text{ then } N_G(x) \cup N_G(y) \cup N_G(u) - \{x, y, v\} \subseteq N_G(v)\}. \quad (2)$$

Almost claw-free graphs (or ACF graphs for short) and quasi-claw-free graphs (or QCF graphs for short) were introduced by Ryjáček in [16] and by Ainouche in [1], respectively. Detailed definitions of these two classes of graphs can be found in [16,1], respectively.

In [2], Ainouche et al. were introduced a new class of graphs that properly include both ACF and QCF graphs. A claw  $G[z, a_1, a_2, a_3]$  of a graph  $G$  is said to be **dominated** if

$$J_G(a_1, a_2) \cup J_G(a_2, a_3) \cup J_G(a_1, a_3) \neq \emptyset. \quad (3)$$

The vertices in  $J_G(a_1, a_2) \cup J_G(a_2, a_3) \cup J_G(a_1, a_3)$  are called the **dominators** of the claw  $[z, a_1, a_2, a_3]$ . A graph  $G$  is with **dominated claw toes** (or DCT for short) if every claw in  $G$  is dominated. It is known [2] that ACF graphs and QCF graphs are all DCT graphs.

Broersma and Vumar [5] recently discovered a different new class of graphs, called  $P_3$ -dominated graphs. A graph  $G$  is  $P_3$ -**dominated** (or P3D for short) if

$$J_G(x, y) \cup J'_G(x, y) \neq \emptyset \quad \text{for any } x, y \in V(G) \text{ with } \text{dist}_G(x, y) = 2. \quad (4)$$

It is known [5] that every QCF graph is a P3D graph, and there are infinitely many DCT graphs that are not P3D, and there are infinitely many P3D graphs that are not DCT.

Beineke [3] and Robertson (unpublished, see Page 74 of [9]) proved that every line graph is a claw-free graph. Matthews and Sumner [15] conjectured that every 4-connected claw-free graph is hamiltonian, and Thomassen [19] conjectured that every 4-connected line graph is hamiltonian. In 1997, Ryjáček [17] proved that these two conjectures are in fact equivalent to each other.

A graph  $G$  is hamiltonian-connected if for every pair of vertices  $u, v \in V(G)$ ,  $G$  has a spanning  $(u, v)$ -path. Zhan [20] proved that every 7-connected line graphs is hamiltonian-connected, and Ryjáček [17] proved that every 7-connected claw-free graphs is hamiltonian. More recently, Zhan [21] proved that every 6-connected line graph without too many vertices of degree 6 is hamiltonian-connected. Kaiser and Vrána [11] proved that every 5-connected line graph with minimum degree at least 6 is hamiltonian.

For line graphs with connectivity at least 4, a number of results have been obtained. Chen et al. [8] first proved that every 4-connected line graph of a claw-free graph is hamiltonian. Kriesell [12] extended this result and showed that every 4-connected line graph of a claw-free graph is hamiltonian connected. In [14,13], the authors improved Kriesell's result by showing that every 4-connected line graph of a ACF graph or a QCF graph is hamiltonian-connected.

Our main result is the following theorem on the hamiltonicity of 3-connected line graphs of DCT graphs and P3D graphs.

**Theorem 1.1.** *Let  $G$  be a connected graph with  $\kappa(L(G)) \geq 3$  such that  $L(G)$  does not have an independent 3-vertex cut.*

- (i) *If  $G$  is a DCT graph, then  $L(G)$  is hamiltonian.*
- (ii) *If  $G$  is a P3D graph, then  $L(G)$  is hamiltonian.*

The next corollary follows immediately from Theorem 1.1.

**Corollary 1.2.** *Each of the following holds.*

- (i) *Every 4-connected line graph of a DCT graph is hamiltonian.*
- (ii) *Every 4-connected line graph of a P3D graph is hamiltonian.*

Our approach will utilize the theorem of Harary and Nash-Williams on the relationship between hamiltonian cycles in the line graph  $L(G)$  and Eulerian subgraphs in  $G$ , and Catlin's reduction method. These will be applied to develop some useful tools in Section 2. The main results will be proved in Section 3.

