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Global Stability in Job Systems

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1. THE MODEL

The main inward factor which determines the evolution of a population with many species (prey–predator, many species models) is the need for food arising directly from the natural instinct of the individuals to survive. Here we shall discuss a population model based on the need of *all* persons of the community to work in the environment considered. After formulating the model we shall discuss its global stability.

Consider a graded system [1] consisting of n grades (stages or levels) $g_1, g_2, ..., g_n$, where each member of the system is restricted to one and only one g_i. These grades represent the specified main and necessary jobs served by the population. Assume that the model is characterized by the following conditions:

(C1) A necessary and sufficient condition for a member to live in the community is that it is not unemployed; namely there are people in the community who need his service. Thus, if a member does not find work that person must leave the system. Let, for each $t \geqslant 0$, $x_i(t)$ be the population belonging to grade g_i $(i = 1, 2, ..., n)$.

(C2) None of the numbers $x_i(0)$ $(i = 1, 2, ..., n)$ is zero.

(C3) The per capita growth rate of the population $x_i(t)$ $(i = 1, 2, ..., n)$ increases if $x_j(t)$ becomes large for at least one $j \neq i$ and

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decreases if $x_i(t)$ becomes large. This means that if the population in g_j $(j \neq i)$ becomes large, then more persons are needed in stage g_i to give their service. But this cannot continue indefinitely; indeed if $x_i(t)$ is large enough with respect to those people who are needed to be in stage g_j, then they cannot find enough work and so some of them will leave the community.

Our model refers to large populations so that factors like death will be avoided.

Under the conditions above which characterize our competition-like job system we can mathematically formulate it as

$$\frac{\dot{x}_i(t)}{x_i(t)} = F_i(x_1(t), x_2(t), ..., x_n(t)), \qquad i = 1, 2, ..., n. \tag{1.1}$$

Here $F_i(z_1, z_2, ..., z_n)$ is a real-valued function which, according to (C3), decreases in z_i and increases in z_j for each $j \neq i$.

In this work we shall assume that F_i is linear, i.e.,

$$F_i(z_1, z_2, ..., z_n) = \sum_{j=1}^{n} a_{ij} z_j, \qquad i = 1, 2, ..., n$$

where a_{ij} are real numbers such that $a_{ii} < 0$ and $a_{ij} > 0$ for each $i = 1, 2, ..., n$ and $j \neq i$. The case where F_i is a general (nonlinear) function we hope will be discussed in a subsequent paper. Indeed, as we shall see later, the linearity of F_i plays an important role in using matrix theory to discuss the model, while nonlinear methods are needed to discuss the general case.

Equation (1.1) now becomes

$$\dot{x}_i(t) = x_i(t) \sum_{j=1}^{n} a_{ij} x_j(t), \qquad i = 1, 2, ..., n, \tag{e}$$

which is the subject of our study.

System (e) is of the form

$$\dot{x}_i(t) = x_i(t)\left(b_i + \sum_{j=1}^{n} a_{ij} x_j(t)\right), \qquad i = 1, 2, ..., n, \tag{1.2}$$

describing the interaction of a population living in a deterministic closed environment. However, observe that the auto-increase coefficients b_i $(i = 1, 2, ..., n)$ do not affect our system. Indeed, $b_i = 0$ in (e). Thus the results obtained and the methods suggested by several authors studying global stability of (1.2) do not cover and cannot apply to (e) (see, e.g., [3, 4] and the references therein). The reason is that in all the cases where (1.2) is examined the existence of a positive isolated equilibrium (namely a vector $x^* = (x_1^*, x_2^*, ..., x_n^*)^T$ such that $x_i^* > 0$ and $b_i + \sum_{j=1}^{n} a_{ij} x_j^* = 0$,

$i = 1, 2, \ldots, n$) is an essential condition. Such a condition cannot be assumed for (e) because of the homogeneity of the total growth rates.

As it is seen the only parameter in the asymptotic behavior for (e) is the coefficient matrix (a_{ij}). We shall show in the present paper that there are only three possible limiting cases for (e) as the time increases. Either the system decreases and becomes empty, or it increases in all stages and comes to include greater and greater population so that satiation will be observed in finite time, or it tends to an equilibrium point. Assuming that the population system is no more than a city, in the first case we have a decentralization from the city; second, we have a centralization to the city, and at last we have an asymptotic (nonzero) equilibrium. Note that the life of the city in the second case is *finite*. This means that in finite time the city will be full of people and a satiation will characterize it.

