Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 207 (2006) 544–565
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

On Hopf algebra structures over free operads

Ralf Holtkamp

University of Bochum, 44780 Bochum, Germany

Received 26 April 2005; accepted 12 December 2005

Available online 10 January 2006

Communicated by Andrei Zelevinsky

Abstract

The operad Lie can be constructed as the operad of primitives Prim $\mathcal{A}s$ from the operad $\mathcal{A}s$ of associative algebras. This is reflected by the theorems of Friedrichs, Poincaré–Birkhoff–Witt and Cartier–Milnor–Moore. We replace the operad $\mathcal{A}s$ by families of free operads $\mathcal{P}$, which include the operad $\mathcal{M}ag$ freely generated by a non-commutative non-associative binary operation and the operad of Stasheff polytopes. We obtain Poincaré–Birkhoff–Witt type theorems and collect information about the operads Prim $\mathcal{P}$, e.g., in terms of characteristic functions.

MSC: 16W30; 17A50; 18D35; 18D50

Keywords: Operad; Hopf algebra; Primitive; Poincaré–Birkhoff–Witt

0. Introduction

Recent developments in Hopf algebra theory show that there are important objects which could be called “non-classical” Hopf algebras. Typical examples are dendriform Hopf algebras [20,29–31]. Dendriform algebras (introduced in [18]) are equipped with two operations $\prec$, $\succ$ whose sum is an associative multiplication. Many of these Hopf algebras provide new links between algebra, geometry, combinatorics, and theoretical physics, e.g., renormalization in quantum field theory in the work of Connes and Kreimer (cf. [6]). Other examples are magma Hopf algebras and infinitesimal Hopf algebras (cf. [1]). Here the usual Hopf algebra axioms have to be changed.

E-mail address: ralf.holtkamp@ruhr-uni-bochum.de.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2005.12.004

Let $\mathcal{M}ag$ be the operad freely generated by a non-commutative non-associative binary operation $\vee^2(x_1, x_2)$ also denoted by $x_1 \cdot x_2$. Let similarly $\mathcal{M}ag_\omega$ be freely generated by n-ary operations $\vee^n$, one for each $2 \leqslant n \in \mathbb{N}$. A basis for the space of n-ary operations is given by reduced planar rooted trees with n leaves. This is the operad of Stasheff polytopes, see [35]. More generally, we consider operads $\mathcal{M}ag_N$, $2 \leqslant N \leqslant \omega$, intermediating between $\mathcal{M}ag$ and $\mathcal{M}ag_\omega$.

In a natural way, Hopf algebra structures over these operads can be considered. In fact it is sufficient that the operad $\mathcal{P}$ fulfills certain coherence conditions. Then the free $\mathcal{P}$-algebra generated by a set X of variables is always a $\mathcal{P}$-Hopf algebra with the diagonal Δ_a (also called co-addition) as a comultiplication. There canonically exists an operad $\operatorname{Prim}\mathcal{P}$, and $\operatorname{Prim}\mathcal{A}s = \mathcal{L}ie$. A table of pairs $\mathcal{P}$, $\operatorname{Prim}\mathcal{P}$ (triples in fact, if coassociativity of the comultiplication is replaced by a different law) is given by Loday and Ronco in [21,22], and the family given by $\mathcal{M}ag_N$ might be added to the list. In the case of $\mathcal{M}ag$, a full set of multilinear primitives was given by Shestakov and Umirbaev in [36], answering a problem posed by Hofmann and Strambach in [14]. An approach to study the primitive elements via Taylor expansions with respect to the so-called algebra of constants was given in [11].

In order to describe the operads $\operatorname{Prim}\mathcal{M}ag_N$, we describe the graded duals of the given $\mathcal{M}ag_N$-Hopf algebras. These duals are equipped with commutative multiplications $\sqcup\!\sqcup$. We show that these commutative algebras are freely generated by the primitive elements (and dually the given generalized Hopf algebras are connected co-free). This is an analogon of the Poincaré–Birkhoff–Witt theorem.

We conclude that the characteristic of the Σ_n-module $\operatorname{Prim}\mathcal{M}ag_N(n)$ is of the form

$$\operatorname{ch}_n\big(\operatorname{Prim}\mathcal{M}ag_N(n)\big) = \frac{1}{n}\sum_{d|n}\mu(d)c[N]'_{n/d}\,p_d^{n/d}.$$

Especially this means that the dimension of $\operatorname{Prim}\mathcal{M}ag_N(n)$ is equal to

$$(n-1)!\,c[N]'_n.$$

For $N = 2$, $c[2]'_k = c'_k$ is the kth log-Catalan number, with $c'_1 = 1$, $c'_2 = 1$, $c'_3 = 4$, $c'_4 = 13$, $c'_5 = 46$, $c'_6 = 166, \ldots.$

In Section 1, we recall basic facts about trees and operads. We introduce the notion of admissibly labeled (planar or abstract) rooted trees, where a sequence of sets M_k contains the allowed labels for vertices of arity k. This is useful to define the operads $\mathcal{M}ag_N$. Related integer sequences and their logarithmic derivatives are also considered.

In Section 2, we use the notion of unit actions on operads (see [19], and cf. [8] for further studies) in a generalized setting. The properties of the unit actions on $\mathcal{M}ag_N$ are needed in Section 3 to define the corresponding $\mathcal{P}$-Hopf algebras. In their definition we do not include objects with various not necessarily associative operations and not necessarily coassociative cooperations. The given definition is general enough to include dendriform Hopf algebras, though, and we sketch the relations to work of Loday, Ronco, and others.

In Section 4 we describe the free unitary $\mathcal{M}ag_\omega$-algebra $K\{X\}_\omega$ together with Δ_a in more detail. We also explicitly describe the graded dual $(K\{X\}_\omega, \Delta_a)$, which is equipped with a shuffle multiplication that is a sum of the shuffles. For $\mathcal{M}ag_N$-Hopf algebras one may pass to the appropriate subalgebras and quotients.

An analogon of the Poincaré–Birkhoff–Witt theorem for the operads $\mathcal{M}ag_N$ is proved in Section 5, see Theorems 25 and 26. Our approach makes use of co-**D** object structures, where

D is the category of unitary magmas. We also discuss the existence of a Cartier–Milnor–Moore theorem and of Eulerian idempotents. Moreover, we note that there are cocommutative $\mathcal{A}s$-Hopf algebras (with the same coalgebra structure) associated to the given $\mathcal{M}ag_N$-Hopf algebras.

In Section 6, we discuss the generating series and characteristic functions for the operads of primitives. Here we focus on $\operatorname{Prim}\mathcal{M}ag$ and $\operatorname{Prim}\mathcal{M}ag_\omega$. For small n, we can present the corresponding Σ_n-modules in terms of irreducible representations.

We also relate our results to recent independent work of Bremner–Hentzel–Peresi [3] and of Pérez-Izquierdo [26] concerning the case of $\mathcal{M}ag$-algebras. We discuss the description of $\operatorname{Prim}\mathcal{M}ag(4)$ by Sabinin operations.

1. Trees and free operads

1.1. Some combinatorics of trees

A finite connected graph $\emptyset \neq T = (\mathrm{Ve}(T), \mathrm{Ed}(T))$, with a distinguished vertex $\rho_T \in \mathrm{Ve}(T)$, is called an abstract rooted tree (with root ρ_T), if for every vertex $\lambda \in \mathrm{Ve}(T)$ there is exactly one path connecting λ and ρ_T. At each vertex there are incoming edges and exactly one outgoing edge. (Here we think of the edges as being oriented towards the root, and we add to the root an outgoing edge that is not connected to any further vertex.) At a given vertex λ, the number n of incoming edges is called the arity ar_λ of λ. We write the set $\mathrm{Ve}(T)$ of vertices as a disjoint union $\bigcup_{n\in\mathbb{N}} \mathrm{Ve}^n(T)$. The vertices of arity 0 are called leaves, and we denote $\mathrm{Ve}^0(T)$ by $\mathrm{Le}(T)$.

An abstract rooted tree T together with a chosen order of incoming edges at each vertex is called a planar rooted tree (or ordered rooted tree), cf. [33,34] as a general reference.

It is well known that the number of planar rooted trees with n vertices is the nth Catalan number

$$c_n = \frac{(2(n-1))!}{n!(n-1)!} = \sum_{l=1}^{n-1} c_l c_{n-l}.$$

The sequence of Catalan numbers is $c_1 = 1$, $c_2 = 1$, $c_3 = 2$, $c_4 = 5$, $c_5 = 14$, $c_6 = 42$, $c_7 = 132$, $c_8 = 429$, $c_9 = 1430, \dots$ with generating series $f(t) = \sum_{n=1}^\infty c_n t^n$ given by $(1-\sqrt{1-4t})/2$.

The numbers c_n also count the number of planar binary rooted trees with n leaves (or $2n-1$ vertices). A tree T is called binary, if $\mathrm{Ve}(T) = \mathrm{Ve}^2(T) \cup \mathrm{Le}(T)$.

Let a_n, $n \geqslant 1$, be a sequence of integers with generating series $f(t) = \sum_{n=1}^\infty a_n t^n$. The logarithmic derivative of $f(t)$ is the series $g(t) := \frac{\partial}{\partial t}\log(1+f(t))$, and we say that the sequence a'_n, $n \geqslant 1$, with $\sum_{n=1}^\infty a'_n t^n = t \cdot g(t)$ is obtained from a_n, $n \geqslant 1$, by logarithmic derivation.

The sequence of log-Catalan numbers c'_n, starting with 1, 1, 4, 13, 46, 166, 610, 2269, 8518, 32206, ... has the generating series $2t/(3\sqrt{1-4t}-1+4t)$.

Since

$$\frac{2}{3\sqrt{1-4t}-1+4t} = \frac{\partial}{\partial t}\log\left(\frac{3-\sqrt{1-4t}}{2}\right),$$

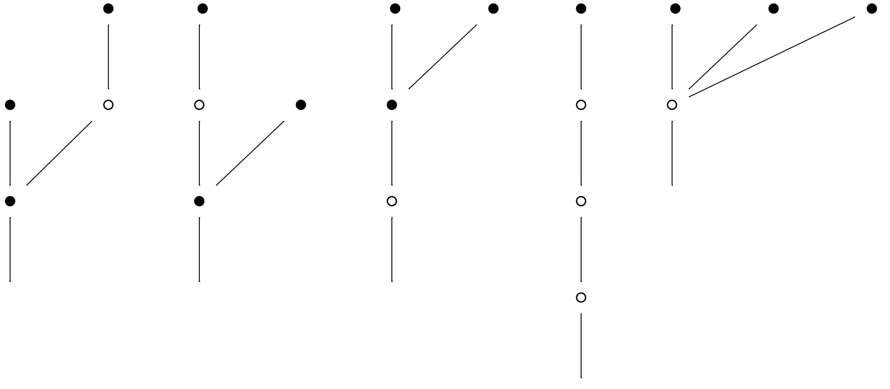
it is obtained by logarithmic derivation from the Catalan numbers c_n, $n \geqslant 1$.

