



On the numerical radius of operators in Lebesgue spaces [☆]

Miguel Martín ^{a,*}, Javier Merí ^a, Mikhail Popov ^{b,1}

^a *Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain*

^b *Department of Mathematics, Chernivtsi National University, str. Kotsjubyn'skogo 2, Chernivtsi, 58012, Ukraine*

Received 4 November 2010; accepted 13 March 2011

Communicated by G. Schechtman

Abstract

We show that the absolute numerical index of the space $L_p(\mu)$ is $p^{-\frac{1}{p}}q^{-\frac{1}{q}}$ (where $\frac{1}{p} + \frac{1}{q} = 1$). In other words, we prove that

$$\sup \left\{ \int |x|^{p-1} |Tx| d\mu : x \in L_p(\mu), \|x\|_p = 1 \right\} \geq p^{-\frac{1}{p}} q^{-\frac{1}{q}} \|T\|$$

for every $T \in \mathcal{L}(L_p(\mu))$ and that this inequality is the best possible when the dimension of $L_p(\mu)$ is greater than one. We also give lower bounds for the best constant of equivalence between the numerical radius and the operator norm in $L_p(\mu)$ for atomless μ when restricting to rank-one operators or narrow operators.

© 2011 Elsevier Inc. All rights reserved.

Keywords: Banach space; Numerical index; Absolute numerical radius; L_p -space; Narrow operator

[☆] First and second authors partially supported by Spanish MICINN and FEDER project no. MTM2009-07498 and Junta de Andalucía and FEDER grants G09-FQM-185 and P09-FQM-4911. Third author supported by Junta de Andalucía and FEDER grant P06-FQM-01438 and by Ukr. Derzh. Tema N 0103Y001103.

* Corresponding author.

E-mail addresses: mmartins@ugr.es (M. Martín), jmeri@ugr.es (J. Merí), misham.popov@gmail.com, popovm@muohio.edu (M. Popov).

URLs: <http://www.ugr.es/local/mmartins> (M. Martín), <http://www.ugr.es/local/jmeri> (J. Merí),

<http://testuvannya.com.ua/mpopov> (M. Popov).

¹ Current address: Department of Mathematics, Miami University, Oxford, OH 45056, USA.

1. Introduction and preliminaries

Let X be a real or complex Banach space. Following the standard notation, by B_X, S_X, X^* and $\mathcal{L}(X)$ we denote the closed unit ball, the unit sphere, the dual space, and the space of all bounded linear operators on X respectively. We write \mathbb{T} for the unit sphere of the base field \mathbb{R} or \mathbb{C} . The *numerical radius* of an operator $T \in \mathcal{L}(X)$ is a semi-norm defined as

$$v(T) = \sup\{|x^*(Tx)|: x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\},$$

which is obviously smaller or equal than the operator norm. The *numerical index* of the space X is the constant

$$n(X) = \inf\{v(T): T \in \mathcal{L}(X), \|T\| = 1\},$$

equivalently, $n(X)$ is the maximum of those $k \geq 0$ such that $k\|T\| \leq v(T)$ for every $T \in \mathcal{L}(X)$. This notion was introduced and studied in the 1970 paper [4], see also the monographs [2,3] and the survey paper [9] for background. Obviously, $0 \leq n(X) \leq 1$, $n(X) > 0$ means that the numerical radius is a norm on $\mathcal{L}(X)$ equivalent to the operator norm and $n(X) = 1$ if and only if numerical radius and operator norm coincide. It is also not hard to see that $n(X^*) \leq n(X)$, being the reversed inequality false in general (see [9, §2] for a detailed account). There are lots of spaces with numerical index 1 (among classical ones, for instance, $L_1(\mu)$ and $C(K)$), and some attractive open problems on them [9]. It is interesting to remark that the numerical index behaves differently in the real and in the complex cases. So, for every complex Banach space one has that $n(X) \geq 1/e$ (and the inequality is the best possible), nevertheless, $n(X) = 0$ for some real Banach spaces X as ℓ_2 or, more in general, for every Hilbert space of dimension greater than 1.

The number of Banach spaces whose numerical index is known is small (see [9, §1] for a recent account) and, therefore, there are many interesting open problems consisting in calculating, or at least estimating, the numerical index of concrete Banach spaces. Among classical spaces, one of the most intriguing open problems is to calculate $n(L_p(\mu))$ for $1 < p < \infty, p \neq 2$. Let us fix the notation and terminology on L_p spaces. Let (Ω, Σ, μ) be any measure space and $1 < p < \infty$. We write $L_p(\mu)$ for the real or complex Banach space of (equivalent classes of) measurable scalar functions x defined on Ω such that

$$\|x\|_p = \left(\int_{\Omega} |x|^p d\mu \right)^{\frac{1}{p}} < \infty.$$

We use the notation ℓ_p^m for the m -dimensional L_p -space. We write $q = p/(p - 1)$ for the conjugate exponent to p . For any $x \in L_p(\mu)$, we denote

$$x^\# = \begin{cases} |x|^{p-1} \text{sign}(x) & \text{in the real case,} \\ |x|^{p-1} \text{sign}(\bar{x}) & \text{in the complex case,} \end{cases}$$

which is the unique element in $L_q(\mu) \equiv L_p(\mu)^*$ such that

$$\|x\|_p^p = \|x^\#\|_q^q \quad \text{and} \quad \int_{\Omega} x x^\# d\mu = \|x\|_p \|x^\#\|_q = \|x\|_p^p.$$

Observe that, with this notation, one has

$$v(T) = \sup \left\{ \left| \int_{\Omega} x^{\#} T x \, d\mu \right| : x \in S_{L_p(\mu)} \right\}$$

for every $T \in \mathcal{L}(L_p(\mu))$. Finally, we consider the constants

$$M_p = \max_{t \in [0,1]} \frac{|t^{p-1} - t|}{1 + t^p} = \max_{t \geq 1} \frac{|t^{p-1} - t|}{1 + t^p} \tag{1}$$

(which is the numerical radius of the operator $T(x, y) = (-y, x)$ defined on the real space ℓ_p^2 , see [11, Lemma 2] for instance) and

$$\kappa_p = \max_{\tau > 0} \frac{\tau^{p-1}}{1 + \tau^p} = \max_{\lambda \in [0,1]} \lambda^{\frac{1}{q}} (1 - \lambda)^{\frac{1}{p}} = \frac{1}{p^{1/p} q^{1/q}} \tag{2}$$

(which is the numerical radius of the operator $T(x, y) = (y, 0)$ defined on the real or complex space ℓ_p^2 , see [11, Lemma 2] for instance).

It has been proved recently that, fixed p , all infinite-dimensional $L_p(\mu)$ spaces have the same numerical index [5–7] (see also [13] for a different approach) and that $n(L_p(\mu)) > 0$ for $p \neq 2$ in the real case [12]. On the way to state the last result, the authors of [12] introduced the so-called absolute numerical radius of an operator on an $L_p(\mu)$ -space as follows. Given a measure space (Ω, Σ, μ) , $1 < p < \infty$ and $T \in \mathcal{L}(L_p(\mu))$, the *absolute numerical radius* of T is the number

$$\begin{aligned} |v|(T) &= \sup \{ |x^*| (|Tx|) \, d\mu : x \in S_{L_p(\mu)}, x^* \in S_{L_p(\mu)^*}, x^*(x) = 1 \} \\ &= \sup \left\{ \int_{\Omega} |x^{\#} T x| \, d\mu : x \in S_{L_p(\mu)} \right\} \\ &= \sup \left\{ \int_{\Omega} |x|^{p-1} |Tx| \, d\mu : x \in S_{L_p(\mu)} \right\}. \end{aligned}$$

It is clear that $|v|$ is a semi-norm on $\mathcal{L}(L_p(\mu))$ satisfying

$$v(T) \leq |v|(T) \leq \|T\| \quad (T \in \mathcal{L}(L_p(\mu))).$$

In [12] it is shown that $n(L_p(\mu))$ is positive by proving that both inequalities above can be reversed up to a positive constant. Namely, it is shown that

$$\frac{1}{2e} \|T\| \leq |v|(T) \quad \text{and} \quad \frac{M_p}{6} |v|(T) \leq v(T)$$

for every $T \in \mathcal{L}(L_p(\mu))$, giving $n(L_p(\mu)) \geq \frac{M_p}{12e} > 0$.