## 2. Preliminaries

Collapsible graphs are introduced by Catlin in [6]. Let  $G$  be a graph and let  $O(G)$  denote the set of all odd degree vertices of  $G$ . A graph  $G$  is **collapsible** if for any even subset  $R$  of  $V(G)$ ,  $G$  has a spanning connected subgraph  $\Gamma_R$  with  $O(\Gamma_R) = R$ . By definition, the graph  $K_1$  is collapsible. A graph  $G$  is **supereulerian** if it has a spanning connected Eulerian subgraph. By definition, every collapsible graph is supereulerian.

Let  $X \subseteq E(G)$  be an edge subset. The contraction  $G/X$  is the graph obtained from  $G$  by identifying the two ends of each edge in  $X$  and then deleting the resulting loops. When  $X = \{e\}$ , we use  $G/e$  for  $G/\{e\}$ . If  $H$  is a subgraph of  $G$ , then we write  $G/H$  for  $G/E(H)$ . Following [4], if  $H_1$  and  $H_2$  are subgraphs of  $G$ ,  $H_1 \cup H_2$  is the union of  $H_1$  and  $H_2$  in  $G$ . Catlin [6] proved that every graph  $G$  has a unique collection of maximal collapsible subgraph  $H_1, H_2, \dots, H_c$ ; and that the contraction  $G/(H_1 \cup H_2 \cup \dots \cup H_c)$  called the **reduction** of  $G$ , does not have any nontrivial collapsible subgraph.

**Theorem 2.1.** *Let  $G$  be a connected graph. Then each of the following holds.*

- (i) (Catlin, Theorem 3 of [6]). Let  $H$  be a collapsible subgraph of  $G$ . Then  $G$  is collapsible if and only if  $G/H$  is collapsible; and  $G$  is supereulerian if and only if  $G/H$  is supereulerian. In particular,  $G$  is collapsible if and only if the reduction of  $G$  is  $K_1$ .
- (ii) (Catlin, Theorem 2 of [6]). If  $\kappa'(G) \geq 4$ , then  $G$  is collapsible.
- (iii) (Catlin, Theorem 4 of [6]). If  $H_1, H_2$  are collapsible subgraphs of  $G$  such that  $V(H_1) \cap V(H_2) \neq \emptyset$ , then  $H_1 \cup H_2$  is a collapsible subgraph of  $G$ .

Let  $G$  be a graph with  $\kappa(L(G)) \geq 3$  and  $L(G)$  is not complete. The **core** of this graph  $G$ , denoted by  $G_0$ , is obtained by deleting all the vertices of degree 1 and contracting exactly one edge  $xy$  or  $yz$  for each path  $xyz$  with  $d_G(y) = 2$ . We name the vertices and edges in  $G_0$  the same as  $G$  if they are not changed in  $G_0$ . If the edge  $xy$  is contracted in  $G_0$ , then we name the new vertex as  $x$  or  $y$ . Utilizing the theorem of Harary and Nash-Williams [10] on the relationship between Hamilton cycles in  $L(G)$  and Eulerian subgraphs in  $G$ , Shao proved the following useful result.

**Theorem 2.2** (Shao, Lemma 1.4.1 and Proposition 1.4.2 of [18]). *Let  $G$  be a connected graph with  $\kappa(L(G)) \geq 3$ , and let  $G_0$  denote the core of  $G$ . Each of the following holds.*

- (i)  $G_0$  is uniquely defined.
- (ii)  $\delta(G_0) \geq \kappa'(G_0) \geq 3$ .
- (iii) If  $G_0$  is supereulerian, then  $L(G)$  is hamiltonian.