In the set of all planar (rooted) trees with n vertices, the number of vertices with even arity is given by c'_n. The corresponding numbers of vertices with odd arity have the generating series

$$\sum_{n=1}^{\infty} nc_n t^n - \sum_{n=1}^{\infty} c'_n t^n = \frac{t}{\sqrt{1-4t}} - \frac{2t}{(3-\sqrt{1-4t})\sqrt{1-4t}}$$
$$= \frac{t(1-\sqrt{1-4t})}{(3-\sqrt{1-4t})\sqrt{1-4t}} = t^2 + 2t^3 + 7t^4 + 24t^5 + 86t^6 + 314t^7 + \cdots$$

(see [32, A026641], [7, p. 258]).

For example, in the set



of planar trees with $n = 4$ vertices, we count seven vertices with odd arity and $c'_4 = 13$ vertices with even arity.

1.2. Admissible labellings

Let M be a set and T a planar (or abstract) tree. Then a labeling of T is a map $\nu : \mathrm{Ve}(T) \to M$. The tree T together with such a labeling is called a labeled tree.

Let a collection $M_0, M_1, M_2, \ldots$ of sets be given, and let $M = \bigcup_{k\in\mathbb{N}} M_k$. A labeling $\nu : \mathrm{Ve}(T) \to M$ of a planar (or abstract) tree T is called admissible, if the restrictions $\nu|\mathrm{Ve}^k(T)$ are maps $\mathrm{Ve}^k(T) \to M_k$, i.e. it holds that:

$$\nu(\lambda) \in M_k \quad \text{if } \mathrm{ar}_\lambda = k.$$

If $M_1 = \emptyset$, only reduced trees can be provided with an admissible labeling. A tree T is called reduced, if $\mathrm{ar}_\lambda \neq 1$ for all $\lambda \in \mathrm{Ve}(T)$.

The set of planar rooted trees $T \in \mathrm{PTree}$ with admissible labeling from $(M_k)_{k\in\mathbb{N}}$ is denoted by $\mathrm{PTree}\{(M_k)_{k\in\mathbb{N}}\}$.

Non-labeled trees occur as trivially labeled trees, i.e. in the case where all M_k, $k \in \mathbb{N}$, are given by a one-element set $\{\circ\}$. Similarly (setting $M_1 = \emptyset$), we consider non-labeled planar reduced trees.

The number $C_n = \#\mathrm{PRTree}^n$ of planar reduced trees with n leaves is called the nth super-Catalan number, also called the nth little Schroeder number. The generating series for the super-Catalan numbers is $\frac{1}{4}(1 + t - \sqrt{1 - 6t + t^2})$ (cf. [32, A001003]). The first 10 super-Catalan numbers are 1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049.

Moreover, for fixed $N \in \mathbb{N}$, we consider the sets $\mathrm{PRTree}[N]^n$ of planar reduced trees which have n leaves and fulfill the property, that for every vertex λ the arity ar_λ is $\leqslant N$. The corresponding integer sequence is denoted by $c[N]_n$. Clearly for $N = 2$, we get binary trees and the sequence c_n, while for $N \to \infty$, we exhaust all planar reduced trees. More exactly, $c[N]_n = C_n$ whenever $n \leqslant N$.

Similar to the definition of log-Catalan numbers, we define the sequences C'_n and $c[N]'_n$ by logarithmic derivation. One can check that these sequences are in fact integer sequences. (This will also follow later on.)

1.3. Free operads

Let K be a field. Let a collection $M = (M_k)_{k \geqslant 2}$ of sets be given, and set $M_0 := \{\circ\}$, $M_1 := \emptyset$. Then the free non-Σ-operad $\underline{\Gamma(M)}$ generated by the collection $(M_k)_{k \geqslant 2}$ can be conveniently described using admissible labellings:

Let $\underline{\Gamma(M)}(0) = 0$, $\underline{\Gamma(M)}(1) = K \cdot |$, where $|$ is the tree consisting of the root. The elements of $\underline{\Gamma(M)(n)}$ are all linear combinations of admissibly labeled (necessarily reduced) planar trees with n leaves.

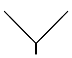
The operad structure on the sequence of vector spaces $\underline{\Gamma(M)}(n)$ is determined by K-linear $\circ_{n,i}$-operations

$$\circ_{n,i} : \underline{\Gamma(M)}(n) \otimes \underline{\Gamma(M)}(m) \to \underline{\Gamma(M)}(m+n-1), \quad \text{all } n, m \geqslant 1,\ 1 \leqslant i \leqslant n,$$

and the unit $|$. Here, if T^1 is a tree with n leaves, and T^2 is a further tree having m leaves, $T^1 \circ_{n,i} T^2$ (for $i = 1, \dots, n$) is given by the substitution of T^2 in T^1 at the ith leaf (obtained by replacing the specified leaf of T^1 by the root of T^2).

Let M_2 consist of one generator α, and let $M_k = \emptyset$ ($k \geqslant 3$). Then all elements of $\underline{\Gamma(M)}$ are linear combinations of planar binary trees. For $n \geqslant 1$, we can identify a basis of $\underline{\Gamma(M)(n)}$ with the set of (non-labeled) planar binary trees with n leaves. Especially,

$$\dim \underline{\Gamma(M)}(n) = c_n.$$

The tree  corresponds to the binary operation α, and we get ternary operations $\alpha \circ_{2,1} \alpha$ and $\alpha \circ_{2,2} \alpha$ as compositions.

This is the non-Σ-operad $\underline{\mathcal{M}ag}$ of (non-unitary) magma algebras.

1.4. Grafting operations

A word $T^1.T^2 \dots T^k$ (or an ordered tuple $(T^1, T^2, \dots, T^k)$, not necessarily non-empty) of planar trees is called a planar forest. Given a forest $T^1.T^2 \dots T^k$ of $k \geqslant 0$ trees, together with a label $\rho \in M_k$, there is a tree $T = \vee_\rho(T^1.T^2 \dots T^k)$ defined by introducing a new root of arity k and grafting the trees $T^1, \dots, T^k$ onto this new root. The new root gets the label ρ, and the specified order determines the order of incoming edges at ρ_T. The tree T is called the grafting of $T^1.T^2 \dots T^k$ over ρ.

The non-Σ-operad $\underline{\mathcal{K}}$ of Stasheff polytopes is the free non-Σ-operad $\underline{\Gamma(M)}$ generated by a collection $(M_k)_{k \geqslant 2}$ of one-element sets. We denote the generator of arity k by $\vee^k$. It corresponds to the grafting operation $\vee$ restricted to planar forests consisting of k trees (and their K-linear combinations). The tree symbolizing this operation is called the k-corolla. The cells of the Stasheff polytope (or associahedron) in dimension $n-2$ can be identified with reduced planar trees with n leaves and form a basis of $\underline{\mathcal{K}}(n)$.

1.5. A family of free operads

For each $N \in \mathbb{N}$, we can also consider the non-Σ-operad $\underline{\mathcal{M}ag_N}$ generated by a collection $(M_k)_{k \geqslant 2}$ given by $\{\vee^k\}$ for $k \leqslant N$ and $\emptyset$ for $k > N$. The trees symbolizing operations are elements of $\mathrm{PRTree}[N]^n$, i.e. they have at most arity N vertices. Clearly $\underline{\mathcal{M}ag_2} = \underline{\mathcal{M}ag}$. There are obvious inclusion maps

$$\underline{\mathcal{M}ag_N} \to \underline{\mathcal{M}ag_{N'}}, \quad N \leqslant N',$$

defined. All these are sub-operads of $\underline{\mathcal{K}}$, which we will also denote by $\underline{\mathcal{M}ag_\omega}$.

Let $\mathcal{M}ag_N$ be the operads given by the symmetrizations of the non-Σ-operads $\underline{\mathcal{M}ag_N}$.

The symmetrization of a non-Σ-operad $\underline{\mathcal{P}}$ is defined by $\mathcal{P}(n) = \underline{\mathcal{P}}(n) \otimes_K K\Sigma_n$ (all n), where Σ_n is the symmetric group. The composition maps are induced by the maps of $\underline{\mathcal{P}}$ (and the maps of the operad $\mathcal{A}s$ with $\mathcal{A}s(n) = K\Sigma_n$). Thus, e.g., the Σ_n-module $\mathcal{M}ag(n)$ is given by c_n copies of the regular representation $K\Sigma_n$, and $\mathcal{M}ag_\omega(n)$ is given by C_n copies of $K\Sigma_n$. Operads that occur as symmetrizations of non-Σ-operads are also called regular operads.

2. Free $\mathcal{M}ag_N$-algebras

Let K be a field of characteristic 0, and let $X = \{x_1, x_2, \ldots\}$ be a finite or countable set of variables. We consider algebras over the operads $\mathcal{M}ag_N$ defined in the previous section. If not specified, N is allowed to be in $\mathbb{N}_{\geqslant 2} \cup \{\omega\}$.

To ensure that the tensor product of $\mathcal{P}$-algebras is provided with the structure of a $\mathcal{P}$-algebra there are several approaches possible (cf. [15,19,24]). The approach of [19] for binary quadratic operads is generalized in the following. For the purpose of this paper, we mostly deal with regular operads.

Definition 1. Let $\mathcal{P}$ be an operad, $\mathcal{P}(0) = 0$, $\mathcal{P}(1) = K = K\,\mathrm{id}$.

Let a 0-ary element η be adjoined to the Σ-space $\mathcal{P}$ by

$$\mathcal{P}'(i) := \begin{cases} \mathcal{P}(i), & i \geqslant 1, \\ K\eta, & i = 0. \end{cases}$$

A unit action on $\mathcal{P}$ is a partial extension of the operad composition onto $\mathcal{P}'$ in the sense that composition maps (fulfilling the associativity, unitary, and invariance conditions) $\mu_{n;m_1,\ldots,m_n}$ are defined on

$$\mathcal{P}'(n) \otimes \mathcal{P}'(m_1) \otimes \cdots \otimes \mathcal{P}'(m_n) \to \mathcal{P}'(m), \quad m := m_1 + \cdots + m_n,$$

for all $m_j \geqslant 0$ ($j = 1, \ldots, n$), for $n \geqslant 2, m > 0$ (or $n \leqslant 1, m \geqslant 0$).

