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Generalized Legendre Polynomials

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Orthogonal polynomials have been extensively investigated. In this paper we construct a sequence of orthogonal Dirichlet polynomials which generalize Legendre polynomials. We show that our new system satisfies a generalized Rodrigues formula. We also extend a well known result concerning the distribution of the zeros of Legendre polynomials.

1. Introduction

The ordinary Legendre polynomials L_n are formed by orthogonalising the sequence $\{x^0, x^1, ..., x^n\}$ with respect to the usual inner product on the interval $[-1, 1]$. The sequence $\tilde{L}_n(x) = L_n(1-2x)$ gives us a corresponding sequence of polynomials which are orthogonal on $[0, 1]$.

In this paper we construct generalized Legendre polynomials γ_n by orthogonalizing the sequence $\{x^{\lambda_1}, x^{\lambda_2}, ..., x^{\lambda_n}\}$ with respect to the usual inner product on the interval $[0, 1]$; i.e., for $f, g \in L^2(0, 1)$ we define

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx.$$

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We assume that the λ_n's are distinct and that $\lambda_n > -\frac{1}{2}$, to ensure that $x^{\lambda_n} \in L^2(0, 1)$.

In the sections that follow we will use a Gram determinant to orthogonalize $\{x^{\lambda_1}, x^{\lambda_2}, ..., x^{\lambda_n}\}$, find a generalized Rodrigues formula for the system, and give properties of the zeros of the system.

2. Orthogonalization of $\{x^{\lambda_1}, x^{\lambda_2}, ..., x^{\lambda_n}\}$

We will denote by $\{\gamma_n\}$ a set of orthogonal Dirichlet polynomials generated by evaluating the Gram determinant associated with the sequence of functions $\{x^{\lambda_1}, x^{\lambda_2}, ..., x^{\lambda_n}\}$ and then simplifying. Of course any orthogonal system is uniquely determined only up to constant factors and so in constructing our sequence $\{\gamma_n\}$ we shall remove constants whenever possible in order to simplify our formulas.

Before we give a formula for γ_n we first create $\tilde{\gamma}_n$, an unsimplified orthogonal system. We take $\tilde{\gamma}_1(x) = x^{\lambda_1}$. For $n \geqslant 2$ the corresponding Gram determinant is

$$\tilde{\gamma}_n(x) = \begin{vmatrix} \langle x^{\lambda_1}, x^{\lambda_1} \rangle & \cdots & \langle x^{\lambda_1}, x^{\lambda_n} \rangle \\ \vdots & & \\ \langle x^{\lambda_{n-1}}, x^{\lambda_1} \rangle & \cdots & \langle x^{\lambda_{n-1}}, x^{\lambda_n} \rangle \\ x^{\lambda_1} & \cdots & x^{\lambda_n} \end{vmatrix}$$

$$= \sum_{s=1}^{n} D_s x^{\lambda_s}$$

where

$$D_s = (-1)^{n+s} \begin{vmatrix} \dfrac{1}{\lambda_1 + \lambda_1 + 1} & \cdots & \dfrac{\widehat{1}}{\lambda_1 + \lambda_s + 1} & \cdots & \dfrac{1}{\lambda_1 + \lambda_n + 1} \\ \vdots & & & & \\ \dfrac{1}{\lambda_{n-1} + \lambda_1 + 1} & \cdots & \dfrac{1}{\lambda_{n-1} + \lambda_s + 1} & \cdots & \dfrac{1}{\lambda_{n-1} + \lambda_n + 1} \end{vmatrix}$$

$$= (-1)^{n+s} \left| \frac{1}{\alpha_i + \beta_j} \right|,$$

where

$$\alpha_i = \lambda_i + 1/2 \qquad 1 \leqslant i \leqslant n-1$$

$$\beta_j = \begin{cases} \lambda_j + 1/2 & 1 \leqslant j \leqslant s-1 \\ \lambda_{j+1} + 1/2 & s \leqslant j \leqslant n-1, \end{cases}$$

and the hat indicates that the column is omitted.