**Theorem 2.3.** *Let  $G$  be a graph with  $\kappa'(G) \geq 3$ . If for every 3-edge-cut  $D$  of  $G$ ,  $G$  has a collapsible subgraph  $H_D$  such that  $E(H_D) \cap D \neq \emptyset$ , then  $G$  is collapsible.*

**Proof.** Let  $\mathcal{C}_3^*(G)$  denote the set of all 3-edge-cuts of  $G$ . For each  $D \in \mathcal{C}_3^*(G)$ , we specify a fixed collapsible subgraph  $H_D$  such that  $E(H_D) \cap D \neq \emptyset$ . Define

$$X = \bigcup_{D \in \mathcal{C}_3^*(G)} E(H_D).$$

By Theorem 2.1(iii), every component of  $G[X]$  is a collapsible subgraph of  $G$ . By the definition of  $X$ , every 3-edge-cut of  $G$  will be contracted in  $G/X$ , and so  $\kappa'(G/X) \geq 4$ . By Theorem 2.1(ii),  $G/X$  is collapsible. As each component of  $G[X]$  is a collapsible subgraph of  $G$ , by repeating applications of Theorem 2.1(i),  $G$  is also collapsible.  $\square$

**Corollary 2.4.** *Let  $G$  be a connected graph with  $\kappa(L(G)) \geq 3$  and let  $G_0$  be the core of  $G$ . If every 3-edge-cut of  $G_0$  intersects a collapsible subgraph of  $G_0$ , then  $L(G)$  is hamiltonian.*

**Proof.** By Theorem 2.2,  $\kappa'(G_0) \geq 3$ . By Theorem 2.3,  $G_0$  is collapsible. As collapsible graphs are supereulerian, by Theorem 2.2(iii),  $L(G)$  is hamiltonian.  $\square$

For an integer  $n \geq 2$ , let  $C_n$  denote a cycle of length  $n$ . Let  $C_4 + e$  denote a graph obtained from  $C_4$  by adding an edge  $e$  joining two adjacent vertices of  $C_4$ ; and  $C_5 + e$  denote a graph obtained from  $C_5$  by adding an edge  $e$  joining two nonadjacent vertices of  $C_5$ .

**Lemma 2.5.** *Each of the following holds.*

- (i) (Catlin, Theorem 3 of [6]). 2-cycles and 3-cycles are collapsible.
- (ii) If  $G$  is collapsible, and if  $e \in E(G)$ , then  $G/e$  is collapsible.
- (iii)  $C_4 + e$  is collapsible.
- (iv)  $C_5 + e$  is collapsible.
- (v) (Lemma 1 of [7]). Both  $K_{3,3}$  and  $K_{3,3} - e$  are collapsible.

**Proof.** The proofs are straightforward and will be omitted.  $\square$

### 3. Proof of the main theorems

We first prove some lemmas that are needed for the proofs of our main theorem. The following lemma follows from the definition of the core of a graph and Lemma 2.5(ii).

**Lemma 3.1.** *Let  $G$  be a graph with  $\kappa(L(G)) \geq 3$ ,  $G_0$  be the core of  $G$  and  $D = \{e_1, e_2, e_3\}$  be a 3-edge-cut of  $G_0$ . If  $D$  is intersecting a collapsible subgraph of  $G$ , then  $D$  is intersecting a collapsible subgraph of  $G_0$ .  $\square$*

Thus to prove that  $D$  is intersecting a collapsible subgraph of  $G_0$ , it suffices to show that  $D$  is intersecting a collapsible subgraph of  $G$ . For notational convenience, for  $\{v_1, v_2, \dots, v_k\} \subseteq V(G)$ , we use  $G[v_1, v_2, \dots, v_k]$  to denote  $G[\{v_1, v_2, \dots, v_k\}]$ , and use  $J(x, y)$  and  $J'(x, y)$  to denote  $J_G(x, y)$  and  $J_{G_0}(x, y)$  respectively.

We need to establish several lemmas for the proof of the main result.