Given a $\mathcal{P}$-algebra $\overline{A}$, we can define structure maps for $A := K1 \oplus \overline{A}$ such that $\eta \in \mathcal{P}'(0)$ is mapped to $1 \in A$, and A is called a unitary $\mathcal{P}$-algebra.

Definition 2. Let $\mathcal{P}$ be a regular operad, $\mathcal{P}(n) = \underline{\mathcal{P}}(n) \otimes_K K\Sigma_n$, where $\underline{\mathcal{P}}$ is generated by fixed sets $(M_k)_{2 \leqslant k \leqslant N}$. Let $\mathcal{P}$ be equipped with a unit action.

If there are operations $\star_n \in \underline{\mathcal{P}}(n) = \underline{\mathcal{P}}(n) \otimes_K K \cdot \mathrm{id} \subseteq \mathcal{P}(n)$, $n \leqslant N$, fulfilling

$$\star_n \circ_{n,i} (\eta) = \star_{n-1}, \quad \text{all } i,$$
$$\star_2 \circ_{2,1} (\eta) = \star_2 \circ_{2,2} (\eta) = \mathrm{id},$$

then we say that the unit action respects the operations $\star_n$, and we define

$$\mu_{n;0,\dots,0}(\star_n \otimes \eta \otimes \cdots \otimes \eta) = \eta.$$

Such a unit action is called coherent, if for all $\mathcal{P}$-algebras $\overline{A}$, $\overline{B}$ it holds that $(\overline{A} \otimes K1) \oplus (K1 \otimes \overline{B}) \oplus (\overline{A} \otimes \overline{B})$ is again a $\mathcal{P}$-algebra with:

For all n, all $p \in M_n$, all $a_i \in A$, $b_i \in B$,

$$p(a_1 \otimes b_1, a_2 \otimes b_2, \dots, a_n \otimes b_n) := \star_n(a_1, a_2, \dots, a_n) \otimes p(b_1, b_2, \dots, b_n)$$

in case that at least one $b_j \in \overline{B}$,

$$p(a_1 \otimes 1, a_2 \otimes 1, \dots, a_n \otimes 1) := p(a_1, a_2, \dots, a_n) \otimes 1,$$

if $p(a_1, a_2, \dots, a_n)$ is defined; and the operations on $(\overline{A} \otimes K1) \oplus (K1 \otimes \overline{B}) \oplus (\overline{A} \otimes \overline{B})$ generated by these p via compositions (and the Σ-action) are obtained by applying the same compositions (or permutations) on both tensor components correspondingly.

In case an operad $\mathcal{Q}$ is a quotient of a regular operad $\mathcal{P}$ with a coherent unit action, we say that $\mathcal{Q}$ is equipped with the induced coherent action if the analogous equations hold for the images of the operations $\star_n$ and p in $\mathcal{Q}$.

Since $\mathcal{M}ag_N$ is a regular operad freely generated by operations $\vee^k$, $k \leqslant N$, we get:

Lemma 3. *Each operad $\mathcal{M}ag_N$ is equipped with a (unique) unit action which respects the operations $\vee^k$, i.e.*

$$\vee^k \circ_{k,i} (\eta) = \vee^{k-1}, \quad \textit{all } i,$$
$$\vee^2 \circ_{2,1} (\eta) = \vee^2 \circ_{2,2} (\eta) = \mathrm{id}.$$

This unit action is coherent.

Remark 4. Let $M^X = (M_k^X)_{k \geqslant 0}$ denote the sequence

$$M_0^X = X, \qquad M_k^X = M_k = \{\vee^k\} \quad \text{for } 2 \leqslant k \leqslant N, \quad \text{and} \quad M_k^X = \emptyset \quad \text{else}.$$

Then the free $\mathcal{M}ag_N$-algebra has the set of admissibly labeled planar trees as a vector space basis. In the case $N = \omega$ these are the reduced planar trees with leaves labeled by X, and for $N \in \mathbb{N}$ only trees with maximum arity N occur.

The free $\mathcal{M}ag_N$-algebra is naturally graded, such that the planar trees with n leaves are homogeneous of degree n. We set

$$K\{X\}_N^{(0)} = K1$$

and identify 1 with the empty tree $\emptyset$ (which we now adjoin to the set of reduced planar trees). For $N \in \mathbb{N}_{\geqslant 2} \cup \{\omega\}$, we denote the free $\mathcal{M}ag_N$-algebra with unit 1 by

$$K\{X\}_N = \bigoplus_{n=0}^{\infty} K\{X\}_N^{(n)}.$$

The admissibly labeled trees with only one vertex are identified with X and form a basis of $K\{X\}_N^{(1)}$. We also set $\vee^k(1, 1, \ldots, 1) = 1$ (all k).

Consequently, the free $\mathcal{M}ag$-algebra with unit 1 can be identified with the space of labeled binary trees $K\{X\} \subset K\{X\}_N$ (all $N \geqslant 2$), equipped with the free binary operation $\cdot = \vee^2$. On $\mathcal{M}ag$, the coherent unit action is just the one given by $1 \cdot a = a \cdot 1 = a$ (for every $\mathcal{M}ag$-algebra A, every $a \in A$), with component-wise multiplication on the tensor product. The operads $\mathcal{A}s$ and $\mathcal{C}om$ are equipped with the induced coherent unit actions.

3. $\mathcal{P}$-Hopf algebras and Prim $\mathcal{P}$

Since Hopf algebras combine operations and cooperations, there is no operad whose algebras or coalgebras are Hopf algebras. To describe them, and also generalizations with not necessarily associative operations and not necessarily associative cooperations, one would use PROPs. Here we are only interested in the case where the set of cooperations is generated by one coassociative cooperation. Thus we do not need the generality of PROPs and stay close to operad theory.

Definition 5. Let $\mathcal{P}$ be an operad equipped with a coherent unit action. Let $A = K1 \oplus \overline{A}$ be a unitary $\mathcal{P}$-algebra and $A \otimes A = K1 \oplus (\overline{A} \otimes K1) \oplus (K1 \otimes \overline{A}) \oplus (\overline{A} \otimes \overline{A})$ be equipped with its unitary $\mathcal{P}$-algebra structure (see Section 2).

Let $\Delta : A \to A \otimes A$ be a K-linear coassociative map (called comultiplication map), such that $\Delta(1) = 1 \otimes 1$ and $\Delta'(a) := \Delta(a) - a \otimes 1 - 1 \otimes a \in \overline{A} \otimes \overline{A}$ for all $a \in \overline{A}$.

Then A together with Δ is called an (augmented) $\mathcal{P}$-bialgebra, if Δ is a morphism of unitary $\mathcal{P}$-algebras, i.e. if $\Delta \circ p_A = p_{A \otimes A} \circ (\Delta \otimes \cdots \otimes \Delta)$, for all $p \in \mathcal{P}$.

Let moreover A be given by $\bigcup_{n \in \mathbb{N}} A_n$, where each A_i is a subspace of A_{i+1}. Then A is called a filtered $\mathcal{P}$-Hopf algebra if $\Delta(A_n) \subseteq \sum_{i=0}^{n} A_i \otimes A_{n-i}$ (all n).

We call A connected graded, if $A = \bigoplus_{n \in \mathbb{N}} A^{(n)}$ for finite dimensional vector spaces $A^{(i)} \subseteq A_i$ such that $A^{(0)} = K$ and

$$\Delta'\big(A^{(n)}\big) \subseteq \sum_{i=1}^{n-1} A^{(i)} \otimes A^{(n-i)} \quad (\text{all } n \geqslant 1).$$

In case that $\mathcal{P}$ is a regular operad equipped with a coherent unit action, we are going to define a comultiplication map on $A = K \oplus F_{\mathcal{P}}(V_X)$. Let us recall that for any operad $\mathcal{P}$, any sequence $(g_1, g_2, \ldots)$ of elements of a $\mathcal{P}$-algebra A' uniquely determines a $\mathcal{P}$-algebra morphism $\gamma = \gamma_{(g_1, g_2, \ldots)} : A \to A'$ with $x_i \mapsto g_i$ (all $x_i \in X$) and $1 \mapsto 1$.

Lemma 6. *Let $\mathcal{P}$ be a regular operad equipped with a coherent unit action and let A be the free unitary $\mathcal{P}$-algebra generated by the vector space V_X with basis X. Then there is a coassociative $\mathcal{P}$-algebra morphism Δ_a defined by*

$$\Delta_a(x) = x \otimes 1 + 1 \otimes x, \quad \textit{all } x \in X.$$

It provides the free unitary $\mathcal{P}$-algebra A with the structure of a connected graded $\mathcal{P}$-Hopf algebra.

If we furthermore assume that the K-linear map $\tau : A \otimes A \to A \otimes A$, $a_1 \otimes a_2 \mapsto a_2 \otimes a_1$ is a $\mathcal{P}$-algebra isomorphism, then Δ_a is cocommutative in the sense that $\tau \circ \Delta_a = \Delta_a$.

Proof. By construction the map Δ_a is a $\mathcal{P}$-algebra morphism and $\Delta'_a(f) \in \overline{A} \otimes \overline{A}$ for all $f \in \overline{A}$. To check coassociativity, we just have to verify that

$$(\Delta_a \otimes \mathrm{id})\big(\Delta_a(x)\big) = (\mathrm{id} \otimes \Delta_a)\big(\Delta_a(x)\big), \quad \text{all } x \in X.$$

Both sides are equal to $x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x$.

The free unitary $\mathcal{P}$-algebra A allows a grading (with $A^{(0)} = K$) respected by all $\mathcal{P}$-operations. Similarly the $\mathcal{P}$-algebra operations on $A \otimes A$ respect the grading (component-wise), and the map Δ_a provides A with the structure of a connected graded $\mathcal{P}$-Hopf algebra.

The last assertion follows directly from the definition of Δ_a. □

Remark 7. We call Δ_a the diagonal or co-addition (in analogy to the usage in [2], where any abelian group-valued representable functor leads to a co-addition on the representing object).

The cocommutative Hopf algebra given by the free unitary $\mathcal{A}s$-algebra $K\langle X\rangle$ together with the diagonal Δ_a is well known.

Dually one can equip the standard tensor coalgebra (i.e. $K\langle X\rangle$ with its deconcatenation coalgebra-structure) with the shuffle multiplication $⧢ = \Delta_a^*$, which is commutative (cf. [28, Chapter 1]).