2. Some Preliminaries

Let us denote by A the coefficient matrix (a_{ij}). By our assumptions, A has the form

$$A = B - sI, \tag{2.1}$$

where B is an $n \times n$ matrix with positive entries (positive matrix), s is a strictly positive real number, and I is the $n \times n$ identity matrix.

Before proceeding to our discussion we borrow from the Perron–Frobenius theory [2, p. 26] the fact that for any $n \times n$ positive matrix W its spectral radius $\rho(W)$ is an eigenvalue and W admits a positive eigenvector w which corresponds to $\rho(W)$. Moreover, any positive eigenvector of W is a (positive) multiple of w.

We must make clear that if $A = B_1 - s_1 I$ is any other presentation of the matrix A, where again B_1 is a positive matrix and s_1 a positive number, then we can easily see it holds that

$$\rho(B) - s = \rho(B_1) - s_1. \tag{2.2}$$

Indeed, by the Perron–Frobenius theory there is a positive vector $z \in \mathbb{R}^n$ such that $Bz = \rho(B)\, z$. Since $B_1 = B - (s - s_1)\, I$, we have $B_1 z = (\rho(B) - s + s_1)\, z$, namely $\rho(B) - s + s_1$ is an eigenvalue of B_1. Since also $\rho(B_1)$ is an eigenvalue we must have $\rho(B_1) \geqslant \rho(B) - s + s_1$. Similarly we prove that $\rho(B) \geqslant \rho(B_1) - s_1 + s$, and so we have (2.2).

The observation above ensures that although the pair (B, s) in the presentation of A is arbitrary, the quantity $\rho(B) - s$ is a constant number which depends only on A and not on the pair (B, s). To follow the relative literature on matrix theory we recall that the matrix $-A$ is called an

M-matrix if the spectral radius $\rho(B)$ does not exceed s, where (B, s) is the pair in (2.1).

Let us denote by $\mathbb{R}_+^m$ the set of all vectors $z \in \mathbb{R}^n$ such that $z_i > 0$.

For now on we shall denote by $\langle x, y \rangle$ the usual inner product for vectors in $\mathbb{R}^n$.

In the sequel we shall always examine solutions $x(t)$ of (e) such that $x(0) \in \mathbb{R}_+^n$. It is clear that in this case we have $x(t) \in \mathbb{R}_+^n$ for all t in the domain of existence of the solution.

3. Convergence to the Origin

We first provide sufficient conditions for any (positive) solution of (e) to be convergent to the origin.

3.1. Theorem. *If* $-A$ *is a nonsingular M-matrix, any positive solution of* (e) *exists on* $[0, +\infty)$ *and the origin* $(0, 0, ..., 0)^{\mathrm{T}}$ *is a global attractor of* (e) *for all positive paths.*

Proof. From the theory of nonnegative matrices it is known that if C is a nonsingular M-matrix then for some positive diagonal matrix D the matrix $CD + DC^{\mathrm{T}}$ is positive definite (see, e.g., [2, p. 136, H_{24}]), where C^{T} is the transpose of C.

Now, letting $C = -A$ and for each $i = 1, 2, ..., n$ $y_i = d_i^{-1} x_i$, where $D = \operatorname{diag}[d_1, d_2, ..., d_n]$ is the matrix mentioned above corresponding to C, we observe that (e) becomes

$$\dot{y}_i = y_i \sum_{j=1}^{n} a_{ij} d_j y_j, \qquad i = 1, 2, ..., n. \tag{3.1}$$

Furthermore we shall show that the function

$$U(y(t)) \equiv u(t) = 2 \sum_{i=1}^{n} y_i(t)$$

is a Liapunov function for (3.1). Indeed, it holds $u(t) \geqslant 0$ and, moreover,

$$\begin{aligned}
\dot{u}(t) &= 2 \sum_{i=1}^{n} \dot{y}_i = 2 \sum_i y_i \sum_j a_{ij} d_j y_j \\
&= \sum_i y_i \sum_j a_{ij} d_j y_j + \sum_i y_i d_i \sum_j a_{ji} y_j \\
&= \langle y, ADy \rangle + \langle y, DA^{\mathrm{T}} y \rangle = \langle y, (AD + DA^{\mathrm{T}}) y \rangle \\
&= -\langle y, (CD + DC^{\mathrm{T}}) y \rangle.
\end{aligned}$$

Now, since $\dot{u}=0$ if and only if $y=0$ and the origin $(0, 0, \ldots, 0)^{\mathrm{T}}$ is an equilibrium for (3.1) it follows that this is a global attractor for (3.1). Consequently this does so for (e) too, and the theorem is proved.