We introduce the definition of the *absolute numerical index* of $L_p(\mu)$ as the number

$$\begin{aligned} |n|(L_p(\mu)) &= \inf \{ |v|(T) : T \in \mathcal{L}(L_p(\mu)), \|T\| = 1 \} \\ &= \max \{ k \geq 0 : k \|T\| \leq |v|(T) \, \forall T \in \mathcal{L}(L_p(\mu)) \} \end{aligned}$$

and the aforementioned result of [12] just says that $|n|(L_p(\mu)) \geq \frac{1}{2e}$. Our first goal in this paper is to calculate the exact value of $|n|(L_p(\mu))$, namely, $|n|(L_p(\mu)) = \kappa_p$ (if the dimension of $L_p(\mu)$ is greater than one) in both the real and the complex cases. In other words, we will prove that,

$$\sup \left\{ \int |x|^{p-1} |Tx| d\mu : x \in L_p(\mu), \|x\|_p = 1 \right\} \geq \kappa_p \|T\|$$

for every $T \in \mathcal{L}(L_p(\mu))$ and that this inequality is the best possible when the dimension of $L_p(\mu)$ is greater than one. As a corollary, we get an improvement of the estimation of $n(L_p(\mu))$ obtained in [12]. Namely, in the real case, we get

$$n(L_p(\mu)) \geq \frac{\kappa_p M_p}{6}.$$

In other words, in the real case,

$$\sup \left\{ \left| \int |x|^{p-1} \text{sign}(x)Tx d\mu \right| : x \in L_p(\mu), \|x\|_p = 1 \right\} \geq \frac{\kappa_p M_p}{6} \|T\|$$

for every $T \in \mathcal{L}(L_p(\mu))$.

Next, we study the numerical radius of rank-one operators on $L_p(\mu)$. We define the *rank-one numerical index* of an arbitrary Banach space X as the number

$$\begin{aligned} n_1(X) &= \inf \{ v(T) : T \in \mathcal{L}(X), \|T\| = 1, T \text{ rank-one} \} \\ &= \max \{ k \geq 0 : k \|T\| \leq v(T) \forall T \in \mathcal{L}(X) \text{ rank-one} \}. \end{aligned}$$

Our results state that for every atomless measure μ ,

$$n_1(L_p(\mu)) \geq \kappa_p^2$$

in both the real and the complex cases. This result is not sharp for values of p close to 2 as, for instance, $n_1(L_2(\mu)) = \frac{1}{2}$ if the dimension of $L_2(\mu)$ is greater than 1. On the other hand, the estimation for $n_1(L_p(\mu))$ tends to 1 as $p \rightarrow 1$ or $p \rightarrow \infty$.

Finally, the last part of the paper is devoted to study numerical radius of the so-called narrow operators on $L_p(\mu)$ when the measure μ is atomless and finite (a class of operators containing compact operators, see Section 4 for the definition and background). Defining the *narrow numerical index* of $L_p(\mu)$ as

$$\begin{aligned} n_{\text{nar}}(L_p(\mu)) &= \inf \{ v(T) : T \in \mathcal{L}(L_p(\mu)), \|T\| = 1, T \text{ narrow} \} \\ &= \max \{ k \geq 0 : k \|T\| \leq v(T) \forall T \in \mathcal{L}(L_p(\mu)) \text{ narrow} \}, \end{aligned}$$

we prove that

$$\begin{aligned} n_{\text{nar}}(L_p(\mu)) &\geq \kappa_p^2 \quad \text{in the complex case,} \\ n_{\text{nar}}(L_p(\mu)) &\geq \max_{\tau > 0} \frac{\kappa_p \tau^{p-1} - \tau}{1 + \tau^p} \quad \text{in the real case.} \end{aligned}$$

Notice that the inequality for the real case gives a positive estimate for $1 < p < \infty$ ($p \neq 2$) which tends to 1 as $p \rightarrow 1$ or $p \rightarrow \infty$.

The outline of the paper is as follows. Section 2 is devoted to show that $|n|(L_p(\mu)) = \kappa_p$. The results on rank-one operators appear in Section 3 and the results on narrow operators are contained in Section 4.

We recall some lattice notation which we will use in the paper. We refer the reader to [1] for abundant information on lattices and positive operators. Let E be a Banach lattice. For any subset $F \subseteq E$ we write $F^+ = \{x \in F: x \geq 0\}$. For two elements $x, y \in E$, by $x \vee y$ (resp. $x \wedge y$) we denote the least upper bound (resp. greatest lower bound) in E of the two-point set $\{x, y\}$, if it exists. A linear operator $T : E \rightarrow E$ is called *positive* provided $T(E^+) \subseteq E^+$, or, in other words, if it sends positive elements to positive elements. An element $y \in E$ is called a *component* of $x \in E$ if $|y| \wedge |x - y| = 0$. In this case we write $y \sqsubseteq x$. Let $z \in E$. An element $x \in E$ is called a *z-step function* if $x = \sum_{k=1}^m a_k z_k$ for some components (z_k) of z .

Let (Ω, Σ, μ) be a measure space. On the real space $L_0(\mu)$ of all (equivalence classes of) Σ -measurable functions, we consider the ordering $x \leq y$ if and only if $x(t) \leq y(t)$ for almost all $t \in \Omega$. For two functions $x, y \in L_0$, $x \vee y$ (resp. $x \wedge y$) is equal to the point-wise maximum (resp. minimum) of these functions. For any $x \in L_0(\mu)$ and $A \in \Sigma$ we denote $x_A = x \mathbf{1}_A$ where $\mathbf{1}_A$ is the characteristic function of A . The expression $A = B \sqcup C$ for sets $A, B, C \in \Sigma$ means that $A = B \cup C$ and $B \cap C = \emptyset$. If E is a sublattice of $L_0(\mu)$ and $x, y \in E$ then $y \sqsubseteq x$ if and only if $y = x \mathbf{1}_A$ for some $A \in \Sigma$ and a **1**-step function is just a simple function and a *z*-step function is the product of z by a simple function. In particular, if $x, z \in E$ are simple (= finite valued) functions with $z \geq 0$ and $\text{supp } x \subseteq \text{supp } z$ then x is a *z*-step function.

2. The absolute numerical index of $L_p(\mu)$

The main aim of this section is to calculate the absolute numerical index of the L_p spaces, as shown in the following result.

Theorem 2.1. *Let $1 < p < \infty$ and let μ be a positive measure such that $\dim(L_p(\mu)) \geq 2$. Then,*

$$|n|(L_p(\mu)) = \kappa_p.$$

It is immediate to check that for positive operators on $L_p(\mu)$, the numerical radius and the absolute numerical radius coincide. Therefore, the following result is a consequence of the above theorem. We state here its proof since it is simple and useful to get a better understanding of the proof of Theorem 2.1.

Proposition 2.2. *Let $1 < p < \infty$ and (Ω, Σ, μ) be a measure space. Then, for every positive operator $T \in \mathcal{L}(L_p(\mu))$ one has*

$$v(T) \geq \kappa_p \|T\|.$$

Proof. Let $T \in \mathcal{L}(L_p(\mu))$ be positive with $\|T\| = 1$, fix $\varepsilon > 0$, and take $x \in S_{L_p(\mu)}$ so that $\|Tx\|^p \geq 1 - \varepsilon$ and $x \geq 0$ (observe that x can be taken positive because $T|x| \geq |Tx|$ due to the positivity of T). Next, fix any $\tau > 0$, set

$$y = x \vee \tau Tx \quad \text{and} \quad A = \{\omega \in \Omega: x(\omega) \geq \tau(Tx)(\omega)\},$$

and observe that

$$\|y\|^p = \int_A x^p d\mu + \int_{\Omega \setminus A} (\tau Tx)^p d\mu \leq 1 + \tau^p \quad \text{and} \quad y^\# = x^{p-1} \vee (\tau Tx)^{p-1}.$$

This, together with the positivity of T , allows us to write

$$\begin{aligned} v(T) &\geq \frac{1}{\|y\|^p} \int_{\Omega} y^\# Ty d\mu \geq \frac{1}{1 + \tau^p} \int_{\Omega} y^\# Ty d\mu \\ &\geq \frac{1}{1 + \tau^p} \int_{\Omega} (\tau Tx)^{p-1} Tx d\mu = \frac{\tau^{p-1}}{1 + \tau^p} \int_{\Omega} (Tx)^p d\mu \\ &\geq \frac{\tau^{p-1}}{1 + \tau^p} (1 - \varepsilon) \end{aligned}$$

for every $\tau > 0$. Taking supremum on $\tau > 0$ and $\varepsilon > 0$, we deduce that $v(T) \geq \kappa_p$, as desired. \square

The proof of Theorem 2.1 depends on the base scalar field. In the real case it needs some auxiliary results which we state here. They carry the main idea for the best possible estimation of the absolute numerical radius in the real case and allow us to apply positivity arguments to any operator as it has been done in the proof of Proposition 2.2.