From (Achieser [1], p. 20) we get that

$$D_s = \frac{(-1)^{n+s} \prod_{1 \leqslant k < j \leqslant n-1} (\alpha_j - \alpha_k)(\beta_j - \beta_k)}{\prod_{i,j=1}^{n-1} (\alpha_i + \beta_j)}$$

$$= \frac{(-1)^{n+s} A \prod_{1 \leqslant k < j \leqslant n-1} (\beta_j - \beta_k)}{\prod_{i,j=1}^{n-1} (\alpha_i + \beta_j)} \qquad \text{(where } A \text{ is independent of } s\text{)}$$

$$= \frac{(-1)^{n+s} A \prod_{1 \leqslant k < j \leqslant n} (\lambda_j - \lambda_k)}{\prod_{i,j=1}^{n-1} (\alpha_i + \beta_j) \prod_{1 \leqslant i < s} (\lambda_s - \lambda_i) \prod_{s < i \leqslant n} (\lambda_i - \lambda_s)}$$

$$= \frac{(-1)^{n+s} AB}{\prod_{i,j=1}^{n-1} (\alpha_i + \beta_j)(-1)^{s-1} \prod_{i=1, i \neq s}^{n} (\lambda_i - \lambda_s)}$$

(where B is independent of s)

$$= \frac{(-1)^{n+1} AB \prod_{i=1}^{n-1} (\lambda_i + \lambda_s + 1)}{\prod_{i=1, i \neq s}^{n} (\lambda_i - \lambda_s) \prod_{i=1}^{n-1} \prod_{j=1}^{n} (\lambda_i + \lambda_j + 1)}$$

$$= \frac{M \prod_{i=1}^{n-1} (\lambda_i + \lambda_s + 1)}{\prod_{i=1, i \neq s}^{n} (\lambda_i - \lambda_s)} \qquad \text{(where } M \text{ is independent of } s\text{).}$$

Hence we are led to define the nth Generalized Legendre Polynomial by

$$\gamma_n(x) = \sum_{s=1}^{n} \frac{\prod_{i=1}^{n-1} (\lambda_i + \lambda_s + 1)}{\prod_{i=1, i \neq s}^{n} (\lambda_i - \lambda_s)} x^{\lambda_s} \tag{1}$$

for $n = 1, 2, \ldots$.

3. Generalized Rodrigues Formula

The Legendre polynomials in standard form satisfy Rodrigues formula (see Szegő [2], page 66):

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} ((x^2 - 1)^n) \qquad n = 0, 1, 2, \ldots.$$

Hence using (0, 1) as the interval of integration we obtain a set of orthogonal polynomials given by

$$\tilde{L}_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^n(1-x)^n), \tag{2}$$

where

$$\tilde{L}_n(x) = L_n(1-2x), \qquad n = 0, 1, 2, \ldots .$$

In order to give a generalized Rodrigues formula, we shall construct a Dirichlet polynomial

$$p_n(x) = \sum_{k=1}^{n} a_k x^{\lambda_k}$$

such that

$$p_n(1) = 0,\ p_n'(1) = 0, \ldots, p_n^{(n-2)}(1) = 0 \qquad n = 2, 3, \ldots .$$

We remark that in the case $\lambda_n = n-1$, we may take

$$p_n(x) = \frac{1}{(n-1)!} (1-x)^{n-1}.$$

The general p_n which we construct will play the same role in our generalized Rodrigues formula as the function $(1/(n-1)!)\ (1-x)^{n-1}$ does in the ordinary Rodrigues formula for $\tilde{L}_{n-1}$ in (2).