**Lemma 3.2.** Let  $e_1 = va_1, e_2 = va_2$  be two adjacent edges in  $G$  satisfying both  $J(a_1, a_2) - \{v\} \neq \emptyset$ , and  $d(v) \geq 3$ . Then  $G_0$  has a collapsible subgraph contains both  $e_1, e_2$ .

**Proof.** Suppose there exists a vertex  $u \in J(a_1, a_2) - \{v\}$ . Then  $G[u, a_1, a_2, v]$  contains a 4-cycle. If  $d_G(u) = 2$ , then by the definition of  $G_0$ , one of the edges incident with  $u$  would be contracted in the process of getting  $G_0$  from  $G$ . Thus the 4-cycle in  $G[u, a_1, a_2, v]$  becomes a 3-cycle  $H_1$  in  $G_0$  with  $E(H_1) \cap E_{G_0}(v) \neq \emptyset$ . By Lemma 2.5(i),  $H_1$  is collapsible, and so the lemma holds in this case.

If  $d_G(u) \geq 3$ , by (1), there exists a vertex  $w \in N_G(u)$  with  $w \in N_G[a_1] \cup N_G[a_2]$ . If  $w \in \{a_1, a_2\}$ , then  $G[u, w]$  contains a cycle of length 2. Thus  $H_2 \subseteq G[u, v, a_1, a_2]$  is isomorphic to a  $C_4 + e$ . By Lemma 2.5(iii),  $H_2$  is collapsible, and so the lemma holds again.

If  $w \notin \{a_1, a_2\}$ , then  $H_3 \subseteq G[u, v, w, a_1, a_2]$  is isomorphic to a  $C_5 + e$ . By Lemma 2.5(iv),  $H_3$  is collapsible. Thus the lemma holds in this final case as well, which completes the proof.  $\square$

**Lemma 3.3.** Suppose that  $G$  is a DCT graph with  $\kappa(L(G)) \geq 3$ , and that  $L(G)$  does not have an independent vertex 3-cut. If for some vertex  $v \in V(G_0), D = \{e_1, e_2, e_3\} \subseteq E_{G_0}(v)$ , then  $G_0$  has a collapsible subgraph that contains at least two edges of  $D$ .

**Proof.** Denote the edge  $e_i = va_i$ , for  $i = 1, 2, 3$ . If  $G_0[v, a_1, a_2, a_3] \not\cong K_{1,3}$ , then  $G_0[v, a_1, a_2, a_3]$  contains a cycle  $H_1$  of length at most 3. By Lemma 2.5(i),  $H_1$  is collapsible, and so the lemma holds.

If  $G_0[v, a_1, a_2, a_3] \cong K_{1,3}$ , then by the definition of a core, we may assume that  $G[v, a_1, a_2, a_3] \cong K_{1,3}$ . By (3) and without loss of generality, we assume that  $J(a_1, a_2) \neq \emptyset$ . If  $J(a_1, a_2) - \{v\} \neq \emptyset$ , by Lemma 3.2,  $G_0$  has a collapsible subgraph contains both  $e_1, e_2$ , and so the lemma follows.

If  $J(a_1, a_2) - \{v\} = \emptyset$ , by  $J(a_1, a_2) \neq \emptyset$ , we have that  $v \in J(a_1, a_2)$ . By (1),  $a_3 \in N[v] \subseteq N[a_1] \cup N[a_2]$ . Then there is a 3-cycle  $H_2$  in  $G[v, a_1, a_2, a_3]$  that contains at least two elements in  $D$ . By Lemma 2.5(i),  $H_2$  is collapsible. By Lemma 3.1,  $D$  is intersecting a collapsible subgraph of  $G_0$ , and so the lemma holds also.  $\square$

**Lemma 3.4.** Suppose that  $G$  is a DCT graph with  $\kappa(L(G)) \geq 3$ , and that  $L(G)$  does not have an independent vertex 3-cut. Then any 3-edge-cut  $D$  of  $G_0$  is intersecting a collapsible subgraph of  $G_0$ .