Lemma 2.3. *Let E be a vector lattice, $z \in E^+$, and $x \in E$ a z -step function with $|x| \leq z$. Then there exist $n \in \mathbb{N}$, $\lambda_j \in [0, 1]$, and $y_j \in E$ with $|y_j| = z$ for $j = 1, \dots, n$ such that $\sum_{j=1}^n \lambda_j = 1$ and*

$$x = \lambda_1 y_1 + \dots + \lambda_n y_n.$$

Proof. Let $x = \sum_{k=1}^m a_k z_k$ with $a_k \in \mathbb{R}$ and $z_k \sqsubseteq z$, and use induction on m . Observe that the hypothesis $|x| \leq z$ implies that $|a_k| \leq 1$ for every $k = 1, \dots, m$. For $m = 1$, one trivially has that $x = \frac{1+a_1}{2} z_1 + \frac{1-a_1}{2} (-z_1)$. For the induction step assume that the assertion is true for a given $m \in \mathbb{N}$ and suppose that $x = \sum_{k=1}^{m+1} a_k z_k$ where $z_k \sqsubseteq z$ and $|a_k| \leq 1$ for $k = 1, \dots, m+1$. Then for $\tilde{x} = \sum_{k=1}^m a_k z_k$ and $\tilde{z} = z - z_{m+1} \in E^+$ we have that $z_k \sqsubseteq \tilde{z}$ for $k = 1, \dots, m$. By the induction assumption there are $n_0 \in \mathbb{N}$, $\tilde{\lambda}_j \in [0, 1]$, and $\tilde{y}_j \in E$ with $|\tilde{y}_j| = \tilde{z}$ for $j = 1, \dots, n_0$ such that $\sum_{j=1}^{n_0} \tilde{\lambda}_j = 1$ and $\tilde{x} = \tilde{\lambda}_1 \tilde{y}_1 + \dots + \tilde{\lambda}_{n_0} \tilde{y}_{n_0}$. Then set $\lambda = \frac{1+a_{m+1}}{2}$ and observe that

$$\begin{aligned} x &= \tilde{x} + a_{m+1} z_{m+1} = \lambda(\tilde{x} + z_{m+1}) + (1 - \lambda)(\tilde{x} - z_{m+1}) \\ &= \lambda(\tilde{\lambda}_1 \tilde{y}_1 + \dots + \tilde{\lambda}_{n_0} \tilde{y}_{n_0} + z_{m+1}) + (1 - \lambda)(\tilde{\lambda}_1 \tilde{y}_1 + \dots + \tilde{\lambda}_{n_0} \tilde{y}_{n_0} - z_{m+1}) \\ &= \lambda(\tilde{\lambda}_1(\tilde{y}_1 + z_{m+1}) + \dots + \tilde{\lambda}_{n_0}(\tilde{y}_{n_0} + z_{m+1})) \\ &\quad + (1 - \lambda)(\tilde{\lambda}_1(\tilde{y}_1 - z_{m+1}) + \dots + \tilde{\lambda}_{n_0}(\tilde{y}_{n_0} - z_{m+1})). \end{aligned}$$

Finally, take $n = 2n_0$ and

$$\begin{aligned} \lambda_j &= \lambda \tilde{\lambda}_j, & y_j &= \tilde{y}_j + z_{m+1} \quad \text{for } j = 1, \dots, n_0, \quad \text{and} \\ \lambda_j &= (1 - \lambda) \tilde{\lambda}_j, & y_j &= \tilde{y}_j - z_{m+1} \quad \text{for } j = n_0 + 1, \dots, 2n_0 \end{aligned}$$

which fulfill the desired conditions. \square

Corollary 2.4. *Let E be a vector lattice, f a positive linear functional on E , $T : E \rightarrow E$ a linear operator, $z \in E^+$, and $x \in E$ a z -step function with $|x| \leq z$. Then, there exists $y \in E$ satisfying $|y| = z$ and $f(|Ty|) \geq f(|Tx|)$.*

Proof. By Lemma 2.3 there are $n \in \mathbb{N}$, $\lambda_j \in [0, 1]$, and $y_j \in E$ with $|y_j| = z$ for $j = 1, \dots, n$ such that $\sum_{j=1}^n \lambda_j = 1$ and $x = \lambda_1 y_1 + \dots + \lambda_n y_n$. Then we can write

$$f(|Tx|) \leq f(\lambda_1 |Ty_1| + \dots + \lambda_n |Ty_n|) = \lambda_1 f(|Ty_1|) + \dots + \lambda_n f(|Ty_n|)$$

and so, $f(|Ty_j|) \geq f(|Tx|)$ for some j . \square

Corollary 2.5. *Let E be a sublattice of $L_0(\mu)$ for some measure space (Ω, Σ, μ) in which the set of all simple functions is dense, $f \in (E^*)^+$, $T \in \mathcal{L}(E)$, $\varepsilon > 0$, $z \in E^+$ and $x \in E$ such that $|x| \leq z$. Then there exists $y \in E$ satisfying $|y| = z$ and $f(|Ty|) \geq f(|Tx|) - \varepsilon$.*

Proof. It follows immediately from Corollary 2.4 and the continuity of $f, |\cdot|$, and T . \square

We are ready to prove the main result.

Proof of Theorem 2.1. To prove that $|n|(L_p(\mu)) \leq p^{-1/p} q^{-1/q}$, it suffices to construct a norm one operator $T_0 \in \mathcal{L}(L_p(\mu))$ with $|v|(T_0) \leq p^{-1/p} q^{-1/q}$. Indeed, we pick disjoint sets $A, B \in \Sigma$ with $0 < \mu(A), \mu(B) < \infty$ (this is possible since $\dim(L_p(\mu)) \geq 2$) and define $T_0 \in \mathcal{L}(L_p(\mu))$ by

$$T_0 x = \mu(A)^{-1/q} \mu(B)^{-1/p} \left(\int_A x \, d\mu \right) \mathbf{1}_B \quad (x \in L_p(\mu)). \tag{3}$$

It is easy to check that $\|T_0\| = 1$. Now we show that $|v|(T_0) \leq \kappa_p$. Given any $x \in S_{L_p(\mu)}$, we set

$$\lambda = \|x_B\|^p = \int_B |x|^p \, d\mu$$

and observe that

$$\|x_A\|^p = \int_A |x|^p \, d\mu \leq 1 - \lambda.$$

Thus,

$$\begin{aligned}
 \int_{\Omega} |x|^{p-1} |T_0 x| d\mu &= \int_{\Omega} |x|^{p-1} \mathbf{1}_B |T_0 x| d\mu \\
 &\leq \left(\int_B |x|^{(p-1)q} d\mu \right)^{1/q} \left(\int_{\Omega} |T_0 x|^p d\mu \right)^{1/p} \\
 &= \left(\int_B |x|^p d\mu \right)^{1/q} \left(\mu(A)^{-p/q} \mu(B)^{-1} \left| \int_A x d\mu \right|^p \mu(B) \right)^{1/p} \\
 &\leq \lambda^{1/q} \left(\mu(A)^{-p/q} \left(\int_A |x| d\mu \right)^p \right)^{1/p} \\
 &\leq \lambda^{1/q} \left(\mu(A)^{-p/q} \mu(A)^{p/q} \|x_A\|^p \right)^{1/p} \\
 &\leq \lambda^{1/q} (1 - \lambda)^{1/p} \leq \kappa_p.
 \end{aligned}$$

Now, we take supremum with $x \in S_{L_p(\mu)}$ to get $|v|(T_0) \leq \kappa_p$ as desired.