4. Convergence to a Positive Finite State

Here we shall discuss the case where $-A$ is a singular M-matrix. Clearly $-A$ does so if and only if A can be written in the form $A=B-\rho(B)\,I$ for some positive matrix B. We shall show the following:

4.1. Lemma. *If $-A$ is a singular M-matrix, there exists exactly one vector $z \in \mathbb{R}^n_+$ $(w \in \mathbb{R}^n_+)$ such that $Az=0$ and $\sum_{i=1}^n z_i=1$ $(A^{\mathrm{T}}w=0$ and $\sum_{i=1}^n w_i=1)$. Moreover, a vector $u \in \mathbb{R}^n_+$ $(v \in \mathbb{R}^n_+)$ is a solution of $Au=0$ $(A^{\mathrm{T}}v=0)$ if and only if $u=\alpha z$ $(v=\beta w)$, where $\alpha>0$ $(\beta>0)$ is a real number.*

Proof. If $-A$ is a singular M-matrix, then its transpose $-A^{\mathrm{T}}$ does so. Thus it is enough to show the existence of z only.

Recall that A has the representation $A=B-\rho(B)\,I$, where B is a positive matrix. Thus its spectral radius $\rho(B)$ is a (positive) eigenvalue of B. Therefore if z is an eigenvector of B which corresponds to $\rho(B)$, this z must have all of its coordinates positive [2, p. 26]. Conversely, any such z corresponds to the eigenvalue $\rho(B)$. Obviously we can assume that $\sum_{i=1}^n z_i=1$. Furthermore, we observe that z satisfies $Az=Bz-\rho(B)\,z=0$.

Q.E.D.

Our main result in this section is the following:

4.2. Theorem. *Assume that $-A$ is a singular M-matrix. Then any (positive) solution $x(t)=(x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ of* (e) *exists on* $[0, +\infty)$ *and it holds*

$$\lim_{t\to\infty} x(t)=\left(\frac{x_1(0)}{z_1}\right)^{w_1}\left(\frac{x_2(0)}{z_2}\right)^{w_2}\cdots\left(\frac{x_n(0)}{z_n}\right)^{w_n} z,$$

where z and w are the vectors given in Lemma 4.1.

Proof. Let z and w be the vectors as above. Define the matrix $C=(c_{ij})$, where

$$c_{ij}=w_i a_{ij} z_j$$

and observe that it satisfies

$$0<\sum_{j\neq i} c_{ij}=-c_{ii}=|c_{ii}|. \tag{4.1}$$

We will show that $C+C^{\mathrm{T}}$ is negative semidefinite. To do this we have to show that $C+C^{\mathrm{T}}$ has all of its eigenvalues nonpositive. Indeed, put $D=C+C^{\mathrm{T}}$ and consider a $\lambda>0$. Then because of (4.1) $D-\lambda I$ has a strong diagonal dominance and by Corollary 5.4 of [6, p. 41] it is nonsingular. Thus all eigenvalues of D are nonpositive and by Theorem 2.3 of [6, p. 21], D is negative semidefinite.

Let us now consider the real valued function $U\colon \mathbb{R}^n_+ \to \mathbb{R}$, where

$$U(y)=2\sum_{i=1}^{n} w_i\left(\frac{y_i}{z_i}-1-\log\frac{y_i}{z_i}\right).$$

First observe that $U(y)\geqslant 0$ and we will show that U is a Liapunov function for the system (e). Indeed, if $x=x(t)$ is a solution of (e) set

$$u(t)=U(x(t)), \qquad t\geqslant 0.$$

Then

$$\begin{aligned}
\dot u(t)&=2\sum_{i=1}^{n} w_i\left(\frac{\dot x_i}{z_i}-\frac{\dot x_i}{x_i}\right)\\
&=2\sum_{i=1}^{n}\frac{w_i}{z_i}(x_i-z_i)\sum_{j=1}^{n} a_{ij}x_j\\
&=2\sum_{i=1}^{n}\frac{w_i}{z_i}(x_i-z_i)\sum_{j=1}^{n} a_{ij}(x_j-z_j)\\
&\quad+2\sum_{i=1}^{n}\frac{w_i}{z_i}(x_i-z_i)\sum_{j=1}^{n} a_{ij}z_j.
\end{aligned}$$