**Proof.** By the definition of  $G_0$ , we may assume that  $D$  is an edge cut of  $G$ . If  $D$  is a peripheral 3-edge-cut of  $G_0$ , by Lemma 3.3,  $G_0$  has a collapsible subgraph that contains at least two edges of  $D$ , and so the lemma holds in this case. Hence we assume that  $D$  is an essential 3-edge-cut of  $G_0$ . By the definition of a core,  $D$  is also an essential 3-edge cut of  $G$ . Let  $D = \{e_1, e_2, e_3\}$ . Then  $D$  is a vertex 3-cut in  $L(G)$ . Since  $L(G)$  does not have an independent 3-cut, by the definition of a line graph, we may assume that  $e_1$  and  $e_2$  are adjacent edges in  $G$ . Thus for some vertices  $v, a_1, a_2 \in V(G), e_1 = va_1, e_2 = va_2$ . By Theorem 2.2(ii),  $d_{G_0}(v) \geq 3$ . Let  $a_3 \in N_{G_0}(v)$ . If  $G_0[v, a_1, a_2, a_3] \not\cong K_{1,3}$ , then there is a 3-cycle  $H_3$  in  $G_0[v, a_1, a_2, a_3]$ . By Lemma 2.5(i),  $H_3$  is collapsible, and so  $D$  is intersecting a collapsible subgraph of  $G$ . This completes the proof of the lemma.  $\square$

Hence we assume that  $G_0[v, a_1, a_2, a_3] \cong K_{1,3}$ . Then by Lemma 3.3,  $E_{G_0}(v)$  intersects a collapsible graph  $H$  which contains at least 2 edges of  $\{va_1, va_2, va_3\}$ . Thus the lemma holds.  $\square$

**Lemma 3.5.** Suppose that  $G$  is a P3D graph with  $\kappa(L(G)) \geq 3$ , and that  $L(G)$  does not have an independent 3 vertex cut. If  $D = \{e_1, e_2, e_3\}$  is a 3-edge-cut of  $G_0$ , then  $G_0$  has a nontrivial collapsible subgraph intersecting  $D$ .

**Proof.** If  $D$  is a peripheral 3-edge-cut of  $G$ , then we assume that  $e_1 = va_1, e_2 = va_2$  for some vertices  $\{v, a_1, a_2\} \subseteq V(G)$ . If  $D$  is an essential 3-edge-cut of  $G$ , since  $L(G)$  does not have an independent 3 vertex cut, we may also assume that  $e_1 = va_1, e_2 = va_2$ .

If  $a_1 = a_2$ , then  $G[v, a_1]$  contains a 2-cycle  $H_1$ . By Lemma 2.5(i),  $H_1$  is collapsible, and so  $D$  is intersecting a collapsible subgraph of  $G$ . By Lemma 3.1,  $D$  is intersecting a collapsible subgraph of  $G_0$  and the lemma holds in this case. Hence we assume  $a_1 \neq a_2$ .

By Theorem 2.2(ii),  $d_{G_0}(v) \geq 3$ . By the definition of a core,  $d_G(v) \geq 3$ . Let  $a_3 \in N(v)$ . If  $G_0[v, a_1, a_2, a_3] \not\cong K_{1,3}$ , then  $G_0[v, a_1, a_2, a_3]$  contains a cycle  $H_3$  of length at most 3 such that at least one of  $e_1, e_2$  must be in  $H_3$ . By Lemma 2.5(i),  $H_3$  is collapsible, and so the lemma obtains.