For the more interesting converse inequality, fix $T \in \mathcal{L}(L_p(\mu))$ with $\|T\| = 1$, $\varepsilon > 0$, and $\tau > 0$, choose $x \in S_{L_p(\mu)}$ so that $\|Tx\|_p^p \geq 1 - \varepsilon$, and set

$$A = \{\omega \in \Omega : |x(\omega)| \geq \tau |(Tx)(\omega)|\} \quad \text{and} \quad B = \Omega \setminus A.$$

We split the rest of the proof depending on the base scalar field.

• *Real case.* Using Corollary 2.5 for $x, z = |x|_A + \tau |Tx|_B$, and $f(u) = \int_{\Omega} |Tx|^{p-1} u d\mu$ ($u \in L_p(\mu)$), choose $y \in L_p(\mu)$ satisfying $|y| = z$ and $f(|Ty|) \geq f(|Tx|) - \varepsilon$. Then

$$\|y\|^p = \|z\|^p \leq 1 + \tau^p \quad \text{and} \quad |y| \geq \tau |Tx|,$$

and therefore, we can write

$$\begin{aligned}
 |v|(T) &\geq \int_{\Omega} \left| \frac{y^\#}{\|y^\#\|} T \left(\frac{y}{\|y\|} \right) \right| d\mu = \frac{1}{\|y\|^p} \int_{\Omega} |y|^{p-1} |Ty| d\mu \\
 &\geq \frac{\tau^{p-1}}{1 + \tau^p} \int_{\Omega} |Tx|^{p-1} |Ty| d\mu \\
 &\geq \frac{\tau^{p-1}}{1 + \tau^p} \left(\int_{\Omega} |Tx|^{p-1} |Tx| d\mu - \varepsilon \right) \\
 &\geq \frac{\tau^{p-1}}{1 + \tau^p} (1 - 2\varepsilon)
 \end{aligned}$$

for every $\tau > 0$. Finally, since the inequality holds for every $\varepsilon > 0$, we obtain that $|v|(T) \geq \max_{\tau > 0} \frac{\tau^{p-1}}{1+\tau^p}$ and so $n(L_p(\mu)) \geq \kappa_p$.

• *Complex case.* Since $|x| < \tau|Tx|$ on B , it is possible to find measurable functions $\theta_1, \theta_2 : B \rightarrow \mathbb{C}$ such that

$$x(\omega) = \frac{1}{2}\theta_1(\omega) + \frac{1}{2}\theta_2(\omega) \quad \text{and} \quad |\theta_j(\omega)| = \tau|(Tx)(\omega)| \quad (\omega \in B, j = 1, 2).$$

Indeed, for $\omega \in B$ define

$$\begin{aligned} \theta_1(\omega) &= \text{sign}(x(\omega))(|x(\omega)| + i(\tau^2|(Tx)(\omega)|^2 - |x(\omega)|^2)^{1/2}), \\ \theta_2(\omega) &= \text{sign}(x(\omega))(|x(\omega)| - i(\tau^2|(Tx)(\omega)|^2 - |x(\omega)|^2)^{1/2}) \end{aligned}$$

if $x(\omega) \neq 0$ and $\theta_1(\omega) = 1, \theta_2(\omega) = -1$ if $x(\omega) = 0$. Then define

$$y_j = x_A + \tilde{\theta}_j \quad (j = 1, 2)$$

where $\tilde{\theta}_j = \theta_j$ on B and $\tilde{\theta}_j = 0$ on A , and observe that

$$x = \frac{1}{2}y_1 + \frac{1}{2}y_2, \quad \|y_j\|^p \leq 1 + \tau^p, \quad \text{and} \quad |y_j| = |x|_A + |\tilde{\theta}_j| \geq \tau|Tx|.$$

Therefore, we can write

$$\begin{aligned} |v|(T) &\geq \frac{1}{2} \frac{1}{\|y_1\|^p} \int_{\Omega} |y_1|^{p-1} |Ty_1| d\mu + \frac{1}{2} \frac{1}{\|y_2\|^p} \int_{\Omega} |y_2|^{p-1} |Ty_2| d\mu \\ &\geq \frac{\tau^{p-1}}{1 + \tau^p} \int_{\Omega} |Tx|^{p-1} \left(\frac{1}{2}|Ty_1| + \frac{1}{2}|Ty_2| \right) d\mu \\ &\geq \frac{\tau^{p-1}}{1 + \tau^p} \int_{\Omega} |Tx|^{p-1} \left| T \left(\frac{1}{2}y_1 + \frac{1}{2}y_2 \right) \right| d\mu \\ &= \frac{\tau^{p-1}}{1 + \tau^p} \int_{\Omega} |Tx|^p d\mu \geq \frac{\tau^{p-1}}{1 + \tau^p} (1 - \varepsilon) \end{aligned}$$

for every $\tau > 0$. Since the inequality holds for every $\varepsilon > 0$, we obtain that $|v|(T) \geq \max_{\tau > 0} \frac{\tau^{p-1}}{1+\tau^p}$ and hence $n(L_p(\mu)) \geq \kappa_p$ which finishes the proof. \square

We can use Theorem 2.1 together with [12, Theorem 1] to improve the estimation of $n(L_p(\mu))$ given for the real case in [12, Corollary 3].

Corollary 2.6. *Let $1 < p < \infty$ and μ be a positive measure. Then, in the real case, one has*

$$n(L_p(\mu)) \geq \frac{M_p \kappa_p}{6}.$$

Remark 2.7. From the proof of Theorem 2.1 we deduce that κ_p is also the best constant of equivalence between the norm and the numerical radius for positive operators (i.e. the inequality in Proposition 2.2 is the best possible). This is because the operator defined on Eq. (3) is clearly positive.

3. The numerical radius of rank-one operators on $L_p(\mu)$

This section is devoted to estimate the numerical radius of rank-one operators on $L_p(\mu)$ for atomless measures μ .

Theorem 3.1. *Let (Ω, Σ, μ) be an atomless measure space. Then for $1 < p < \infty$ one has*

$$\kappa_p \geq n_1(L_p(\mu)) \geq \kappa_p^2.$$

We need the following easy observation.

Remark 3.2. Let (Ω, Σ, μ) be an atomless measure space and let f_1, \dots, f_n be simple functions on Ω . Then, given any $\lambda \in [0, 1]$, there exists a partition $\Omega = A \sqcup B$ into measurable subsets such that

$$\int_A f_j d\mu = \lambda \int_{\Omega} f_j d\mu \quad \text{and} \quad \int_B f_j d\mu = (1 - \lambda) \int_{\Omega} f_j d\mu$$

for every $j = 1, \dots, n$. To see that this is true, let C_0, C_1, \dots, C_m be a partition of Ω with $0 < \mu(C_k) < \infty$ for $k = 1, \dots, m$, such that all the functions f_1, \dots, f_n are null on C_0 and constant on every C_k . Then, for each $k = 1, \dots, m$, take $A_k, B_k \in \Sigma$ satisfying $C_k = A_k \sqcup B_k$, $\mu(A_k) = \lambda\mu(C_k)$, and $\mu(B_k) = (1 - \lambda)\mu(C_k)$, and observe that the sets given by

$$A = C_0 \cup \bigcup_{k=1}^m A_k \quad \text{and} \quad B = \bigcup_{k=1}^m B_k$$

form the desired partition of Ω .

Proof of Theorem 3.1. The first inequality follows from the fact that in the proof of Theorem 2.1 it is constructed a positive and rank-one operator T_0 (see (3)) such that $\|T_0\| = 1$ and $v(T_0) = |v|(T_0) \leq \kappa_p$. Therefore, $n_1(L_p(\mu)) \leq \kappa_p$.