But $\sum_{j=1}^n a_{ij}z_j=(Az)_i=0$ and therefore we have

$$\begin{aligned}
\dot u(t)&=2\sum_{i=1}^{n}\frac{w_i}{z_i}(x_i-z_i)\sum_{j=1}^{n} a_{ij}(x_j-z_j)\\
&=2\sum_{i=1}^{n}\left(\frac{x_i}{z_i}-1\right)w_i\sum_{j=1}^{n} a_{ij}z_j\left(\frac{x_j}{z_j}-1\right)\\
&=2\sum_{i=1}^{n}\left(\frac{x_i}{z_i}-1\right)\sum_{j=1}^{n} c_{ij}\left(\frac{x_j}{z_j}-1\right)\\
&=2\langle y, Cy\rangle=\langle y,(C+C^T)\,y\rangle, \qquad \text{where } y_i=\frac{x_i}{z_i}-1.
\end{aligned}$$

As we have shown above, the matrix $C+C^{\mathrm{T}}$ is negative semidefinite and thus $\dot u(t)\leqslant 0$. Therefore U is a Liapunov function for (e).

Furthermore, since $\lim_{y\to+\infty} U(y)=+\infty$ it follows that any (positive) solution of (e) is bounded. Thus it is uniformly continuous and bounded, namely precompact.

Now, let us consider the equation

$$\dot{U}(x)|_{(e)}=0, \tag{4.2}$$

namely the equation

$$\langle y, Cy\rangle = 0, \qquad \text{with } y_i = x_i/z_i - 1. \tag{4.3}$$

Since any y which satisfies (4.3), maximizes the quantity $\langle y, Cy\rangle$ (recall that $C+C^{\mathrm{T}}$ is negative semidefinite) it must hold

$$0=\frac{\partial U}{\partial y_i}, \qquad i=1, 2, ..., n. \tag{4.4}$$

If there is such a y that satisfies $y_{i_0} \langle y_i$, $i \neq i_0$ then from (4.4) and (4.1) we get

$$0=\frac{\partial U}{\partial y_{i_0}}=\sum_{i\neq i_0} c_{i_0 i}(y_i - y_{i_0}) > 0,$$

a contradiction. Thus any solution y of (4.4) must have all of its coordinates equal to a constant, or otherwise the set

$$E=\{x \in \mathbb{R}^n \colon \dot{U}(x)|_{(e)}=0\}$$

consists exactly of those $x \in \mathbb{R}^n$ such that

$$x = kz \qquad \text{for some } k>0.$$

Finally, observe that E is invariant along solutions of (e) because of $Az=0$. Apply now the well-known LaSalle's invariance principle [5, p. 30] to get that any solution $x(t)$ of (e) satisfies

$$x(t) \to E \cap U^{-1}(l), \qquad \text{as} \quad t \to +\infty, \tag{4.5}$$

for some $l \geqslant 0$, where the convergence is in the Hausdorff sense. Since l depends only on $x(t)$, if k_0 is the (unique) real root of $ke^{l+1}=e^k$, then (4.5) gives

$$\lim_{t\to\infty} x(t)=k_0 z. \tag{4.6}$$

We shall obtain k_0 in terms of the initial value $x(0)$. To this end observe that the function

$$v(t)=\sum_{i=1}^{n} w_i \log x_i(t)$$

is constant. Indeed $\dot{v}(t) = \langle A^T w, x(t) \rangle = 0$. Thus $v(t) = v(0)$, $t \geqslant 0$, or $x_1^{w_1}(t) \cdots x_n^{w_n}(t) = x_1^{w_1}(0) \cdots x_n^{w_n}(0)$. This and (4.6) give, by passing to the limit, that

$$k_0 = \left(\frac{x_1(0)}{z_1}\right)^{w_1} \left(\frac{x_2(0)}{z_2}\right)^{w_2} \cdots \left(\frac{x_n(0)}{z_n}\right)^{w_n}, \tag{4.7}$$

which proves the theorem.