In the following, we assume that  $G_0[v, a_1, a_2, a_3] \cong K_{1,3}$ . By the definition of  $G_0$ , we have  $G[v, a_1, a_2, a_3] \cong K_{1,3}$ . Hence  $\text{dist}_G(a_i, a_j) = 2$  for any  $i, j$  with  $1 \leq i < j \leq 3$ . By (4),

$$J(a_i, a_j) \cup J'(a_i, a_j) \neq \emptyset, \quad \text{for any } i, j \text{ with } 1 \leq i < j \leq 3.$$

If  $J(a_i, a_j) - \{v\} \neq \emptyset$  for some  $i, j$ , then by Lemma 3.2,  $G_0$  has a collapsible subgraph  $L$  contains both  $va_i$  and  $va_j$ . It follows that  $L$  contains at least one edge in  $\{e_1, e_2\}$ , and so the lemma must hold.

If  $J(a_i, a_j) - \{v\} = \emptyset$  and  $v \in J(a_i, a_j)$  for some  $i, j$ , then by (1), the vertex  $a_k \in N(v) - \{a_i, a_j\}$  must be also adjacent to  $a_i$  or  $a_j$ . Therefore  $H_4 = G[v, a_i, a_k]$  or  $H_4 = G[v, a_j, a_k]$  must contain a 3-cycle. By Lemma 2.5(i),  $H_4$  is collapsible, and at least one of  $e_1, e_2$  must be in  $H_4$ . Therefore  $D$  is intersecting a collapsible subgraph of  $G$ . By Lemma 3.1,  $D$  is intersecting a collapsible subgraph of  $G_0$ , and so the lemma follows.

Hence we assume that  $J(a_i, a_j) = \emptyset$  for any  $i, j$  with  $1 \leq i < j \leq 3$ . Since  $G$  is a P3D graph, by (4), we conclude that  $J'(a_i, a_j) \neq \emptyset$  for any  $i, j$  with  $1 \leq i < j \leq 3$ .

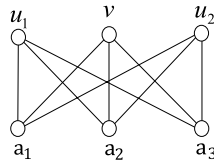


Fig. 1.  $K_{3,3}$ .

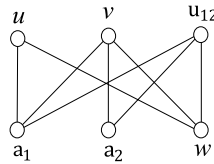


Fig. 2.  $K_{3,3} - e$ .

We complete the proof of this lemma by arguing in each of the following two cases.

Case 1.  $v \in J'(a_i, a_j)$  for any  $i, j$  with  $1 \leq i < j \leq 3$ .

Then  $v \in J'(a_1, a_2)$ . By (2) and since  $a_3 \in N(v) - (N[a_1] \cup N[a_2])$ , we must have  $N(a_1) \cup N(a_2) \cup N(v) - \{a_1, a_2, a_3\} \subseteq N_G(a_3)$ . Since  $a_i \in V(G_0)$ , by Theorem 2.2(ii),  $d_{G_0}(a_i) \geq 3$ . Hence  $d_G(a_i) \geq 2$  for  $i \in \{1, 2\}$ , and so there exists a vertex  $u_1 \in N(a_1) - \{v\}$  such that  $u_1 a_3 \in E(G)$ . (See Fig. 1). Similarly, as  $v \in J'(a_1, a_3)$ , we must also have  $u_1 a_2 \in E(G)$ . Hence  $\{a_1, a_2, a_3\} \subseteq N_G(u_1)$ .

If  $d_G(a_1) = 2$ , then the 4-cycle  $va_1u_1a_2v$  in  $G$  will be contracted to a 3-cycle in  $G_0$ . By Lemma 2.5(i), every 3-cycle is collapsible. Hence the lemma must hold.

Therefore, we assume that  $d_G(a_1) > 2$ , and there exists  $u_2 \in N(a_1) - \{v, u_1\}$ . Arguing as above with  $u_1$  replaced by  $u_2$ , we conclude that  $\{a_1, a_2, a_3\} \subseteq N_G(u_2)$ . It follows that  $G[v, u_1, u_2, a_1, a_2, a_3]$  contains a  $K_{3,3}$  as a spanning subgraph (see Fig. 1). By Lemma 2.5(v),  $K_{3,3}$  is collapsible. Thus every edge of  $D$  is in a collapsible subgraph of  $G$ . It follows by Theorem 2.1(iii) and by the definition of a core that  $D$  lies in a collapsible of  $G_0$ . This completes the proof for Case 1.