We now prove the more interesting second inequality. Let $T \in \mathcal{L}(L_p(\mu))$ be a rank-one operator of norm one, that is,

$$Tz = \left(\int_{\Omega} x^{\#} z d\mu \right) y \quad (z \in L_p(\mu))$$

for some fixed $x, y \in S_{L_p(\mu)}$ (observe that $x^\# \in S_{L_q(\mu)}$ and that every element in $S_{L_q(\mu)}$ can be written in this form). Without loss of generality, we may and do assume that x, y are simple functions. Fix $\tau > 0, \lambda \in [0, 1]$ and set

$$\theta = \text{sign}\left(\int_{\Omega} x^\# y d\mu\right) \in \mathbb{T} \quad \text{and} \quad \delta = \left|\int_{\Omega} x^\# y d\mu\right|.$$

Using Remark 3.2, choose a partition $\Omega = A \sqcup B$ so that

$$\begin{aligned} \int_A x^\# y d\mu &= \lambda \int_{\Omega} x^\# y d\mu = \lambda\theta\delta, & \|x_A\|^p &= \|y_A\|^p = \lambda, \\ \int_B x^\# y d\mu &= (1-\lambda) \int_{\Omega} x^\# y d\mu = (1-\lambda)\theta\delta, & \text{and } \|x_B\|^p &= \|y_B\|^p = 1-\lambda. \end{aligned} \quad (4)$$

Then define $z = \lambda^{-\frac{1}{p}} x_A + \bar{\theta}(1-\lambda)^{-\frac{1}{p}} \tau y_B$ and observe that

$$\|z\|^p = \lambda^{-1} \|x_A\|^p + (1-\lambda)^{-1} \tau^p \|y_B\|^p = 1 + \tau^p.$$

Therefore, we can write

$$v(T) \geq \left| \int_{\Omega} \frac{z^\#}{\|z^\#\|} T\left(\frac{z}{\|z\|}\right) d\mu \right| = \frac{1}{\|z\|^p} \left| \int_{\Omega} z^\# T z d\mu \right| = \frac{1}{1 + \tau^p} \left| \int_{\Omega} x^\# z d\mu \right| \left| \int_{\Omega} z^\# y d\mu \right|. \quad (5)$$

Besides, using the fact that $(u + v)^\# = u^\# + v^\#$ for disjointly supported elements $u, v \in L_p(\mu)$, it is clear that $z^\# = \lambda^{-\frac{1}{q}} x_A^\# + \theta(1-\lambda)^{-\frac{1}{q}} \tau^{p-1} y_B^\#$. Using this and (4) it is easy to check that

$$\begin{aligned} \left| \int_{\Omega} x^\# z d\mu \right| &= \left| \lambda^{-\frac{1}{p}} \int_A x^\# x d\mu + \bar{\theta}(1-\lambda)^{-\frac{1}{p}} \tau \int_B x^\# y d\mu \right| \\ &= \left| \lambda^{-\frac{1}{p}} \lambda + \bar{\theta}(1-\lambda)^{-\frac{1}{p}} \tau (1-\lambda)\theta\delta \right| \\ &= \lambda^{\frac{1}{q}} + (1-\lambda)^{\frac{1}{q}} \tau\delta \geq \lambda^{\frac{1}{q}} \end{aligned}$$

and

$$\begin{aligned} \left| \int_{\Omega} z^\# y d\mu \right| &= \left| \lambda^{-\frac{1}{q}} \int_A x^\# y d\mu + \theta(1-\lambda)^{-\frac{1}{q}} \tau^{p-1} \int_B y^\# y d\mu \right| \\ &= \left| \lambda^{-\frac{1}{q}} \lambda\theta\delta + \theta(1-\lambda)^{-\frac{1}{q}} \tau^{p-1} (1-\lambda) \right| \\ &= \lambda^{\frac{1}{p}} \delta + (1-\lambda)^{\frac{1}{p}} \tau^{p-1} \\ &\geq (1-\lambda)^{\frac{1}{p}} \tau^{p-1}. \end{aligned}$$

This, together with (5), tells us that

$$v(T) \geq \lambda^{\frac{1}{q}}(1 - \lambda)^{\frac{1}{p}} \frac{\tau^{p-1}}{1 + \tau^p}$$

for every $\tau > 0$ and every $\lambda \in [0, 1]$. Finally, since $\max_{\tau > 0} \frac{\tau^{p-1}}{1 + \tau^p} = \max_{\lambda \in [0, 1]} \lambda^{\frac{1}{q}}(1 - \lambda)^{\frac{1}{p}} = \kappa_p$, one obtains that $v(T) \geq \kappa_p^2$ which finishes the proof. \square

Note that for $p \rightarrow 1$ and $p \rightarrow \infty$ this gives the best possible estimation of the order of $n_1(L_p(\mu))$ because $\kappa_p^2 \rightarrow 1$. Nevertheless, for $p = 2$ one has $\kappa_2^2 = 1/4$, while the best estimation is $1/2$, as the following easy result shows.

Proposition 3.3. *Let H be a real Hilbert space of dimension greater than one. Then $n_1(H) = \frac{1}{2}$.*

Proof. We fix a rank-one operator $T \in L(H)$ with $\|T\| = 1$. Then, T has the form $Tx = (x | x_1)x_2$ for some elements $x_1, x_2 \in S_H$. If $|(x_1 | x_2)| = 1$ then $v(T) \geq |(Tx_1 | x_1)| = 1$ and we are done. If otherwise $|(x_1 | x_2)| < 1$, take $x = \frac{x_1 + \theta x_2}{\|x_1 + \theta x_2\|} \in S_H$ for $\theta \in \{-1, 1\}$ and observe that

$$\begin{aligned} v(T) &\geq |(Tx | x)| = \left| \frac{(x_1 + \theta x_2 | x_1)(x_2 | x_1 + \theta x_2)}{\|x_1 + \theta x_2\|^2} \right| \\ &= \left| \frac{[1 + \theta(x_2 | x_1)]^2}{\|x_1 + \theta x_2\|^2} \right| \\ &= \left| \frac{[1 + \theta(x_2 | x_1)]^2}{2[1 + \theta(x_2 | x_1)]} \right| \\ &= \left| \frac{1 + \theta(x_2 | x_1)}{2} \right|. \end{aligned}$$

By just choosing the suitable $\theta \in \{-1, 1\}$ one obtains $v(T) \geq 1/2$ and so $n_1(H) \geq \frac{1}{2}$.

For the converse inequality, observe that if we take x_1, x_2 orthogonal, then for each $x \in S_H$ one has that $(x | x_1)^2 + (x | x_2)^2 \leq 1$ and, therefore,

$$|(Tx | x)| = |(x | x_1)||x | x_2| = \frac{(x | x_1)^2 + (x | x_2)^2}{2} - \frac{1}{2} (|(x | x_1)| - |(x | x_2)|)^2 \leq \frac{1}{2}$$

which implies $v(T) \leq \frac{1}{2}$ and so $n_1(H) \leq \frac{1}{2}$. \square

4. The numerical radius of narrow operators

In Section 3 we obtained an estimate for the numerical radius of rank-one operators in $L_p(\mu)$, it is natural to ask if it is possible to obtain a similar estimate for finite-rank operators. The aim of this section is to prove that it is so. In fact, we will do the work for the wider class of narrow operators. Let us recall the relevant definitions. An operator $T \in \mathcal{L}(E, X)$ on a (real or complex) Köthe function space E on a finite measure space (Ω, Σ, μ) acting to a Banach space X is *narrow* if for each $A \in \Sigma$ and each $\varepsilon > 0$ there is an $x \in E$ such that $x^2 = \mathbf{1}_A$, $\int_{\Omega} x \, d\mu = 0$

and $\|Tx\| < \varepsilon$. The conditions $x^2 = \mathbf{1}_A$, $\int_{\Omega} x d\mu = 0$ mean that there exists a decomposition $A = A^+ \sqcup A^-$ into sets of equal measure with $x = \mathbf{1}_{A^+} - \mathbf{1}_{A^-}$. This concept was introduced in [15] and developed in some other papers [8,10,14] (see also the expository paper [16]). Note that if $A \in \Sigma$ is an atom then $T\mathbf{1}_A = 0$ for any narrow operator $T \in \mathcal{L}(E, X)$, thus, the notion of narrow operator is nontrivial only for atomless measure spaces (Ω, Σ, μ) . For a more general consideration of narrow operators we refer the reader to [14]. If the norm of E is absolutely continuous, then for every Banach space X every compact (AM -compact, Dunford–Pettis...) operator $T \in \mathcal{L}(E, X)$ is narrow [14,15]. For $E = L_p(\mu)$ this is easy to see using the technique of the Rademacher system. Indeed, consider any Rademacher system (r_n) on A [15]. Since (r_n) is a weakly null sequence, we have that $\|Tr_n\| \rightarrow 0$ as $n \rightarrow \infty$. However, the converse is not true: there exists a narrow projection $P \in \mathcal{L}(L_p[0, 1])$ of norm one onto a subspace of $L_p[0, 1]$ isometric to $L_p[0, 1]$ [15].

Our estimate for the numerical radius of narrow operators in $L_p(\mu)$ depends on the base scalar field. For the complex case we obtain the same estimate as we did for rank-one operators.