We start by defining a function $\tilde{p}_n(x)$. Let

$$\tilde{p}_n(x) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & & & \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \cdots & \lambda_n^{n-2} \\ x^{\lambda_1} & x^{\lambda_2} & \cdots & x^{\lambda_n} \end{vmatrix}$$

Note that the coefficient of x^{λ_n} is non-zero, since it is a Vandermonde determinant. We get that

$$\tilde{p}_n(x) = \sum_{s=1}^{n} E_s x^{\lambda_s},$$

where

$$E_s = (-1)^{n+s} \begin{vmatrix} 1 & 1 & \cdots & \widehat{1} & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_s & \cdots & \lambda_n \\ \vdots & & & & & \\ \lambda_1^{n-2} & \lambda_2^{n-2} & \cdots & \lambda_s^{n-2} & \cdots & \lambda_n^{n-2} \end{vmatrix}$$

$$= (-1)^{n+s} \prod_{1 \leqslant j < k \leqslant n-1} (\mu_k - \mu_j) \text{ where } \mu_p = \begin{cases} \lambda_p & 1 \leqslant p < s \\ \lambda_{p+1} & s \leqslant p \leqslant n-1 \end{cases}$$

$$= \frac{(-1)^{n+s} \prod_{1 \leqslant j < k \leqslant n} (\lambda_k - \lambda_j)}{\prod_{1 \leqslant i < s} (\lambda_s - \lambda_i) \prod_{s < i \leqslant n} (\lambda_i - \lambda_s)}$$

$$= \frac{A}{\prod_{i=1, i \neq s}^{n} (\lambda_i - \lambda_s)},$$

where A is independent of s.

We now define $p_n(x)$ *by*

$$p_n(x) = \sum_{s=1}^{n} \frac{x^{\lambda_s}}{\prod_{i=1, i \neq s}^{n} (\lambda_i - \lambda_s)} \qquad n = 1, 2, ..., \tag{3}$$

and p_n has the required properties.

We can now give a generalized Rodrigues formula for Legendre polynomials:

$$\gamma_n(x) = \sum_{s=1}^{n} \frac{\prod_{i=1}^{n-1} (\lambda_i + \lambda_s + 1)\, x^{\lambda_s}}{\prod_{i=1, i \neq s}^{n} (\lambda_i - \lambda_s)}$$

$$= x^{-\lambda_{n-1}} \frac{d}{dx} x^{\lambda_{n-1}+1} \cdots x^{-\lambda_2} \frac{d}{dx} x^{\lambda_2+1} x^{-\lambda_1} \frac{d}{dx} x^{\lambda_1+1} p_n(x)$$

$$= \left(\prod_{k=1}^{n-1} D_{\lambda_k} \right) (p_n(x)), \tag{4}$$

where $D_\lambda(f) = x^{-\lambda}(d/dx)(x^{\lambda+1} f)$. Equation (4) is our generalized Rodrigues formula. We remark that in the case $\lambda_n = n-1$ our formula is identical with (2). To see this, note that in this case Eq. (3) gives $p_n(x) = (1/(n-1)!)(1-x)^{n-1}$ and

$$\left(\prod_{k=1}^{n-1} D_{\lambda_k} \right) p_n(x) = \left(\prod_{k=1}^{n-1} D_{\lambda_{n-1-k}} \right) p_n(x) = \frac{d^{n-1}}{dx^{n-1}} x^{n-1}(1-x)^{n-1},$$

since $D_\lambda(x^r) = (r + \lambda + 1)\, x^r$ so that $D_\alpha D_\beta = D_\beta D_\alpha$ on p_n.

In the proof that follows we use our Rodrigues formula to present an alternative verification that $\{\gamma_n\}$ forms an orthogonal system. It is of interest that even in the ordinary case where $\lambda_n = n-1$, this proof differs from that found in the standard texts where integration by parts is used.