Case 2.  $v \notin J'_G(a_i, a_j)$  for some  $i, j$  with  $1 \leq i < j \leq 3$ .

Without loss of generality, we assume that  $v \notin J'(a_1, a_2)$ . By (2),  $G$  has a vertex  $u_{12} \in J'(a_1, a_2)$ . If  $d_G(u_{12}) = 2$ , then the 4-cycle  $va_1u_{12}a_2v$  will be contracted to a 3-cycle which is a collapsible graph in  $G_0$ , and so the lemma holds in this case.

Hence we assume that  $d_G(u_{12}) \geq 3$ . Then there exists a vertex  $w \in N(u_{12})$ . If  $w \notin N(u_{12}) - N[a_1] \cup N[a_2]$ , then  $G[a_i, u_{12}]$  or  $G[a_i, w, u_{12}]$  contains a cycle of length 2 or 3 for some  $i \in \{1, 2\}$ . It follows that  $G[v, a_1, a_2, u_{12}]$  contains a subgraph  $H_6$  isomorphic to  $C_4 + e$  or a  $C_5 + e$ . By Lemma 2.5,  $H_6$  is collapsible, and so  $D$  is intersecting a collapsible subgraph of  $G$ . By Lemma 3.1,  $D$  intersects a collapsible subgraph of  $G_0$ , and so lemma holds. Hence we assume  $w \in N(u_{12}) - N[a_1] \cup N[a_2]$ .

By Claim 3 and by (2),  $v \in N(a_1) \cup N(a_2) \cup N(u_{12}) - \{a_1, a_2, w\} \subseteq N(w)$ . Thus  $v \in N_G(w)$ . By Theorem 2.2(ii),  $d_{G_0}(a_1) \geq 3$ . Hence  $d_G(a_1) \geq 2$ . If  $d_G(a_1) = 2$ , then the 4-cycle  $va_1u_{12}a_2v$  will be contracted to a 3-cycle in  $G_0$ , and so the lemma holds in this case. Therefore, we assume that  $d_G(a_1) \geq 3$ . Let  $u \in N(a_1)$ . If  $u \in \{v, u_{12}\}$ , then  $G[v, a_1, a_2, u_{12}]$  has a subgraph isomorphic to  $C_4 + e$ . By Lemma 2.5(iii) and by Lemma 3.1,  $D$  intersects a collapsible subgraph of  $G_0$ , and so the lemma holds.

If  $u \notin \{v, u_{12}\}$ , by (2),  $u \in N(a_1) \cup N(a_2) \cup N(u_{12}) - \{a_1, a_2, w\} \subseteq N(w)$ . (see Fig. 2). Then  $G[u, v, w, a_1, a_2, u_{12}]$  contains a spanning subgraph isomorphic to  $K_{3,3} - e$ . By Lemma 2.5(v) and by Lemma 3.1,  $D$  intersects a collapsible subgraph of  $G_0$ . Thus the lemma holds in any cases. This completes the proof of Lemma 3.5.  $\square$

**Proof of Theorem 1.1.** Let  $G$  be a connected DCT graph or P3D graph with  $\kappa(L(G)) \geq 3$ . If  $\kappa'(G_0) \geq 4$ , then by Theorem 2.1,  $G_0$  is supereulerian. By Theorem 2.2(iii),  $L(G)$  is hamiltonian.

Since  $\kappa(L(G)) \geq 3$  and by Theorem 2.2,  $\kappa'(G_0) \geq 3$ . By Theorem 2.3, it suffices to show that every 3-edge-cut  $D$  of  $G_0$  lies in a collapsible subgraph of  $G_0$ . But this follows from Lemmas 3.4 and 3.5 directly.  $\square$

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