Theorem 4.1. *Let (Ω, Σ, μ) be an atomless finite measure space. Then, for every $1 < p < \infty$ one has*

$$n_{\text{nar}}(L_p(\mu)) \geq \kappa_p^2 \quad \text{in the complex case,}$$

$$n_{\text{nar}}(L_p(\mu)) \geq \max_{\tau > 0} \frac{\kappa_p \tau^{p-1} - \tau}{1 + \tau^p} \quad \text{in the real case.}$$

Notice that the inequality for the real case gives a positive estimate for $1 < p < \infty$ ($p \neq 2$) which tends to 1 as $p \rightarrow 1$ or $p \rightarrow \infty$.

To prove this result we need the following lemmas which suggest that a narrow operator behaves almost as a rank-one operator when it is restricted to a suitable finite-dimensional subspace of large dimension.

Lemma 4.2. *Let (Ω, Σ, μ) be an atomless finite measure space, $1 \leq p < \infty$, $T \in \mathcal{L}(L_p(\mu))$ a narrow operator, $x \in L_p(\mu)$ a simple function, $Tx = y$, and $\Omega = D_1 \sqcup \dots \sqcup D_\ell$ any partition and $\varepsilon > 0$. Then there exists a partition $\Omega = A \sqcup B$ such that*

- (i) $\|x_A\|^p = \|x_B\|^p = 2^{-1} \|x\|^p$.
- (ii) $\mu(D_j \cap A) = \mu(D_j \cap B) = \frac{1}{2} \mu(D_j)$ for each $j = 1, \dots, \ell$.
- (iii) $\|Tx_A - 2^{-1}y\| < \varepsilon$ and $\|Tx_B - 2^{-1}y\| < \varepsilon$.

Proof. Let $x = \sum_{k=1}^m a_k \mathbf{1}_{C_k}$ for some $a_k \in \mathbb{K}$ and $\Omega = C_1 \sqcup \dots \sqcup C_m$. For each $k = 1, \dots, m$ and $j = 1, \dots, \ell$ define sets $E_{k,j} = C_k \cap D_j$ and, using the definition of narrow operator, choose $u_{k,j} \in L_p(\mu)$ so that

$$u_{k,j}^2 = \mathbf{1}_{E_{k,j}}, \quad \int_{\Omega} u_{k,j} d\mu = 0, \quad \text{and} \quad |a_k| \|Tu_{k,j}\| < \frac{2\varepsilon}{m\ell}.$$

Then set

$$E_{k,j}^+ = \{t \in E_{k,j}: u_{k,j}(t) \geq 0\}, \quad E_{k,j}^- = E_{k,j} \setminus E_{k,j}^+$$

which satisfy $\mu(E_{k,j}^+) = \mu(E_{k,j}^-) = \frac{1}{2}\mu(E_{k,j})$, and define

$$A = \bigcup_{k=1}^m \bigcup_{j=1}^{\ell} E_{k,j}^+ \quad \text{and} \quad B = \bigcup_{k=1}^m \bigcup_{j=1}^{\ell} E_{k,j}^-.$$

Let us show that the partition $\Omega = A \sqcup B$ has the desired properties. Indeed, observe that

$$\begin{aligned} \|x_A\|^p &= \sum_{k=1}^m \sum_{j=1}^{\ell} |a_k|^p \mu(E_{k,j}^+) = \sum_{k=1}^m |a_k|^p \sum_{j=1}^{\ell} \frac{1}{2} \mu(E_{k,j}) \\ &= \frac{1}{2} \sum_{k=1}^m |a_k|^p \mu(C_k) = \frac{1}{2} \|x\|^p \end{aligned}$$

and that one obviously has $\|x_B\|^p = \|x_A\|^p$, thus (i) is proved.

Since $E_{k,j}^+ \subseteq E_{k,j} \subseteq D_j$, for each $j_0 \in \{1, \dots, \ell\}$ we have that

$$D_{j_0} \cap A = \bigcup_{k=1}^m \bigcup_{j=1}^{\ell} (D_{j_0} \cap E_{k,j}^+) = \bigcup_{k=1}^m E_{k,j_0}^+$$

and hence

$$\mu(D_{j_0} \cap A) = \sum_{k=1}^m \mu(E_{k,j_0}^+) = \frac{1}{2} \sum_{k=1}^m \mu(E_{k,j_0}) = \frac{1}{2} \sum_{k=1}^m \mu(C_k \cap D_{j_0}) = \frac{1}{2} \mu(D_{j_0}).$$

Analogously it is proved that $\mu(D_j \cap B) = \frac{1}{2}\mu(D_j)$ for every $j \in \{1, \dots, \ell\}$ which finishes (ii).

To prove (iii) observe that

$$x_A - x_B = \sum_{k=1}^m \sum_{j=1}^{\ell} a_k (\mathbf{1}_{E_{k,j}^+} - \mathbf{1}_{E_{k,j}^-}) = \sum_{k=1}^m \sum_{j=1}^{\ell} a_k u_{k,j}$$

and hence,

$$\|T(x_A - x_B)\| \leq \sum_{k=1}^m \sum_{j=1}^{\ell} |a_k| \|Tu_{k,j}\| < 2\varepsilon.$$

Therefore, one has that

$$\left\| Tx_A - \frac{1}{2}y \right\| = \frac{1}{2} \|2Tx_A - Tx_A - Tx_B\| = \frac{1}{2} \|T(x_A - x_B)\| < \varepsilon.$$

Analogously, one obtains $\|Tx_B - \frac{1}{2}y\| < \varepsilon$ finishing the proof of (iii). \square

Lemma 4.3. Let (Ω, Σ, μ) be an atomless finite measure space, $1 \leq p < \infty$, let $T \in \mathcal{L}(L_p(\mu))$ be a narrow operator, and let $x, y \in L_p(\mu)$ be simple functions such that $Tx = y$. Then for each $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a partition $\Omega = A_1 \sqcup \dots \sqcup A_{2^n}$ such that for each $k = 1, \dots, 2^n$ one has

- (1) $\|x_{A_k}\|^p = 2^{-n} \|x\|^p$.
- (2) $\|y_{A_k}\|^p = 2^{-n} \|y\|^p$.
- (3) $\|Tx_{A_k} - 2^{-n}y\| < \varepsilon$.

Proof. Let $y = \sum_{j=1}^{\ell} b_j \mathbf{1}_{D_j}$ for some $b_j \in \mathbb{K}$ and $\Omega = D_1 \sqcup \dots \sqcup D_{\ell}$. We proceed by induction on n . Suppose first that $n = 1$ and use Lemma 4.2 to find a partition $\Omega = A \sqcup B$ satisfying properties (i)–(iii). Then (i) and (iii) mean (1) and (3) for $A_1 = A, A_2 = B$. Besides, observe that (2) follows from (ii):

$$\|y_{A_1}\|^p = \sum_{j=1}^{\ell} |b_j|^p \mu(A_1 \cap D_j) = \sum_{j=1}^{\ell} |b_j|^p \frac{1}{2} \mu(D_j) = \frac{1}{2} \|y\|^p$$

and analogously $\|y_{A_2}\|^p = 2^{-1} \|y\|^p$.

For the induction step suppose that the statement of the lemma is true for $n \in \mathbb{N}$ and find a partition $\Omega = A_1 \sqcup \dots \sqcup A_{2^n}$ such that for every $k = 1, \dots, 2^n$ the following hold:

$$\|x_{A_k}\|^p = 2^{-n} \|x\|^p, \quad \|y_{A_k}\|^p = 2^{-n} \|y\|^p, \quad \text{and} \quad \|Tx_{A_k} - 2^{-n}y\| < \varepsilon. \tag{6}$$

Then, for each $k = 1, \dots, 2^n$ use Lemma 4.2 for x_{A_k} instead of x, Tx_{A_k} instead of y , the decomposition

$$\Omega = \bigsqcup_{k=1}^{2^n} \bigsqcup_{j=1}^{\ell} (D_j \cap A_k)$$

instead of $\Omega = D_1 \sqcup \dots \sqcup D_{\ell}$ and $\frac{\varepsilon}{2}$ instead of ε , and find a partition $\Omega = A(k) \sqcup B(k)$ satisfying properties (i)–(iii). Namely, for each $k = 1, \dots, 2^n$ we have that:

- (i) $\|x_{(A_k \cap A(k))}\|^p = \|x_{(A_k \cap B(k))}\|^p = 2^{-1} \|x_{A_k}\|^p$.
- (ii) $\mu(D_j \cap A_k \cap A(k)) = \mu(D_j \cap A_k \cap B(k)) = \frac{1}{2} \mu(D_j \cap A_k)$ for each $j = 1, \dots, \ell$.
- (iii) $\|Tx_{(A_k \cap A(k))} - 2^{-1}Tx_{A_k}\| < \frac{\varepsilon}{2}$ and $\|Tx_{(A_k \cap B(k))} - 2^{-1}Tx_{A_k}\| < \frac{\varepsilon}{2}$.