We have

$$\gamma_n = \left(\prod_{k=1}^{n-1} D_{\lambda_k}\right) p_n$$
$$= D_{\lambda_s}\left(\left(\prod_{k=1,\, k\neq s}^{n-1} D_{\lambda_k}\right) p_n\right).$$

Since $p_n(1) = p_n'(1) \cdots = p_n^{(n-2)}(1) = 0$ we get that

$$\left(\prod_{k=1,\, k\neq s}^{n-1} D_{\lambda_k}\right) p_n \Bigg|_{x=1} = 0.$$

If $s < n$ and $n \geqslant 2$ then

$$\int_0^1 x^{\lambda_s}\gamma_n(x)\, dx = \int_0^1 x^{\lambda_s}\left\{D_{\lambda_s}\left(\prod_{k=1,\, k\neq s}^{n-1} D_{\lambda_k}\right) p_n(x)\right\} dx$$
$$= \int_0^1 \frac{d}{dx}\left\{\left(x^{\lambda_s+1}\prod_{k=1,\, k\neq s}^{n-1} D_{\lambda_k}\right) p_n(x)\right\} dx$$
$$= x^{\lambda_s+1}\left(\prod_{k=1,\, k\neq s}^{n-1} D_{\lambda_k}\right) p_n \Bigg|_0^1$$
$$= 0.$$

Since γ_n is a linear combination of $x^{\lambda_1}, x^{\lambda_2}, ..., x^{\lambda_n}$, it follows that $\{\gamma_n\}$ is an orthogonal system.

4. Zeros of $\{\gamma_n\}$

In this section we obtain general results concerning the zeros of generalized polynomials, i.e., polynomials formed using x^{λ_n} instead of x^n, and then show that $\{\gamma_n\}$, the generalized Legendre polynomials, have a property of zeros found in ordinary Legendre polynomials.

THEOREM 4.1. *Let* $P(x) = \sum_{k=1}^{n} a_k x^{\lambda_k}$, *where* $\lambda_k \in \mathscr{R}$. *If* $P \not\equiv 0$, *then* P *has at most* $n-1$ *zeros in* $\mathscr{R}^+ = (0, \infty)$.

Proof. If $n=1$ then $P(x)=a_1 x^{\lambda_1}$ which has no zeros in $\mathscr{R}^+$. Assume the theorem is true for n; i.e., any function of the form

$$p(x)=a_1 x^{\lambda_1}+a_2 x^{\lambda_2}+\cdots+a_n x^{\lambda_n}$$

has at most $n-1$ zeros in $\mathscr{R}^+$. Let

$$q(x)=a_1 x^{\lambda_1}+a_2 x^{\lambda_2}+\cdots+a_n x^{\lambda_n}+a_{n+1} x^{\lambda_{n+1}}.$$

Then

$$\begin{aligned} q(x)&=x^{\lambda_1}(a_1+a_2 x^{\lambda_2-\lambda_1}+\cdots+a_{n+1} x^{\lambda_{n+1}-\lambda_1}) \\ &=x^{\lambda_1} q_1(x), \end{aligned}$$

where $q(x)$ and $q_1(x)$ have the same zeros in $\mathscr{R}^+$. Call their number s (taking account of multiplicity). By Rolle's theorem, $q_1'(x)$ has at least $s-1$ zeros. By inductive hypothesis, $s-1\leqslant n-1$ so that $s\leqslant n$. Hence $q(x)$ has at most n zeros in $\mathscr{R}^+$. ▌

COROLLARY 4.2. *Assume $\{\lambda_k\}$ are distinct. Let $w_k\in\mathscr{R}$, $1\leqslant k\leqslant n$ and $x_1<x_2\cdots<x_n\in\mathscr{R}^+$. Then $\exists a_1, a_2, ..., a_n\in\mathscr{R}$ such that $p(x_k)=w_k$ $k=1, 2, ..., n$ where $p(x)=\sum_{k=1}^n a_k x^{\lambda_k}$.*