5. Convergence to $+\infty$

Consider now the case where

$$\rho(B) > s.$$

Clearly $\rho(B) - s = \lambda$ is a positive eigenvalue of A and by Perron–Frobenius theory A has a positive eigenvector z which corresonds to λ, i.e.,

$$Az = \lambda z.$$

Fix such a z and take a solution $x(t) = (x_1(t), ..., x_n(t))^T$ of system (e) for which $x_i(0) > 0$ $(i = \overline{1, n})$. Let us denote by $[0, T_x)$ the maximal existence interval of $x(t)$ and by $[0, T_1)$ $(\subseteq [0, T_x))$ the maximal interval such that $x_i(t) > 0$ $(0 \leqslant t < T_1,\ i = \overline{1, n})$. Such a T_1 exists, since $x_i(0) > 0$, $i = \overline{1, n}$.

Define a function u: $[0, \infty) \to \mathbb{R}$ of the type

$$u(t) = \min_{1 \leqslant i \leqslant n} \frac{x_i(t)}{z_i}, \qquad 0 \leqslant t < T_1.$$

For any fixed $t \in [0, T_1)$, there exists an index $i_0 = i_0(t)$ such that

$$u(t) = \frac{x_{i_0}(t)}{z_{i_0}},$$

and thus, we have

$$D^- u(t) = \limsup_{h \to 0+} \frac{u(t) - u(t-h)}{h} \geqslant \lim_{h \to 0+} \frac{x_{i_0}(t) - x_{i_0}(t-h)}{h z_{i_0}} = \frac{\dot{x}_{i_0}(t)}{z_{i_0}}.$$

From (e), we obtain

$$D^- u(t) \geqslant \frac{x_{i_0}(t)}{z_{i_0}} \sum_{j=1}^n a_{i_0 j} z_j \frac{x_j(t)}{z_j} \geqslant \sum_{j=1}^n a_{i_0 j} z_j u^2(t) = \lambda z_{i_0} u^2(t).$$

Thus

$$D^- u(t) \geqslant \gamma u^2(t), \qquad 0 \leqslant t < T_1, \tag{5.1}$$

where $\gamma = \lambda \cdot \min_{1 \leqslant i \leqslant n} \{z_i\}$ (>0).

Now let us consider the following initial-value problem:

$$\begin{aligned} \dot{v}(t) &= \gamma v^2(t) \\ v(0) &= u(0). \end{aligned} \tag{5.2}$$

This initial-value problem has exactly one solution

$$v(t) = \frac{v(0)}{1 - \gamma v(0)\, t}$$

with the maximal existence interval $[0, 1/\gamma v(0))$. But (5.1) and (5.2) imply that

$$u(t) = \min_{1 \leqslant i \leqslant n} \frac{x_i(t)}{z_i} \geqslant v(t) = \frac{v(0)}{1 - \gamma v(0)\, t} > 0, \tag{5.3}$$

on $0 \leqslant t < \min[T_1, 1/\gamma v(0))$. From this, it follows that $T_1 = \min\{T_x, 1/\gamma v(0)\}$, i.e., $x_i(t) > 0$, $(i = \overline{1, n})$, on $0 \leqslant t < \min\{T_x, 1/\gamma v(0)\}$ and inequality (5.3) is valid on this interval.

We shall show that $T_x \leqslant 1/\gamma v(0)$. To do this assume that $T_x > 1/\gamma v(0)$. Then from (5.3), we get

$$+\infty > \max_{0 \leqslant s \leqslant 1/\gamma v(0)} \min_{1 \leqslant i \leqslant n} \frac{x_i(t)}{z_i} \geqslant u(t) \geqslant \frac{v(0)}{1 - \gamma v(0)\, t} \to +\infty, \qquad \text{as } t \to \frac{1}{\gamma v(0)},$$

which is a contradiction, and so $T_x \leqslant 1/\gamma v(0)$.

Since $[0, T_x)$ ($\subseteq [0, 1/\gamma v(0))$) is the maximal existence interval of solution $x(t)$, we have $\sum_{i=1}^n x_i(t) \to +\infty$, as $t \to T_x$.