Let us show that the partition

$$\Omega = (A_1 \cap A(1)) \sqcup \dots \sqcup (A_{2^n} \cap A(2^n)) \sqcup (A_1 \cap B(1)) \sqcup \dots \sqcup (A_{2^n} \cap B(2^n))$$

has the desired properties for $n + 1$:

Property (1): using (i) and (6), one obtains

$$\|x_{(A_k \cap A(k))}\|^p = \|x_{(A_k \cap B(k))}\|^p = 2^{-1} \|x_{A_k}\|^p = 2^{-(n+1)} \|x\|^p.$$

Property (2): for each $k = 1, \dots, 2^n$ use (ii) and (6) to obtain

$$\begin{aligned} \|y_{(A_k \cap A(k))}\|^p &= \sum_{j=1}^{\ell} |b_j|^p \mu(D_j \cap A_k \cap A(k)) \\ &= \frac{1}{2} \sum_{j=1}^{\ell} |b_j|^p \mu(D_j \cap A_k) \\ &= \frac{1}{2} \|y_{A_k}\|^p = 2^{-(n+1)} \|y\|^p \end{aligned}$$

and analogously $\|y_{(A_k \cap B)}\|^p = 2^{-(n+1)} \|y\|^p$.

Property (3): for each $k = 1, \dots, 2^n$ use (iii) and (6) to write

$$\begin{aligned} \|Tx_{(A_k \cap A(k))} - 2^{-(n+1)}y\| &\leq \|Tx_{(A_k \cap A(k))} - 2^{-1}Tx_{A_k}\| + \frac{1}{2}\|Tx_{A_k} - 2^{-n}y\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and analogously $\|Tx_{(A_k \cap B(k))} - 2^{-(n+1)}y\| < \varepsilon$, which completes the proof. \square

Lemma 4.4. *Let (Ω, Σ, μ) be an atomless finite measure space, $1 \leq p < \infty$, let $T \in \mathcal{L}(L_p(\mu))$ be a narrow operator, and let $x, y \in L_p(\mu)$ be simple functions such that $Tx = y$. Then for each $n \in \mathbb{N}$, each number λ of the form $\lambda = \frac{j}{2^n}$ where $j \in \{1, \dots, 2^n - 1\}$ and each $\varepsilon > 0$ there exists a partition $\Omega = A \sqcup B$ such that:*

- (A) $\|x_A\|^p = \lambda \|x\|^p$.
- (B) $\|y_B\|^p = (1 - \lambda) \|y\|^p$.
- (C) $\|Tx_A - \lambda y\| < \varepsilon$.

Proof. Use Lemma 4.3 to choose a partition $\Omega = A_1 \sqcup \dots \sqcup A_{2^n}$ satisfying properties (1)–(3) with ε/j instead of ε . Then, setting

$$A = \bigsqcup_{k=1}^j A_k \quad \text{and} \quad B = \bigsqcup_{k=j+1}^{2^n} A_k,$$

one obtains

$$\begin{aligned} \|x_A\|^p &= \sum_{k=1}^j \|x_{A_k}\|^p = \sum_{k=1}^j 2^{-n} \|x\|^p = \lambda \|x\|^p, \\ \|y_B\|^p &= \sum_{k=j+1}^{2^n} \|y_{A_k}\|^p = \sum_{k=j+1}^{2^n} 2^{-n} \|y\|^p = (1 - \lambda) \|y\|^p, \end{aligned}$$

and

$$\|Tx_A - \lambda y\| = \left\| \sum_{k=1}^j Tx_{A_k} - \sum_{k=1}^j 2^{-n} y \right\| \leq \sum_{k=1}^j \|Tx_{A_k} - 2^{-n} y\| < j \frac{\varepsilon}{j} = \varepsilon$$

as desired. \square

Proof of Theorem 4.1. Let $T \in \mathcal{L}(L_p(\mu))$ be a narrow operator of norm one. Fix $\varepsilon > 0, \tau > 0, n \in \mathbb{N}$ and $\lambda \in]0, 1[$ of the form $\lambda = \frac{j}{2^n}$ where $j \in \{1, \dots, 2^n - 1\}$. Pick a simple function $x \in S_{L_p(\mu)}$ so that $y = Tx$ satisfies $\|y\|^p \geq 1 - \varepsilon$. Without loss of generality we may assume that y is a simple function since one can approximate T by a sequence of narrow operators with the desired property (indeed, take a sequence of simple functions (y_m) converging to y and define $T_m = T - x^\# \otimes (y - y_m)$. Then, $T_m(x) = y_m, \|T_m - T\| \leq \|y - y_m\|$, and T_m is narrow for every $m \in \mathbb{N}$ since it is the sum of a rank-one operator and a narrow one [15, Proposition 6 on p. 59]).

Use Lemma 4.4 to find a partition $\Omega = A \sqcup B$ satisfying (A)–(C) and use (B) and (C) to obtain the following estimate:

$$\left| \int_B y^\# Tx_A d\mu - \lambda(1 - \lambda)\|y\|^p \right| = \left| \int_B y^\# Tx_A d\mu - \lambda \int_B y^\# y d\mu \right| \leq \|Tx_A - \lambda y\| < \varepsilon. \tag{7}$$

Then for $\theta \in \mathbb{T}$ define $z_\theta = \lambda^{-\frac{1}{p}} x_A + \theta(1 - \lambda)^{-\frac{1}{p}} \tau y_B$ and observe, using (A) and (B) of Lemma 4.4, that

$$\|z_\theta\|^p = \lambda^{-1} \|x_A\|^p + (1 - \lambda)^{-1} \tau^p \|y_B\|^p \leq 1 + \tau^p.$$

Besides, using the fact that $(u + v)^\# = u^\# + v^\#$ for disjointly supported elements $u, v \in L_p(\mu)$, it is clear that $z_\theta^\# = \lambda^{-\frac{1}{q}} x_A^\# + \bar{\theta}(1 - \lambda)^{-\frac{1}{q}} \tau^{p-1} y_B^\#$. Using this and (7) we can write

$$\begin{aligned} (1 + \tau^p)v(T) &\geq \max_{\theta \in \mathbb{T}} \left| \int_\Omega z_\theta^\# T z_\theta d\mu \right| \\ &= \max_{\theta \in \mathbb{T}} \left| \lambda^{-1} \int_A x^\# Tx_A d\mu + \theta \lambda^{-\frac{1}{q}} (1 - \lambda)^{-\frac{1}{p}} \tau \int_A x^\# Ty_B d\mu \right. \\ &\quad \left. + \bar{\theta} \lambda^{-\frac{1}{p}} (1 - \lambda)^{-\frac{1}{q}} \tau^{p-1} \int_B y^\# Tx_A d\mu + (1 - \lambda)^{-1} \tau^p \int_B y^\# Ty_B d\mu \right| \\ &\geq \max_{\theta \in \mathbb{T}} \left| \lambda^{-1} \int_A x^\# Tx_A d\mu + (1 - \lambda)^{-1} \tau^p \int_B y^\# Ty_B d\mu \right. \\ &\quad \left. + \theta \lambda^{-\frac{1}{q}} (1 - \lambda)^{-\frac{1}{p}} \tau \int_A x^\# Ty_B d\mu + \bar{\theta} \lambda^{\frac{1}{q}} (1 - \lambda)^{\frac{1}{p}} \tau^{p-1} \|y\|^p \right| \end{aligned}$$

$$\begin{aligned}
 & \left| -\lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1} \int_B y^\# T x_A d\mu - \lambda(1-\lambda)\|y\|^p \right| \\
 \geq & \max_{\theta \in \mathbb{T}} \left| \lambda^{-1} \int_A x^\# T x_A d\mu + (1-\lambda)^{-1}\tau^p \int_B y^\# T y_B d\mu \right. \\
 & \left. + \theta \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \int_A x^\# T y_B d\mu + \bar{\theta} \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p \right| \\
 & - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon \\
 \geq & \max_{\theta \in \mathbb{T}} \left| \theta \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \int_A x^\# T y_B d\mu + \bar{\theta} \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p \right| \\
 & - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon.
 \end{aligned} \tag{8}$$