Proof. Consider the linear transformation $T\colon \mathscr{R}^n\to\mathscr{R}^n$ defined by

$$T(a_1, a_2, ..., a_n)=\begin{bmatrix} x_1^{\lambda_1} & x_1^{\lambda_2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1} & x_2^{\lambda_2} & \cdots & x_2^{\lambda_n} \\ \vdots & & & \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \cdots & x_n^{\lambda_n} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

By the previous theorem, if $T(a_1, a_2, ..., a_n)=(0, 0, ..., 0)$, then $a_1=a_2=\cdots=a_n=0$ (since $\sum_{k=1}^n a_k x^{\lambda_k}$ can have at most $n-1$ zeros if it is non-trivial). Therefore T is one-to-one and hence onto; i.e., given $w=(w_1, ..., w_2, ..., w_n)\in\mathscr{R}^n$, $\exists a=(a_1, a_2, ..., a_n)$ such that $T(a)=w$. Taking $p(x)=\sum_{k=1}^n a_k x^{\lambda_k}$ gives the result. ▌

COROLLARY 4.3. *Let $x_1<x_2\cdots<x_{n-1}\in\mathscr{R}^+$. Then there is a non-trivial function $p(x)=\sum_{k=1}^n a_k x^{\lambda_k}$ such that $p(x_i)=0$ for $i=1, 2, ..., n-1$ and $p(x)$ has no other zeros in $\mathscr{R}^+$.*

Proof. Choose $x_n>x_{n-1}$. By 4.2 there is a function $p(x)=\sum_{k=1}^n a_k x^{\lambda_k}$ such that $p(x_i)=0$ for $1\leqslant i\leqslant n-1$ and $p(x_n)=1$. Since $p(x)$ has at most $n-1$ zeros in $\mathscr{R}^+$, it follows that $p(x)$ has no other zeros in $\mathscr{R}^+$. ▌

This brings us to our main result. We are now in a position to show that the generalized Legendre polynomial γ_n has exactly $n-1$ simple zeros in $(0, 1)$ and no other zeros in $\mathscr{R}^+$. In fact we prove a stronger result.

Suppose w is a positive measurable function on the interval (a, b) where $0 \leqslant a < b$. We now define

$$\langle f, g \rangle_w = \int_a^b f(x)\, g(x)\, w(x)\, dx.$$

Let the sequence $\{\phi_n\}$ be an orthogonalization of $\{x^{\lambda_1}, x^{\lambda_2}, ..., x^{\lambda_n}\}$ with respect to this inner product.

THEOREM 4.4. *The function ϕ_n as defined above has exactly $n-1$ simple zeros in (a, b) and no other zeros in $\mathscr{R}^+$.*

Proof. The result is clearly true for $n=1$. Assume $n \geqslant 2$. By definition, $\langle \phi_n, x^{\lambda_s} \rangle_w = 0$ for $1 \leqslant s < n$. Let $x_1, x_2, ..., x_k$ be the points in (a, b) where ϕ_n changes sign. Suppose (for the sake of contradiction) that $k < n-1$. Then by Corollary 4.3 there is a non-trivial function $q(x) = \sum_{s=1}^{k+1} a_s x^{\lambda_s}$ such that $q(x_i) = 0$ for $1 \leqslant i \leqslant k$. By Theorem 4.1, $q(x)$ has no other zeros in $\mathscr{R}^+$. Hence $q(x)$ changes sign at $x_1, x_2, ..., x_k$. Consequently, $\phi_n q w$ has constant sign in (a, b) and $\phi_n q w \not\equiv 0$. Therefore

$$\int_a^b \phi_n(x)\, q(x)\, w(x)\, dx \neq 0.$$

However, $k+1 < n$ so that $\langle \phi_n, q \rangle_w = 0$. This gives a contradiction. Therefore $k = n-1$. Hence $x_1, x_2, ..., x_{n-1}$ are simple zeros of ϕ_n and ϕ_n has no other zeros in $\mathscr{R}^+$. ∎

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