Definition 8. For any connected graded $\mathcal{P}$-Hopf algebra $A = \bigoplus_{n\in\mathbb{N}} A^{(n)}$, we define the vector space $A^{*g} = \bigoplus_{n\in\mathbb{N}} (A^{*g})^{(n)} = \bigoplus_{n\in\mathbb{N}} (A^{(n)})^*$, where $V^* = \mathrm{Hom}_K(V, K)$ for any vector space V. We call A^{*g} the graded dual of A.

We denote by $\Delta^* : A^{*g} \otimes A^{*g} \to A^{*g}$ the K-linear map given by $(\Delta^*(f_1 \otimes f_2))(a) = \sum f_1(a_{(1)}) f_2(a_{(2)})$, where $f_1, f_2 \in A^{*g}$ and $a \in A$ with $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

The maps Δ^* and Δ are adjoint with respect to the canonical bilinear form $\langle , \rangle : A^{*g} \times A \to K$, i.e.

$$\big\langle \Delta^*(f_1 \otimes f_2), a\big\rangle = \big\langle f_1 \otimes f_2, \Delta(a)\big\rangle,$$

where $\langle f_1 \otimes f_2, g_1 \otimes g_2\rangle = \langle f_1, g_1\rangle \otimes \langle f_2, g_2\rangle$.

Lemma 9. *The space A^{*g} together with the operations p^*, $p \in \mathcal{P}$, is a $\mathcal{P}$-coalgebra.*

The space A^{*g} together with $\Delta^* : A^{*g} \otimes A^{*g} \to A^{*g}$ is a graded $\mathcal{A}s$-algebra with unit $1 \in (A^{*g})^{(0)} = K$. Furthermore, a cocommutative Δ leads to a commutative multiplication Δ^*.

Proof. This is just the classical result, modified using the fact that a $\mathcal{P}$-algebra structure on A induces a $\mathcal{P}$-coalgebra structure on A^{*g}. □

We consider primitive elements, i.e. the elements f such that $\Delta(f) = f \otimes 1 + 1 \otimes f$, with respect to the comultiplication Δ_a on the free $\mathcal{P}$-algebra $F_{\mathcal{P}}(V_X)$ (over an operad $\mathcal{P}$ with coherent unit action).

The following lemma reflects that the composition of primitive elements is again primitive and gives the definition of the operad of primitives (compare also [11,19]). We recall that the space of elements in $F_{\mathcal{P}}(V_{\{x_1,x_2,\dots,x_n\}})^{(n)}$ which are multilinear (i.e. have degree 1 with respect to each variable x_i) can be identified with $\mathcal{P}(n)$, for any operad $\mathcal{P}$.

Lemma 10. *Let $\mathcal{P}$ be as in Lemma* 6. *The Σ_n-spaces* $\operatorname{Prim}\mathcal{P}(n)$ *of multilinear primitive elements in $F_{\mathcal{P}}(V_{\{x_1,x_2,\dots,x_n\}})^{(n)}$ define a sub-operad* $\operatorname{Prim}\mathcal{P}$ *of $\mathcal{P}$, with free algebra functor $F_{\operatorname{Prim}\mathcal{P}} =$* $\operatorname{Prim} F_{\mathcal{P}}$.

Proof. (1) Let $A = K \oplus F_{\mathcal{P}}(V_X)$ together with Δ_a be the connected graded $\mathcal{P}$-Hopf algebra defined in Lemma 6. The image of an element $p(x_1, x_2, \dots)$ of A under Δ_a is given by

$$\Delta_a\bigl(p(x_1, x_2, \dots)\bigr) = p_{A\otimes A}(x_1 \otimes 1 + 1 \otimes x_1, x_2 \otimes 1 + 1 \otimes x_2, \dots).$$

This can be expanded into a sum. To simplify notation, we treat the case where $p \in M_n$ is a generating operation. Two of the summands are $p(x_1, x_2, \dots) \otimes 1$ and $\star_n(1, 1, \dots) \otimes p(x_1, x_2, \dots) = 1 \otimes p(x_1, x_2, \dots)$, because the unit action is coherent. Let $n \geqslant 2$, and let us collect all summands with first tensor component x_1, say. Therefore we have to compute

$$p_{A\otimes A}\bigl(x_1 \otimes 1, 1 \otimes x_2, 1 \otimes x_3, \dots, 1 \otimes x_n\bigr) = \underbrace{\star_n(x_1, 1, \dots, 1)}_{x_1} \otimes p(1, x_2, x_3, \dots, x_n).$$

Similarly, in the Δ_a-image of every multilinear $p(x_1, x_2, \dots, x_n)$, the term $x_j \otimes p(\dots, x_{j-1}, 1, x_{j+1}, \dots)$ collects all summands with first tensor component x_j. Moreover this implies that every multilinear primitive $p(x_1, x_2, \dots, x_n)$ is mapped to 0 if any k variables are replaced by 1, $1 \leqslant k \leqslant n - 1$.

(2) Given a sequence $(g_1, g_2, \dots)$ of primitive elements of A, we consider the $\mathcal{P}$-algebra morphism $\gamma = \gamma_{(g_1,g_2,\dots)} : A \to A$ with $x_i \mapsto g_i$ (all $x_i \in X$). Then

$$(\Delta_a \circ \gamma)(x_i) = g_i \otimes 1 + 1 \otimes g_i = \bigl((\gamma \otimes \gamma) \circ \Delta_a\bigr)(x_i).$$

Now let $p(x_1, x_2, \dots, x_n)$ be a multilinear primitive element and let $g := \gamma(p(x_1, x_2, \dots, x_n))$. We claim that g is primitive. Since Δ_a and γ are $\mathcal{P}$-algebra morphisms,

$$\begin{aligned}\Delta_a(g) &= (\Delta_a \circ \gamma \circ p)(x_1, x_2, \dots) = p_{A\otimes A}\bigl(\Delta_a(g_1), \Delta_a(g_2), \dots\bigr)\\ &= p_{A\otimes A}(g_1 \otimes 1 + 1 \otimes g_1, g_2 \otimes 1 + 1 \otimes g_2, \dots).\end{aligned}$$

As in step (1), we can expand into a sum where two of the summands are

$$\underbrace{p(g_1, g_2, \dots)}_{g} \otimes 1 \quad \text{and} \quad 1 \otimes \underbrace{p(g_1, g_2, \dots)}_{g}.$$

The other summands yield 0 by step (1), because $\gamma(p(1, x_2, x_3, \ldots, x_n)), \ldots, \gamma(p(1, 1, \ldots, 1, x_n))$ are 0.

(3) The vector space $\mathrm{Prim}\,\mathcal{P}(n)$ is a Σ_n-space: This follows from step (2), when we consider the maps γ for $(g_1, g_2, \ldots, g_n)$ any permutation of $(x_1, x_2, \ldots, x_n)$. In the case of arbitrary primitive elements $g_i = q_i(x_1, x_2, \ldots)$ that are homogeneous of a degree m_i, step (2) shows that there are well-defined composition maps for $\mathrm{Prim}\,\mathcal{P}$. Thus we have constructed an operad $\mathrm{Prim}\,\mathcal{P}$. The vector space $\mathrm{Prim}(F_{\mathcal{P}}(V_X))$ is a $\mathrm{Prim}\,\mathcal{P}$-subalgebra of $F_{\mathcal{P}}(V_X)$ which is free on V_X. $\square$

Remark 11. The operad $\mathrm{Prim}\,\mathcal{A}s$ is $\mathcal{L}ie$ (cf. [19]). This follows from the theorem of Friedrichs (cf. [28]), which states that Lie polynomials are exactly the polynomials in non-commuting associative variables which are primitive under Δ_a. The theorem of Cartier–Milnor–Moore (cf. [27, Appendix B]), together with the theorem of Poincaré–Birkhoff–Witt, shows that the category of cocommutative connected graded Hopf algebras is equivalent to the category of Lie algebras.

The operad $\mathcal{D}end$ of dendriform algebras of [18] can be provided with a coherent unit action such that $| \prec y = 0, x \prec | = x, x \succ | = 0, | \succ y = y$, see [19]. The free $\mathcal{D}end$-algebra $(K\,\mathrm{YTree}^\infty, \prec, \succ)$ can be provided with the structure of a $\mathcal{D}end$-Hopf algebra (see [20]). It is exactly the $\mathcal{D}end$-Hopf algebra structure given by Δ_a. Ronco [29] has determined the primitive elements, the operad $\mathrm{Prim}\,\mathcal{D}end$ is the operad of $\mathcal{B}race$-algebras, special pre-Lie algebras equipped with n-ary operations $\langle \ldots \rangle : A^{\otimes n} \to A$ for each n. For the operad $\mathcal{D}end$ they play the role of $\mathcal{L}ie$-algebras in the analogues of Cartier–Milnor–Moore and Poincaré–Birkhoff–Witt theorems [4,30]. Further results in this direction can be found in [22].

4. $\mathcal{M}ag_N$-Hopf algebras with duals related to shuffles

We consider the connected graded $\mathcal{M}ag_\omega$-Hopf algebra $A = K\{X\}_\omega$ with Δ_a, and we consider $K\{X\}_N$ as a sub-$\mathcal{M}ag_N$-Hopf algebra for each N.

Lemma 12. *The $\mathcal{M}ag_N$-Hopf algebras $K\{X\}_N$, $2 \leqslant N \leqslant \omega$, are cocommutative* (*in the sense of Lemma* 6).

The following lemma generalizes the formula for $N = 2$ given in [11].

Lemma 13. *Let T be a tree in $K\{X\}_\omega$. For I in* $\mathrm{Le}(T)$ *let I^c denote the complement of I in* $\mathrm{Le}(T)$. *Then the image $\Delta_a(T)$ of T is given by the formula*

$$\sum_{I \subseteq \mathrm{Le}(T)} \mathrm{red}(T|I) \otimes \mathrm{red}\bigl(T|I^c\bigr),$$

where the construction of $\mathrm{red}(T|I)$ *can be sketched as follows*: *All vertices and edges of T that do not lie on some path from a leaf in I to the root are removed from T. Then the necessary contractions are made to obtain a reduced* (*admissibly labeled*) *tree* $\mathrm{red}(T|I)$.