Let us prove the last step in the formula above. Indeed, we write

$$\begin{aligned}
 a &= \lambda^{-1} \int_A x^\# T x_A d\mu + (1-\lambda)^{-1}\tau^p \int_B y^\# T y_B d\mu, \\
 b &= \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \int_A x^\# T y_B d\mu, \\
 c &= \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p
 \end{aligned}$$

and observe that what we need to prove is

$$\max_{\theta \in \mathbb{T}} |a + \theta b + \bar{\theta} c| \geq \max_{\theta \in \mathbb{T}} |\theta b + \bar{\theta} c|.$$

This inequality is easy. Fixed $\theta_0 \in \mathbb{T}$ it is clear that

$$\max_{\theta \in \mathbb{T}} |a + \theta b + \bar{\theta} c| \geq \max\{|a + (\theta_0 b + \bar{\theta}_0 c)|, |a - (\theta_0 b + \bar{\theta}_0 c)|\} \geq |\theta_0 b + \bar{\theta}_0 c|$$

and the arbitrariness of θ_0 gives the desired inequality.

From this point we study the real and the complex case separately. For the complex case, we continue the estimation in (8) as follows

$$\begin{aligned}
 (1 + \tau^p)v(T) \geq & \max_{\theta \in \mathbb{T}} \left| \theta \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \int_A x^\# T y_B d\mu + \bar{\theta} \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p \right| \\
 & - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon
 \end{aligned}$$

$$\begin{aligned}
 &= \left| \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \int_A x^\# T y_B d\mu \right| + |\lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p| \\
 &\quad - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon \\
 &\geq \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon \\
 &\geq \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}(1-\varepsilon) - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon.
 \end{aligned}$$

By the arbitrariness of ε we can write

$$v(T) \geq \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}} \frac{\tau^{p-1}}{1+\tau^p}$$

for every $\tau > 0$ and every $\lambda \in]0, 1[$ of the form $\lambda = \frac{j}{2^n}$ where $j \in \{1, \dots, 2^n - 1\}$. Since the dyadic numbers are dense in $[0, 1]$ and $\max_{\lambda \in [0,1]} \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}} = \kappa_p = \max_{\tau > 0} \frac{\tau^{p-1}}{1+\tau^p}$, the last inequality implies $v(T) \geq \kappa_p^2$ which finishes the proof in the complex case.

In the real case, using (A) and (B) of Lemma 4.4, it is easy to check that

$$\begin{aligned}
 \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \left| \int_A x^\# T y_B d\mu \right| &\leq \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \|x_A^\#\|_q \|y_B\|_p \\
 &\leq \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}} = \tau
 \end{aligned}$$

which, together with (8) and the choice of y , implies that

$$\begin{aligned}
 (1+\tau^p)v(T) &\geq \left| \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \int_A x^\# T y_B d\mu + \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p \right| \\
 &\quad - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon \\
 &\geq \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}\|y\|^p - \lambda^{-\frac{1}{q}}(1-\lambda)^{-\frac{1}{p}}\tau \left| \int_A x^\# T y_B d\mu \right| \\
 &\quad - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon \\
 &\geq \lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1}(1-\varepsilon) - \tau - \lambda^{-\frac{1}{p}}(1-\lambda)^{-\frac{1}{q}}\tau^{p-1}\varepsilon.
 \end{aligned}$$

Hence, by the arbitrariness of ε we deduce that

$$v(T) \geq \frac{\lambda^{\frac{1}{q}}(1-\lambda)^{\frac{1}{p}}\tau^{p-1} - \tau}{1+\tau^p}$$

for every $\tau > 0$ and every $\lambda \in]0, 1[$ of the form $\lambda = \frac{j}{2^n}$ where $j \in \{1, \dots, 2^n - 1\}$. Taking supremum on λ , one has

$$v(T) \geq \frac{\kappa_p \tau^{p-1} - \tau}{1+\tau^p}$$

for each $\tau > 0$, completing the proof. \square

5. Open problems

Problem 5.1. Calculate the numerical index of $L_p(\mu)$ for $1 < p < \infty$, $p \neq 2$. As we commented in the introduction, there are some estimations in the real case, but in the complex case the knowledge about $n(L_p(\mu))$ is almost negligible. As a conjecture, we think that $n(L_p(\mu)) = M_p$ in the real case and $n(L_p(\mu)) = \kappa_p$ in the complex case.

Problem 5.2. Calculate the rank-one numerical index of $L_p(\mu)$. We conjecture that $n_1(L_p(\mu)) = \kappa_p$ in both the real and the complex cases (if the dimension of $L_p(\mu)$ is greater than 1).

Problem 5.3. Is it true that the numerical index of $L_p(\mu)$ coincides with $n_{\text{nar}}(L_p(\mu))$? Let us comment that for $Z = L_p([0, 1], \ell_2)$ one has $n(Z) = n(\ell_2) < 1$ and $n_{\text{nar}}(Z) = 1$ since Z has the so-called Daugavet property. On the other hand, it is not difficult to show that $n(\ell_p)$ coincides with the numerical index of compact operators on ℓ_p .

Acknowledgments

The authors would like to thank Rafael Payá and Beata Randrianantoanina for fruitful conversations concerning the matter of this paper.

References

- [1] C.D. Aliprantis, O. Burkinshaw, Positive Operators, Springer, Dordrecht, 2006.
- [2] F.F. Bonsall, J. Duncan, Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras, London Math. Soc. Lecture Note Ser., vol. 2, Cambridge University Press, 1971.
- [3] F.F. Bonsall, J. Duncan, Numerical Ranges II, London Math. Soc. Lecture Note Ser., vol. 10, Cambridge University Press, 1973.
- [4] J. Duncan, C. McGregor, J. Pryce, A. White, The numerical index of a normed space, J. Lond. Math. Soc. (2) (1970) 481–488.
- [5] E. Ed-dari, On the numerical index of Banach spaces, Linear Algebra Appl. 403 (2005) 86–96.
- [6] E. Ed-dari, M. Khamsi, The numerical index of the L_p space, Proc. Amer. Math. Soc. 134 (2006) 2019–2025.
- [7] E. Ed-dari, M. Khamsi, A. Aksoy, On the numerical index of vector-valued function spaces, Linear Multilinear Algebra 55 (2007) 507–513.
- [8] V. Kadets, N. Kalton, D. Werner, Unconditionally convergent series of operators and narrow operators on L_1 , Bull. Lond. Math. Soc. 37 (2005) 265–274.
- [9] V. Kadets, M. Martín, R. Payá, Recent progress and open questions on the numerical index of Banach spaces, Rev. R. Acad. Cienc. Ser. A. Mat. 100 (2006) 155–182.
- [10] V.M. Kadets, M.M. Popov, Some stability theorems about narrow operators on $L_1[0, 1]$ and $C(K)$, Mat. Fiz. Anal. Geom. 10 (2003) 49–60.
- [11] M. Martín, J. Merí, A note on the numerical index of the L_p space of dimension two, Linear Multilinear Algebra 57 (2009) 201–204.
- [12] M. Martín, J. Merí, M. Popov, On the numerical index of real $L_p(\mu)$ -spaces, Israel J. Math., doi:10.1007/s11856-011-0064-y, in press.
- [13] M. Martín, J. Merí, M. Popov, B. Randrianantoanina, Numerical index of absolute sums of Banach spaces, J. Math. Anal. Appl. 375 (2011) 207–222.
- [14] O.V. Maslyuchenko, V.V. Mykhaylyuk, M.M. Popov, A lattice approach to narrow operators, Positivity 13 (2009) 459–495.
- [15] A.M. Plichko, M.M. Popov, Symmetric function spaces on atomless probability spaces, Dissertationes Math. (Rozprawy Mat.) 306 (1990) 1–85.
- [16] M. M. Popov, Narrow operators (a survey), in: Proceedings of Function Spaces IX. – Proc. Conf. Warsaw, 2009, in press.