We shall show that if for an index $i_0 \in \{1, 2, ..., n\}$, $\limsup_{t \to T_x} x_{i_0}(t) = +\infty$, then $\liminf_{t \to +T_x} x_{i_0}(t) = +\infty$. To do this assume that i_0 is a fixed index such that there exists a sequence (t_k), with $t_k \to T_x$ $(k \to +\infty)$, and

$$\lim_{k \to +\infty} x_{i_0}(t_k) = +\infty,$$

but $m = \liminf_{t \to T_x} x_{i_0}(t) < \infty$. In that case there is a sequence (τ_k) such that $\tau_k \to T_x$ $(k \to +\infty)$, $x_{i_0}(\tau_k) = m + 1$, and $\dot{x}_{i_0}(\tau_k) \leqslant 0$ for all $k = 1, 2, ...$. Taking into account the fact that $\sum_{i=1}^n x_i(t) \to +\infty$, $t \to T_x$, we conclude that there is an index $j_0 \neq i_0$ such that $x_{j_0}(\tau'_k) \to +\infty$, for some subsequence (τ'_k) of (τ_k). Then from (e) we get the contradiction

$$0 \geqslant \dot{x}_{i_0}(\tau'_k) = x_{i_0}(\tau'_k)(a_{i_0 i_0} x_{i_0}(\tau'_k)$$

$$+ \sum_{j \neq i_0} a_{i_0 j} x_j(\tau'_k)) \to +\infty, \qquad \text{as} \quad k \to +\infty.$$

Thus $x_{i_0}(t) \to +\infty$, $(t \to T_x)$, whenever $\limsup_{t \to T_x} x_{i_0}(t) = +\infty$. But $\sum_{i=1}^n x_i(t) \to +\infty$ $(t \to T_x)$, implies the existence of an index i_0 such that $\limsup_{t \to T_x} x_i(t) = +\infty$. From the arguments above we get that for any $M > 0$ and index $i \neq i_0$ there exists $\tau \in (0, T_x)$ such that $x_{i_0}(t) > M/a_{ii_0}$, for $t \in (\tau, T_x)$. Then we have on one hand that

$$\dot{x}_i(t) \geqslant x_i(t)(a_{ii} x_i(t) + a_{ii_0} x_{i_0}(t)), \qquad t \in (0, T_x), \tag{5.4}$$

and, on the other hand,

$$\dot{x}_i(t) \geqslant x_i(t)(M + a_{ii} x_i(t)), \qquad t \in (\tau, T_x). \tag{5.5}$$

Assume that $\limsup_{t \to T_x} x_i(t) < +\infty$. Then $x_i(t)$ is bounded and by (5.4) it follows that $\lim_{t \to T_x} \dot{x}_i(t) = +\infty$. Thus $\lim_{t \to T_x} x_i(t) \equiv l$ exists. We shall show that $l = +\infty$. Indeed, otherwise we have $l < +\infty$. From (5.5) it follows that $x_i(t) \geqslant y_i(t)$, $t \in (\tau, T_x)$, where $y_i(t)$ is the solution of the Cauchy problem

$$\dot{y}_i = y_i(M + a_{ii} y_i)$$
$$y_i(\tau) = x_i(\tau).$$

Thus

$$l = \lim_{t \to T_x} x_i(t) \geqslant \frac{M x_i(\tau)}{(M + a_{ii} x_i(\tau))\, e^{-M(T_x - \tau)} - a_{ii} x_i(\tau)}. \tag{5.6}$$

Now take $k = 1, 2, ...,$ in the place of M and let (τ_k) be a sequence, where τ_k corresponds to k as above. Since $(x_i(\tau_k))$ is bounded we can assume that it converges, say, to r, which by $\lim_{t \to T_x} \dot{x}_i(t) = +\infty$ it must be nonzero. Then from (5.6) we get that

$$l \geqslant \lim_{k \to +\infty} \frac{k x_i(\tau_k)}{(k + a_{ii} x_i(\tau_k))\, e^{-k(T_x - \tau)} - a_{ii} x_i(\tau_k)} = +\infty.$$

Thus $l = +\infty$ and so $\lim_{t \to T_x} x_i(t) = +\infty$. This means that $\limsup_{t \to T_x} x_i(t) = +\infty$ and, following the same procedure as for i_0 above, we conclude that $\lim_{t \to T_x} x_i(t) = +\infty$.

Consequently we have proved the following theorem.

5.1. THEOREM. *If* $\rho(B) > s$, *then any solution* $x(t) = (x_1(t), ..., x_n(t))^{\mathrm{T}}$ $(x_i(0) > 0,\ i = \overline{1, n})$ *of* (e) *has a time life* T_x *no greater than* $\{[\rho(B) - s] \min z_i \cdot \min(x_i(0)/z_i)\}^{-1}$ *and for any* $i = \overline{1, n}$ *we have* $0 < x_i(t) \to +\infty$, *as* $t \to T_x$.