Proof. If $\mathrm{Le}(T) = \emptyset$, the formula says that $\Delta_a(1) = 1 \otimes 1$. If T consists of one leaf labeled by x_k, then

$$\Delta_a(T) = x_k \otimes 1 + 1 \otimes x_k = \mathrm{red}\bigl(T|\mathrm{Le}(T)\bigr) \otimes \mathrm{red}(T|\emptyset) + \mathrm{red}(T|\emptyset) \otimes \mathrm{red}\bigl(T|\mathrm{Le}(T)\bigr).$$

Assume now that T has at least two leaves. We may iteratively apply the $\mathcal{M}ag_\omega$-algebra morphism property

$$\Delta_a \circ \vee^n = \vee^n_{A\otimes A} \circ (\Delta_a \otimes \cdots \otimes \Delta_a)$$

of Δ_a and expand $\vee^{n_1}_{A\otimes A}(\vee^{n_2}_{A\otimes A} \ldots (\ldots, x_k \otimes 1 + 1 \otimes x_k, \ldots), \ldots)$ distributively to show that $\Delta_a(T)$ is given by summands where the tree T is splitted as indicated, according to the choice of $I \subseteq \mathrm{Le}(T)$. $\square$

Definition 14. Given two planar admissibly labeled trees T^1, T^2 with n_1, n_2 leaves, and a planar tree T with $n_1 + n_2$ leaves, we say that T is a shuffle of T^1 and T^2 in $K\{X\}_\omega$, if

$$\mathrm{red}(T|I) = T^1, \qquad \mathrm{red}\big(T|I^c\big) = T^2$$

for some subset $I \subseteq \mathrm{Le}(T)$.

Similarly defined is the notion of a shuffle in $K\{X\}_N$. If T^1, T^2 both belong to $K\{X\}_N$, $N \in \mathbb{N}$, one can consider their shuffles in $K\{X\}_N$ and the (larger) set of shuffles in $K\{X\}_\omega$.

For any tree T in $K\{X\}_\omega$ we define a K-linear map $\partial_T : K\{X\}_\omega \to K\{X\}_\omega$ by

$$\Delta_a(f) = \sum_{T \in \mathrm{PRTree}\{M^X\}} T \otimes \partial_T(f),$$

where $f \in K\{X\}_\omega$. Especially, for $T = 1_{K\{X\}_\omega} = \emptyset$, $\partial_\emptyset = \mathrm{id}$.

Remark 15. The map ∂_T may be called a generalized differential operator. For any $x_k \in X$, ∂_{x_k} is the unique mapping $D : K\{X\}_\omega \to K\{X\}_\omega$ with

$$D(x_l) := \begin{cases} 1, & k = l, \\ 0, & k \neq l, \end{cases}$$

satisfying the Leibniz rule

$$D\big(\vee^n(v_1, \ldots, v_n)\big) = \sum_{i=1}^{n} \vee^n\big(v_1, \ldots, v_{i-1}, D(v_i), v_{i+1}, \ldots, v_n\big) \quad \text{for all } v_i.$$

Proposition 16.

(i) *For $S, T \in \mathrm{PRTree}\{M^X\}$, $T = \vee^p(T^1.T^2 \ldots T^p)$, we have*

$$\partial_S(T) = \sum_{\vee^p(S^1.S^2 \ldots S^p) = S} \vee^p\big(\partial_{S^1}\big(T^1\big) \ldots \partial_{S^p}\big(T^p\big)\big),$$

where the sum is over all not necessarily non-empty trees $S^1, S^2, \ldots, S^p$ such that $\vee^p(S^1.S^2 \ldots S^p) = S$.

(ii) *Let $x_k \in X$, and let $W(k,n)$ be the set of reduced trees with n leaves that are all labeled by x_k. Then*:

$$\sum_{S \in W(k,n)} \partial_S = \frac{1}{n!}(\partial_{x_k})^n.$$

Proof. Since Δ_a is a $\mathcal{M}ag_\omega$-algebra morphism, we get assertion (i). To prove the equation in assertion (ii), we apply both sides to $T = \vee^p(T^1 \dots T^p)$. The assertion is trivial for $n = 0$, $n = 1$, or if T has less than n leaves. On the one hand, we have

$$\frac{1}{n!}(\partial_{x_k})^n(T) = \frac{1}{n!} \sum_{i_1+\cdots+i_p=n} \binom{n}{i_1, \dots, i_p} \vee^p\big((\partial_{x_k})^{i_1}\big(T^1\big) \dots (\partial_{x_k})^{i_p}\big(T^p\big)\big).$$

On the other hand, by (i),

$$\sum_{S \in W(k,n)} \partial_S(T) = \sum_{i_1+\cdots+i_p=n} \sum_{S^j \in W(k,i_j)} \vee^p\big(\partial_{S^1}\big(T^1\big) \dots \partial_{S^p}\big(T^p\big)\big),$$

and assertion (ii) follows by induction. □

Lemma 17. *For f homogeneous of degree n, f is primitive if and only if $\partial_T(f) = 0$ for all monomials $T \in K\{X\}_\omega$ with $1 \leqslant \deg T < (n+1)/2$.*

Proof. By definition f is primitive if and only if $\partial_T(f) = 0$ for all monomials $T \in K\{X\}_\omega$. Using the cocommutativity of Δ_a, the criterion follows. □

The commutator operation $[x,y] := \vee^2(x,y) - \vee^2(y,x) \in \operatorname{Prim}\mathcal{M}ag_\omega(2)$ is only the first in a large number of primitive operations, see Example 31.

In view of Lemma 9, we can describe the graded dual $K\{X\}_\omega^{*g}$ of the $\mathcal{M}ag_\omega$-Hopf algebra $K\{X\}_\omega$ equipped with Δ_a as follows. In analogy to the classical case, see Remark 7, the commutative associative binary operation corresponding to Δ_a is called planar tree shuffle multiplication (or planar shuffle product), and the coefficients $\binom{T}{T^1, T^2}$ are called planar binomial coefficients, see [10].

Proposition 18. *The vector spaces $K\{X\}_\omega$ and $K\{X\}_\omega^{*g}$ can be identified by mapping the basis given by trees T on the corresponding dual basis elements δ_T. Then the commutative associative multiplication Δ_a^* is given by the binary operation $\sqcup\!\sqcup : K\{X\}_\omega \otimes K\{X\}_\omega \to K\{X\}_\omega$ induced by*

$$T^1 \otimes T^2 \mapsto \sum_{\text{all shuffles } T \text{ of } T^1 \text{ and } T^2} \binom{T}{T^1, T^2} T \quad \text{for tree monomials } T^1, T^2,$$

where $\binom{T}{T^1, T^2} \geqslant 1$ is the number of subsets $I \subseteq \operatorname{Le}(T)$ with

$$\operatorname{red}(T|I) = T^1, \qquad \operatorname{red}\big(T|I^c\big) = T^2.$$

Especially $1 \sqcup\!\sqcup 1 = 1$.

Remark 19.

(i) In the case where T^1 has k leaves labeled bijectively by $x_1, \dots, x_k$, and T^2 has $n-k$ leaves labeled bijectively by $x_{k+1}, \dots, x_n$, then no coefficient $\binom{T}{T^1, T^2} > 1$ can occur. We get a generalization of the well-known shuffle multiplication of permutations $\sqcup\!\sqcup : K\Sigma_k \times K\Sigma_{n-k} \to K\Sigma_n$.

(ii) The graded dual of the $\mathcal{M}ag_N$-Hopf algebra $K\{X\}_N$ equipped with Δ_a is the quotient of $(K\{X\}_\omega, \sqcup\!\sqcup)$ with respect to the projection $K\{X\}_\omega \to K\{X\}_N$ which is the identity on the subspace $K\{X\}_N$ and 0 on its complement. By abuse of notation, we denote all these shuffle multiplications by $\sqcup\!\sqcup$.

Example 20. The product $x_1 \sqcup\!\sqcup x_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_n$ is given by

$$\sum_{T \in \mathrm{PRTree}^n} \sum_{\sigma \in \Sigma_n} T^\sigma,$$

where T^σ is the admissibly labeled tree with first leaf labeled by $x_{\sigma(1)}$, second by $x_{\sigma(2)}$, and so on. This can be shown by induction (the case $n = 1$ being trivial) using the fact that every term of $x_1 \sqcup\!\sqcup x_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_n$ occurs in a unique way as a shuffle of a term of $x_1 \sqcup\!\sqcup x_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup x_{n-1}$ and x_n.

5. An analogon of Poincaré–Birkhoff–Witt

As a vector space (in fact as a coalgebra) the free $\mathcal{A}s$-algebra on V is isomorphic to the free $\mathcal{C}om$-algebra generated by all Lie polynomials, i.e. by the primitive elements. This is the Poincaré–Birkhoff–Witt theorem.

To describe the operads $\mathrm{Prim}\,\mathcal{M}ag_N$, we are going to use an analogon of this theorem.

First, we need to describe the primitive elements as irreducible elements with respect to the shuffle multiplication, and we also need a description of the operation $(\vee^2)^* : K\{X\}_\omega \to K\{X\}_\omega \otimes K\{X\}_\omega$.

Proposition 21. *Let $f \in K\{X\}_N^{(n)}$ be homogeneous of degree $n \geqslant 1$.*

(i) *If f is primitive, then $\langle f, g_1 \sqcup\!\sqcup g_2 \rangle = 0$ for all homogeneous $g_1, g_2 \in K\{X\}_N$ of degree $\geqslant 1$.*

(ii) *If S is an admissibly labeled tree of degree $1 \leqslant k \leqslant n-1$, then $\partial_S(f) = 0$ if and only if $\langle f, S \sqcup\!\sqcup g \rangle = 0$ for all $g \in K\{X\}_N^{(n-k)}$.*

(iii) *If the homogeneous element f of degree n is orthogonal to all shuffle products $S \sqcup\!\sqcup T$ of trees $S \in K\{X\}_N^{(k)}$, $T \in K\{X\}_N^{(n-k)}$ with $1 \leqslant k < (n+1)/2$, then f is primitive.*

Proof. We can consider elements of $K\{X\}_\omega$. By definition, $\langle \Delta_a(f), g_1 \otimes g_2 \rangle = \langle f, g_1 \sqcup\!\sqcup g_2 \rangle$. Clearly, if f is primitive, $\langle f, g_1 \sqcup\!\sqcup g_2 \rangle = \langle f, g_1 \rangle \langle 1, g_2 \rangle + \langle 1, g_1 \rangle \langle f, g_2 \rangle = 0$.

If $\partial_S(f) \neq 0$ for some tree monomial S of degree $1 \leqslant k \leqslant n-1$, then by definition

$$0 \neq \langle \Delta_a(f), S \otimes g \rangle = \langle f, S \sqcup\!\sqcup g \rangle \quad \text{for some } g \in K\{X\}_\omega^{(n-k)}.$$

If f is orthogonal to all shuffle products $S \sqcup\!\sqcup S'$, S, S' trees with $\deg S = k \geqslant 1$, then

$$\Delta_a(f) = \sum_{T \in \mathrm{PRTree}\{M^X\} - \{S\}} T \otimes \partial_T(f)$$

and $\partial_S(f) = 0$. Then assertion (iii) follows by Lemma 17. $\square$

Definition 22. Let $N \in \mathbb{N}_{\geqslant 2} \cup \{\omega\}$. The maps given by $(\vee^k)^* : K\{X\}_N^{*g} \to (K\{X\}_N^{*g})^{\otimes k}$ are simply denoted by ∇_k, for each $k \leqslant N$, see Lemma 9.