6. SOME COMMENTS AND A SPECIAL CASE

Above we examined Eq. (e) and we took a complete picture of the asymptotic behavior of its (positive) solutions. We see that if $\rho(B) \leqslant s$, then any solution of (e) is defined on $[0, +\infty)$, and it has a finite limit as the time increases. On the other hand, if $\rho(B) > s$ then any solution tends to $+\infty$ as the time tends to a finite time. Thus in this case the system has *finite life* with life length equal to T_x. An estimation of T_x is given in the proof of Theorem 5.1.

The conditions stated above involve the spectral radius of the martrix B and therefore the spectral set of A. We shall show here that in the case $n = 2$ we can get information for system (e) without having the set of eigenvalues of A but the sign of the determinant of A. Specifically, consider the system

$$\begin{aligned} \dot{x}_1 &= x_1(-a_{11}x_1 + a_{12}x_2) \\ \dot{x}_2 &= x_2(a_{21}x_1 - a_{22}x_2) \end{aligned} \tag{6.1}$$

and we shall show the following:

6.2. THEOREM. *If A is the coefficient matrix of* (6.1) *and* $x(t) = (x_1(t), x_2(t))^{\mathrm{T}}$ *a solution with* $x_i(0) > 0$, *then it holds*:

(i) *If* $\det A < 0$, *then for some* $T_x > 0$ *with*

$$T_x \leqslant \frac{1}{\lambda \min\{k, 1\} \cdot \min\{x_1(0)/k, x_2(0)\}},$$

where $\lambda = \frac{1}{2}(-a_{11} - a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}})$ *and* $k = (1/2a_{21})(a_{22} - a_{11} + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}})$, *we have*

$$\lim_{t \to T_x} x_i(t) = +\infty, \qquad i = 1, 2.$$

(ii) *If* $\det A > 0$, *then* $\lim_{t \to +\infty} x_i(t) = 0$, *and*

(iii) *if* $\det A = 0$, *then*

$$\lim_{t \to \infty} x(t) = \frac{1}{a_{11} + a_{12}} \left(\frac{(a_{11} + a_{12})\, x_1(0)}{a_{12}} \right)^{a_{21}/(a_{11} + a_{21})} \times \left(\frac{(a_{11} + a_{12})\, x_2(0)}{a_1} \right)^{a_{11}/(a_{11} + a_{21})} \begin{pmatrix} a_{12} \\ a_{11} \end{pmatrix}.$$

Proof. Write $A = B - sI$, where $s > 0$ and B is a positive 2×2 matrix and let $\Delta(\lambda) = \det(B - \lambda I)$.

(i) If $\rho(B) < s$, then, since $\Delta(s) = \det A < 0$ and $\lim_{\lambda \to +\infty} \Delta(\lambda) = +\infty$ there is a certain $s_1 > s > \rho(B)$ with $\Delta(s_1) = 0$, a contradiction because $\rho(B)$ is the greater eigenvalue of B. Thus $\rho(B) > s$ and the result follows from Theorem 5.1 (Fig. 1).

(ii) Here $\Delta(s) = \det A > 0$. If $s < \rho(B)$, then, since $\lim_{\lambda \to \pm\infty} \Delta(\lambda) = +\infty$ we have either the existence of a certain $s_1 > \rho(B)$ with $\Delta(s_1) = 0$, or $\rho(B)$ is of multiplicity 2. Both of these facts are impossible, because B is positive. Thus $s > \rho(B)$ and the result follows from Theorem 3.1 (Fig. 2).

(iii) Since $\Delta(s) = \det A = 0$ it follows that s is a positive eigenvalue of B such that $s \leqslant \rho(B)$. Then $\rho(B) \cdot s = \det B = b_{11} b_{22} - b_{12} b_{21} > 0$. But $b_{11} = s - a_{11}$, $b_{22} = s - a_{22}$, $b_{12} = a_{12}$, and $b_{21} = a_{21}$; thus $s(s - (a_{12} + a_{22})) > 0$, namely $s > a_{12} + a_{21}$. Now $\det(A - \lambda I) = 0$ if and only if $\lambda = 0$, or $\lambda = -(a_{12} + a_{21})$. Hence $\det(B - (s + \lambda) I) = 0$ if and only if $s + \lambda = s$, or $s + \lambda = s - (a_{12} + a_{21})$. Since $s > s + \lambda > 0$, it follows that $s = \rho(B)$. Now the result follows from Theorem 4.2 (Fig. 3).

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