Especially, we consider ∇_2 as a map $K\{X\}_N \to K\{X\}_N \otimes K\{X\}_N$ for each N.

Lemma 23. *We consider the unitary $\mathcal{C}om$-algebras $A = (K\{X\}_N, \sqcup\!\sqcup)$, $N \in \mathbb{N}_{\geqslant 2} \cup \{\omega\}$. By $\bar{A}$ we denote the augmentation ideal. Then*:

(i) *The K-linear map $\nabla_2 : A \to A \otimes A$ is a morphism of $\mathcal{C}om$-algebras.*
(ii) *The K-linear map ∇_2 is non-coassociative. If T is an admissibly labeled tree, then*

$$\nabla_2(T) = \begin{cases} T \otimes 1 + 1 \otimes T + T^1 \otimes T^2, & T = \vee^2(T^1.T^2), \\ T \otimes 1 + 1 \otimes T, & \mathrm{ar}_\rho \neq 2 \ (\rho \text{ the root}). \end{cases}$$

(iii) *If $\nabla_2'(f) := \nabla_2(f) - f \otimes 1 - 1 \otimes f$, then $\nabla_2'(f) \in \bar{A} \otimes \bar{A}$ for $f \in \bar{A}$.*

*In other words, these unitary $\mathcal{C}om$-algebras are co-**D** objects, where **D** is the category of unitary magmas.*

Proof. (1) The chosen unit actions on $\mathcal{M}ag_N$ and $\mathcal{C}om$ have the property that the operations on the tensor product $A \otimes A$ are defined component-wise. We have to check that

$$\nabla_2 \circ \sqcup\!\sqcup = \underbrace{(\sqcup\!\sqcup \otimes \sqcup\!\sqcup) \circ \tau_2}_{\sqcup\!\sqcup_{A \otimes A}} \circ (\nabla_2 \otimes \nabla_2).$$

By looking at the graded dual, this is equivalent to the equation

$$\Delta \circ \vee^2 = \left(\vee^2 \otimes \vee^2\right) \circ \tau_2 \circ (\Delta \otimes \Delta).$$

The latter equation is fulfilled, because $(\vee^2 \otimes \vee^2) \circ \tau_2$ is $\vee^2_{A \otimes A}$ and $\Delta : A \to A \otimes A$ is a morphism of $\mathcal{M}ag$-algebras.

(2) Since ∇_2 is determined by the equation

$$\left\langle \vee^2(f_1.f_2), g \right\rangle = \left\langle f_1 \otimes f_2, \nabla_2(g) \right\rangle$$

we conclude that, for T an admissibly labeled tree,

$$\nabla_2(T) = T \otimes 1 + 1 \otimes T + \sum_{T = \vee^2(T^1.T^2)} T^1 \otimes T^2.$$

It also follows that ∇_2 is not coassociative. We note that, for $f, g \in A$,

$$\langle f, g\rangle = \langle \vee^2(1.f), g\rangle = \langle 1 \otimes f, \nabla_2(g)\rangle = \langle f \otimes 1, \nabla_2(g)\rangle.$$

Thus assertion (iii) follows.

(3) The categorical coproduct for $\mathcal{C}om$-algebras is the tensor product $\otimes$. By (i) and (iii) this means that ∇_2 provides the unitary $\mathcal{C}om$-algebras $A = (K\{X\}_N, ⧢)$ with the structure of a co-magma object, with counit given by the augmentation map. □

Remark 24. The vector space $K\{X\}_N$ equipped with ∇_2 is a non-associative coalgebra in the sense of [13].

Theorem 25. *Let, for $N \in \mathbb{N}_{\geqslant 2} \cup \{\omega\}$, $A = K\{X\}_N$ be the unitary free $\mathcal{M}ag_N$-algebra. Let $W = \operatorname{Prim} A$ be its space of primitive elements, graded by $W^{(k)} = \operatorname{Prim} A^{(k)}$.*

(i) *The unitary $\mathcal{C}om$-algebra $(A, ⧢)$ is freely generated by $W = \operatorname{Prim} A$ with respect to the shuffle multiplication.*
(ii) *Let $n \geqslant 2$, and let $B^{(n)}$ be the orthogonal complement of $\operatorname{Prim} A^{(n)}$ in the vector space $A^{(n)}$. If $f_1, \dots, f_r$ is a basis (consisting of homogeneous elements) of $\bigoplus_{k=1}^{n-1} \operatorname{Prim} A^{(k)}$, then a basis of $B^{(n)}$ is given by the elements*

$$f_{i_1} ⧢ f_{i_2} ⧢ \cdots ⧢ f_{i_k}, \quad 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant r, \quad \textit{with} \sum_{j=1}^{k} \deg f_{i_j} = n.$$

Proof. (1) We have shown in Proposition 21 that the homogeneous primitive elements of A are also the homogeneous ⧢-irreducible elements of A, i.e. the elements f not of the form $g_1 ⧢ g_2$ (for f, g_1, g_2 homogeneous of degree $\geqslant 1$). Thus the unitary $\mathcal{C}om$-algebra morphism $\pi : K1 \oplus F_{\mathcal{C}om}(W) \to (A, ⧢)$ is surjective, and for every proper subspace U of W, $K1 \oplus F_{\mathcal{C}om}(U)$ cannot be isomorphic to $(A, ⧢)$.

(2) We have to show that π is injective, i.e. that $(A, ⧢)$ is a free $\mathcal{C}om$-algebra.

By Lemma 23, there exists a morphism $\nabla_2 : A \to A \otimes A$ of $\mathcal{C}om$-algebras, which provides the unitary $\mathcal{C}om$-algebras $(A, ⧢)$ with the structure of a co-**D** object in the category of $\mathcal{C}om$-algebras, where **D** is the category of unitary magmas.

Over a field K of characteristic 0, all connected (i.e. $A^{(0)} = K$) $\mathcal{C}om$-algebras that are equipped with the structure of a unital co-magma are free. This is the Leray theorem, see [25]. Thus π is an isomorphism.

(3) Since the space $\operatorname{Prim} A^{(n)}$ is orthogonal (with respect to $\langle , \rangle$) to the shuffle products in $A^{(n)}$, see Proposition 21, assertion (ii) follows from assertion (i). □

Dualizing the statement of Theorem 25(i) and its proof, we obtain:

Theorem 26. *The $\mathcal{M}ag_N$-Hopf algebras given by the free $\mathcal{M}ag_N$-algebras equipped with Δ_a are cofree co-nilpotent $\mathcal{C}om$-coalgebras, co-generated by their primitive elements.*

Remark 27.

(i) Given an arbitrary graded connected $\mathcal{M}ag_N$-Hopf algebra A, one can consider it as a $\operatorname{Prim}\mathcal{M}ag_N$-algebra, and its primitive elements as a ($\operatorname{Prim}\mathcal{M}ag$-)subalgebra. For K of characteristic 0, the situation is analogous to the classical Cartier–Milnor–Moore theorem (cf. [27, Appendix B]), and the Hopf algebra A is of the form $U(\operatorname{Prim} A)$. Of course the concrete description of the functor U depends on a description of the $\operatorname{Prim}\mathcal{M}ag$-operations.
Theorem 25(ii) shows that one may search for orthogonal projectors

$$e_n^{(1)}, e_n^{(2)}, \dots, e_n^{(n)} : K\{X\}_N^{(n)} \to K\{X\}_N^{(n)}$$

which are similar to the Eulerian idempotents (cf. [17]). The idempotent $e_n^{(i)}$ projects elements of $K\{X\}_N^{(n)}$ into the subspace of i-factor shuffle products of primitive elements. Here any series like the exponential series of [9] which maps primitive elements on group-like elements is useful.

(ii) Theorem 26 suggests the definition of a family of cocommutative $\mathcal{A}s$-Hopf algebras H_N related to the given $\mathcal{M}ag_N$-Hopf algebras. The idea is to have a free Lie algebra on a space C of graded generators associated to H_N by the Cartier–Milnor–Moore theorem, such that the cocommutative coalgebra structure of H_N is the same as the one on the corresponding $\mathcal{M}ag_N$-Hopf algebra.
One might consider the $\operatorname{Prim}\mathcal{M}ag_N$-operations $[\dots[x_n, x_{\delta 1}], x_{\delta 2}]\dots x_{\delta(n-1)}]$, $\delta \in \Sigma_{n-1}$, $n \geqslant 2$, and choose C such that these operations evaluated on C yield all primitive elements.

6. The generating series and representations

The generating series $f^{\mathcal{P}}(t)$ of an operad $\mathcal{P}$ is the series $\sum_{n\geqslant 1} \frac{\dim \mathcal{P}(n)}{n!} t^n$.

The classical Poincaré–Birkhoff–Witt theorem implies that the generating series of $\operatorname{Prim}\mathcal{A}s = \mathcal{L}ie$ is

$$f^{\mathcal{L}ie}(t) = \sum_{n\geqslant 1} \frac{(n-1)!}{n!} t^n = -\log(1-t),$$

because its composition with $f^{\mathcal{C}om} = \exp(t) - 1$ is $f^{\mathcal{A}s}(t) = 1/(1-t) - 1$.

Similarly, it is implied by Theorem 25 that the generating series of the operads $\operatorname{Prim}\mathcal{M}ag_N$ are the logarithms $\log(1+t)$ of the generating series of the corresponding $\mathcal{M}ag_N$. In fact this holds on the level of characteristic functions in the ring Λ of symmetric functions, equipped with the plethysm $\circ$. For the theory of symmetric functions, see [23]. Let p_r, $r \geqslant 1$, denote the $\mathbb{Q}$-basis of Λ given by the power sum symmetric functions $\sum_{i\geqslant 1} x_i^r$.

Corollary 28.

(i) *The characteristic of the Σ_n-module* $\operatorname{Prim}\mathcal{M}ag(n)$ *is given by*

$$\operatorname{ch}_n\bigl(\operatorname{Prim}\mathcal{M}ag(n)\bigr) = \frac{1}{n} \sum_{d|n} \mu(d) c'_{n/d} p_d^{n/d}$$

and the characteristic function $\mathrm{ch}(\mathcal{P}) = \sum_{n\geqslant 1} \mathrm{ch}_n(\mathcal{P}(n))$ *for* $\mathrm{Prim}\,\mathcal{M}ag$ *is given by*

$$\sum_{d\geqslant 1} \frac{\mu(d)}{d} \sum_{k\geqslant 1} \frac{c'_k}{k} p_d^k.$$

Here c'_k *is the kth* log-*Catalan number.*

(ii) *Especially it holds that the dimension of* $\mathrm{Prim}\,\mathcal{M}ag(n)$ *is given by*

$$\dim \mathrm{Prim}\,\mathcal{M}ag(n) = (n-1)!\,c'_n$$

and the generating series $f^{\mathrm{Prim}\,\mathcal{M}ag}(t)$ *is given by*

$$\log\left(\frac{3-\sqrt{1-4t}}{2}\right).$$

(iii) *The analogous assertions of* (i) *and* (ii) *hold for the operads* $\mathrm{Prim}\,\mathcal{M}ag_N$, $N \in \mathbb{N}_{\geqslant 2} \cup \{\omega\}$: *The sequence* c'_n *has to be replaced by the sequence* $c[N]'_n$. *Especially, for* $\mathcal{M}ag_\omega$ *the logarithmic derivative* C'_n *of the super-Catalan numbers* C_n *has to be taken. The generating series* $f^{\mathrm{Prim}\,\mathcal{M}ag_\omega}(t)$ *is given by*

$$\log\left(\frac{5+t-\sqrt{1-6t+t^2}}{4}\right).$$

Proof. We may modify a computation given in [12]. While the classical Poincaré–Birkhoff–Witt theorem implies that $\mathrm{ch}(\mathcal{L}ie) = \mathrm{Log}(1+\mathrm{ch}(\mathcal{A}s))$, Theorem 25 implies that $\mathrm{ch}(\mathrm{Prim}\,\mathcal{M}ag) = \mathrm{Log}(1+\mathrm{ch}(\mathcal{M}ag))$. Here the operation Log on symmetric functions is given by

$$\mathrm{Log}(1+f) = \sum_{d\geqslant 1} \frac{\mu(d)}{d} \sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} p_d^n \circ f.$$

Clearly $\mathrm{ch}(\mathcal{M}ag)$ is given by $\sum_{k\geqslant 1} c_k p_1^k$, thus $\sum_{n\geqslant 1} \frac{(-1)^{n+1}}{n} p_d^n \circ \mathrm{ch}(\mathcal{M}ag)$ is equal to

$$\log\left(1 + p_d \circ \sum_{k\geqslant 1} c_k p_1^k\right) = \log\left(1 + \sum_{k\geqslant 1} c_k p_d^k\right).$$

By definition of the log-Catalan number c'_k as the coefficient of t^{k-1} in the logarithmic derivative of $\sum_{k\geqslant 1} c_k t^k$, we get that

$$\log\left(1 + \sum_{k\geqslant 1} c_k t^k\right) = \sum_{k\geqslant 1} \frac{c'_k}{k} t^k.$$

Hence we get the asserted expressions for $\mathrm{ch}(\mathrm{Prim}\,\mathcal{M}ag)$ and $\mathrm{ch}_n(\mathrm{Prim}\,\mathcal{M}ag)$ in (i).

To show assertion (ii), one may repeat the same computation for generating functions instead of characteristic functions, or apply the rank morphism that maps $p_1 \mapsto t$ and $p_n \mapsto 0, n > 1$.

The same arguments, with Catalan numbers c_n replaced by the sequence $c[N]_n$, apply to $\mathrm{Prim}\,\mathcal{M}ag_N$, $N \in \mathbb{N}_{\geqslant 2} \cup \{\omega\}$. $\square$

Remark 29. The occurrence of log-Catalan numbers in dimension formulas for homogeneous elements of $\mathrm{Prim}\,K\{X\}$ has independently also been observed by Bremner, Hentzel, and Peresi in [3]. They consider a set X of r_i generators of multi-degree e_i, $i = 1, \ldots, N$, and show that the Witt dimension formula

$$\frac{1}{|\mathbf{n}|} \sum_{\substack{k,\mathbf{d} \\ k\mathbf{d}=\mathbf{n}}} \mu(k) c'_{|\mathbf{d}|} \binom{|\mathbf{d}|}{d_1, \ldots, d_N} r_1^{d_1} \cdots r_N^{d_N}$$

yields the dimension of the space of homogeneous primitive elements of multi-degree $\mathbf{n} = (n_1, n_2, \ldots, n_N)$.

Remark 30. Using Corollary 28 it is easy for small n to describe the representation of Σ_n given by $\mathrm{Prim}\,\mathcal{M}ag_N(n)$ in terms of irreducible representations. In the basis of Schur functions, one checks that ch_3 is $s_3 + 3s_{2,1} + s_{1,1,1}$ for $\mathrm{Prim}\,\mathcal{M}ag$, and $2s_3 + 5s_{2,1} + 2s_{1,1,1}$ for $\mathrm{Prim}\,\mathcal{M}ag_\omega$. We get $3s_4 + 10s_{3,1} + 6s_{2,2} + 10s_{2,1,1} + 3s_{1,1,1,1}$ for $\mathrm{Prim}\,\mathcal{M}ag(4)$, and $8s_4 + 25s_{3,1} + 16s_{2,2} + 25s_{2,1,1} + 8s_{1,1,1,1}$ for $\mathrm{Prim}\,\mathcal{M}ag_\omega(4)$.

These representations of Σ_n are not given by copies of the $(n-1)!$-dimensional representation $\mathcal{L}ie(n)$. But they occur as the representations of primitives associated to the cocommutative Hopf algebras mentioned in Remark 27(ii), because the homogeneous components of degree n of these Hopf algebras are $c[N]_n$ copies of the regular representation (in the $\mathcal{M}ag_N$-case). Thus one may obtain this type of representations, starting with free $\mathcal{A}s$-algebra generators, as was pointed out to us by J.-C. Novelli. Here one needs $e[N]_n$ generators in degree n, where the generating series $e(t)$ and $c(t)$ for $e[N]_n$ and $c[N]_n$ are related by $c(t) = 1/(1 - e(t)) - 1$.

A comment made by F. Chapoton is that it would be nice to have presentations of the operads $\mathrm{Prim}\,\mathcal{M}ag_N$ as Hadamard products, and one may ask for a possible (anti-)cyclic operad structure, see [5].

An intrinsic characterization of $\mathrm{Prim}\,\mathcal{M}ag_N$ by generators and relations would naturally be interesting, a question that was posed (for primitive elements in free $\mathcal{M}ag$-algebras) by Umirbaev and Shestakov in [36], see also [11].

Example 31. The commutator operation $\vee^2(x, y) - \vee^2(y, x) \in \mathrm{Prim}\,\mathcal{M}ag(2) = \mathrm{Prim}\,\mathcal{M}ag_\omega(2)$ is denoted by $[x, y]$. Let $\langle x, y, z\rangle := (x, z, y) - (x, y, z)$, where (x, y, z) is the associator operation $\vee^2(\vee^2(x, y), z) - \vee^2(x, \vee^2(y, z))$. Furthermore, let $\{x, y, z\} := (x, z, y) + (x, y, z)$.

The space $\mathrm{Prim}\,\mathcal{M}ag(3)$ has dimension 8 and is generated (as a Σ_3-module) by the operations $\langle x, y, z\rangle$, $\{x, y, z\}$, and $[[x, y], z]$, which fulfill

$$\langle x, z, y\rangle = -\langle x, y, z\rangle, \qquad \{x, z, y\} = \{x, y, z\}, \qquad \big[[y, x], z\big] = -\big[[x, y], z\big], \quad \text{and}$$

$$\text{the relation} \quad \sum_{a,b,c \text{ cyclic}} \langle a, b, c\rangle = \sum_{a,b,c \text{ cyclic}} \big[[a, b], c\big],$$

which is called the non-associative Jacobi relation (cf. [19]) or Akivis relation (cf. [36]). Here the operation $\langle x, y, z\rangle + [[y, z], x]$ generates only a 2-dimensional Σ_3-module.

The space $\operatorname{Prim}\mathcal{M}ag_3(3) = \cdots = \operatorname{Prim}\mathcal{M}ag_\omega(3)$ has dimension 14 and is generated by the operations above together with the operations $(x, y, z)_t := \vee^2(\vee^2(x, y), z) - \vee^3(x, y, z)$.

The space $\operatorname{Prim}\mathcal{M}ag(4)$ of dimension of $3! \cdot c'_3 = 78$ can be generated by the operations p and q given by $p(x, t, y, z) := (xt, y, z) - x \cdot (t, y, z) - t \cdot (x, y, z)$ and $q(x, t, y, z) := (x, ty, z) - y \cdot (x, t, z) - t \cdot (x, y, z)$.

Definition 32. (Cf. [26,36].) A vector space V (over a field K of characteristic 0) together with $(m+2)$-ary operations $\langle \mathbf{x}|y|z\rangle$, all $m \geqslant 0$, and $(m+n)$-ary operations $\Phi(\mathbf{x}|\mathbf{y})$, all $m \geqslant 1, n \geqslant 2$, is called a Sabinin algebra, if relations hold which can be abbreviated—using Sweedler's notation for $\Delta_a(x_1.x_2 \ldots x_r)$—by:

$$\langle \mathbf{x}|a|b\rangle = -\langle \mathbf{x}|b|a\rangle,$$

$$\Phi(\mathbf{x}|\mathbf{y}) = \Phi(x_{\tau 1}, \ldots, x_{\tau m}|y_{\delta 1}, \ldots, y_{\delta n}), \quad \text{all } \tau \in \Sigma_m,\ \delta \in \Sigma_n,$$

$$\langle x_1, \ldots x_r, a, b, z_{r+1}, \ldots z_m|c|d\rangle - \langle x_1, \ldots x_r, b, a, z_{r+1}, \ldots z_m|c|d\rangle$$
$$+ \sum \big\langle \mathbf{x}_{(1)}\langle \mathbf{x}_{(2)}|a|b\rangle z_{r+1} \ldots z_m \big| c \big| d\big\rangle = 0, \quad \text{all } r,$$

$$\sum_{a,b,c \text{ cyclic}} \Big(\langle \mathbf{x}c|a|b\rangle + \sum \big\langle \mathbf{x}_{(1)} \big| \langle \mathbf{x}_{(2)}|a|b\rangle \big| c\big\rangle\Big) = 0.$$

Remark 33. We used p and q to generate $\operatorname{Prim}\mathcal{M}ag(4)$. More generally, Shestakov and Umirbaev [36] show that the recursively defined operations

$$P(x_1, \ldots, x_m; y_1, \ldots, y_n; z) = (\mathbf{x}, \mathbf{y}, z) - {\sum}' \mathbf{x}_{(1)}\mathbf{y}_{(1)} P(\mathbf{x}_{(2)}; \mathbf{y}_{(2)}; z)$$

(together with $[x, y]$) give a complete set of primitive operations. Here products of more than two arguments are evaluated from left to right, and ${\sum}'$ indicates that there are no terms with $\mathbf{x}_{(i)} = \mathbf{y}_{(i)} = 1$. Moreover they show that the space B of primitive elements together with

$$\langle y|z\rangle := -[y, z], \quad \text{and} \quad \text{for } m \geqslant 1: \quad \langle \mathbf{x}|y|z\rangle := P(\mathbf{x}; z; y) - P(\mathbf{x}; y; z), \quad \text{and}$$

$$\text{for } m \geqslant 1, n \geqslant 2: \quad \Phi(\mathbf{x}|\mathbf{y}) := \sum_{\tau,\delta \in \Sigma} \frac{P(x_{\tau 1}, \ldots, x_{\tau m}; y_{\delta 1}, \ldots, y_{\delta(n-1)}; y_{\delta n})}{m!\,n!}$$

is a Sabinin algebra $G(B)$. Especially this means that these (multilinear) Sabinin operations form a sub-operad $\mathcal{S}ab$ of $\operatorname{Prim}\mathcal{M}ag$.

We denote $m!\,n!\,\Phi(\mathbf{x}|\mathbf{y})$ also by $\{\mathbf{x}|\mathbf{y}\}$.

Conjecture 34. *The operads $\mathcal{S}ab$ and* Prim *$\mathcal{M}ag$ are the same.*

This conjecture is enforced by recent results of Pérez-Izquierdo [26], which show that Sabinin algebras V have a universal enveloping ($\mathcal{M}ag$-)algebra with a Poincaré–Birkhoff–Witt basis (given by monomials on V) and that there is a Cartier–Milnor–Moore theorem for Sabinin algebras. The free $\mathcal{M}ag$-algebra also has a Poincaré–Birkhoff–Witt basis formed by primitive elements (by [36] or Theorem 26).

We know that $\operatorname{Prim}\mathcal{M}ag(n)=\mathcal{S}ab(n)$ for $n=2,3$. In the degree 4 component of $\operatorname{Prim}\mathcal{M}ag$ of dimension 78 the following Sabinin relation holds:

$$\sum_{a,b,c\text{ cyclic}}\langle x,c|a|b\rangle=\sum_{a,b,c\text{ cyclic}}\big(\big\langle x,[a,b],c\big\rangle+\big[\langle x,a,b\rangle,c\big]\big),$$

where $\langle x,t|y|z\rangle:=p(x,t,z,y)-p(x,t,y,z)$. By a computation we find that the Σ_4-subspace given by the iterated commutators and the operations $[\langle x,t,y\rangle,z]$, $[\{x,t,y\},z]$, $\langle[x,t],y,z\rangle$, $\langle x,[t,y],z\rangle$, $\{[x,t],y,z\}$, and $\{x,[t,y],z\}$ has dimension 65. Three more basis elements of $\operatorname{Prim}\mathcal{M}ag(4)$ may be given by $\langle x,t|y|z\rangle$, $\langle y,z|x|t\rangle$, and $\langle z,t|y|x\rangle$. Together with the (independent) $6+4$ operations given by $\{xt|yz\}$ and $\{x|tyz\}$, we arrive at a description of $\operatorname{Prim}\mathcal{M}ag(4)$ by Sabinin operations, confirming that $\operatorname{Prim}\mathcal{M}ag(4)=\mathcal{S}ab(4)$.

Remark 35. The commutative version of the operad $\mathcal{M}ag$ is the operad $\mathcal{C}mg$ of commutative magma algebras. It is the free operad generated by a free binary commutative operation. The trees symbolizing operations are abstract instead of planar trees. If we replace planar trees by abstract trees, we get 'commutative versions' $\mathcal{C}mg_N$ of the operads $\mathcal{M}ag_N$. In this paper we have focused on the operads $\operatorname{Prim}\mathcal{M}ag_N$. The same techniques can be applied to describe operads $\operatorname{Prim}\mathcal{C}mg_N$, and we will consider these operads in a subsequent paper. For $\mathcal{P}=\operatorname{Prim}\mathcal{C}mg$, we get that $\mathcal{P}(2)=0$ and $\mathcal{P}(3)$ is 2-dimensional, generated by the associator operation $(x_1,x_2,x_3)=(x_1\cdot x_2)\cdot x_3-x_1\cdot(x_2\cdot x_3)=-(x_3,x_2,x_1)$ subject to the relation $(x_1,x_2,x_3)+(x_2,x_3,x_1)+(x_3,x_1,x_2)=0$. In terms of irreducible representations (or Schur functions), ch_3 is $s_{2,1}$ and ch_4 is $s_4+s_{3,1}+s_{2,2}$ (with dimension 6).

Acknowledgments

This paper is based on my Habilitationsschrift [16]. I thank F. Chapoton, L. Gerritzen, J.-L. Loday, J.-C. Novelli, and I.P. Shestakov for helpful discussions and remarks. I also thank the referee for helpful suggestions.

References

[1] M. Aguiar, Infinitesimal Hopf algebras, in: N. Andruskiewitsch, et al. (Eds.), New Trends in Hopf Algebra Theory, in: Contemp. Math., vol. 267, Amer. Math. Soc., 2000, pp. 1–29.
[2] G.M. Bergman, A.O. Hausknecht, Cogroups and Co-Rings in Categories of Associative Rings, Math. Surveys Monogr., vol. 45, Amer. Math. Soc., 1996.
[3] M.R. Bremner, I.R. Hentzel, L.A. Peresi, Dimension formulas for the free nonassociative algebra, Comm. Algebra 33 (2005) 4063–4081.
[4] F. Chapoton, Un théorème de Cartier–Milnor–Moore–Quillen pour les bigèbres dendriformes et les algèbres braces, J. Pure Appl. Algebra 168 (2002) 1–18.
[5] F. Chapoton, On some anticyclic operads, Algebr. Geom. Topology 5 (2005) 53–69.
[6] A. Connes, D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, Comm. Math. Phys. 199 (1998) 203–242.
[7] E. Deutsch, L. Shapiro, A survey of the Fine numbers, Discrete Math. 241 (2001) 241–265.
[8] K. Ebrahimi-Fard, L. Guo, Coherent unit actions on operads and Hopf algebras, preprint, math.RA/0503342, 2005.
[9] L. Gerritzen, Planar rooted trees and non-associative exponential series, Adv. Appl. Math. 33 (2) (2004) 342–365.
[10] L. Gerritzen, Planar binomial coefficients, preprint, math.RA/0502380.
[11] L. Gerritzen, R. Holtkamp, Hopf co-addition for free magma algebras and the non-associative Hausdorff series, J. Algebra 265 (2003) 264–284.

[12] E. Getzler, Operads and moduli spaces of genus 0 Riemann surfaces, in: Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 199–230.
[13] G. Griffing, The cofree nonassociative coalgebra, Comm. Algebra 16 (1988) 2387–2414.
[14] K.H. Hofmann, K. Strambach, Topological and analytic loops, in: O. Chein, et al. (Eds.), Quasigroups and Loops: Theory and Applications, Heldermann Verlag, Berlin, 1990, pp. 205–262.
[15] R. Holtkamp, A pseudo-analyzer approach to formal group laws not of operad type, J. Algebra 237 (2001) 382–405.
[16] R. Holtkamp, On Hopf algebra structures over operads, Habilitationsschrift, Bochum, 2004.
[17] J.-L. Loday, Série de Hausdorff, idempotents Eulériens et algèbres de Hopf, Expo. Math. 12 (1994) 165–178.
[18] J.-L. Loday, Dialgebras, in: Dialgebras and Related Operads, in: Lecture Notes in Math., vol. 1763, Springer, 2001, pp. 7–66.
[19] J.-L. Loday, Scindement d'associativité et algèbres de Hopf, in: Actes des journées mathématiques à la mémoire de Jean Leray, Nantes, 2002, in: Séminaire et Congrès, vol. 9, SMF, 2004, pp. 155–172.
[20] J.-L. Loday, M. Ronco, Hopf algebra of the planar binary trees, Adv. Math. 139 (1998) 299–309.
[21] J.-L. Loday, M. Ronco, Algèbres de Hopf colibres, C. R. Acad. Sci. Paris Sér. I 337 (2003) 153–158.
[22] J.-L. Loday, M. Ronco, On the structure of cofree Hopf algebras, preprint, math.QA/0405330; J. Reine Angew. Math., in press.
[23] I.G. Macdonald, Symmetric Functions and Hall Polynomials, Clarendon Press, Oxford, 1995.
[24] I. Moerdijk, On the Connes–Kreimer construction of Hopf algebras, in: Contemp. Math., vol. 271, 2001, pp. 311–321.
[25] J.-M. Oudom, Théorème de Leray dans la catégorie des algèbres sur une opérade, C. R. Acad. Sci. Paris Sér. I 329 (1999) 101–106.
[26] J.M. Pérez-Izquierdo, Algebras, hyperalgebras, nonassociative bialgebras and loops, preprint, 2004.
[27] D. Quillen, Rational homotopy theory, Ann. of Math. (2) 90 (1969) 205–295.
[28] C. Reutenauer, Free Lie Algebras, London Math. Soc. Monogr., Oxford Univ. Press, New York, 1993.
[29] M. Ronco, Primitive elements in a free dendriform algebra, in: N. Andruskiewitsch, et al. (Eds.), New Trends in Hopf Algebra Theory, in: Contemp. Math., vol. 267, Amer. Math. Soc., 2000, pp. 245–263.
[30] M. Ronco, A Milnor–Moore Theorem for dendriform Hopf algebras, C. R. Acad. Sci. Paris Sér. I 332 (2000) 109–114.
[31] M. Ronco, Eulerian idempotents and Milnor–Moore theorem for certain non-cocommutative Hopf algebras, J. Algebra 254 (1) (2002) 152–172.
[32] N. Sloane (Ed.), The On-Line Encyclopedia of Integer Sequences, 2004, http://www.research.att.com/~njas/sequences/index.html.
[33] R.P. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Univ. Press, Cambridge, 1997.
[34] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Univ. Press, Cambridge, 1999.
[35] J.D. Stasheff, From operads to physically inspired theories, in: J.-L. Loday, et al. (Eds.), Operads, Proceeding of Renaissance Conferences, Amer. Math. Soc., 1997.
[36] I.P. Shestakov, U.U. Umirbaev, Free Akivis algebras, primitive elements, and hyperalgebras, J. Algebra 250 (2) (2002) 533–548.