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Crystal duality and Littlewood–Richardson rule of extremal weight crystals ☆

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ABSTRACT

We consider a category of $\mathfrak{gl}_\infty$-crystals, whose objects are disjoint unions of extremal weight crystals of non-negative level with certain finite conditions on the multiplicity of connected components. We show that it is a monoidal category under tensor product of crystals and the associated Grothendieck ring is anti-isomorphic to an Ore extension of the character ring of integrable lowest weight $\mathfrak{gl}_\infty$-modules with respect to derivations shifting the characters of fundamental weight modules. A Littlewood–Richardson rule of extremal weight crystals of non-negative level is described explicitly in terms of classical Littlewood–Richardson coefficients.

1. Introduction

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra associated with a symmetrizable Kac–Moody algebra $\mathfrak{g}$. In [11], Kashiwara introduced a class of integrable $U_q(\mathfrak{g})$-modules called extremal weight modules, which is a natural generalization of integrable highest weight or lowest weight modules. There exist a global crystal base and a crystal base of an extremal weight module, and the crystal base of an extremal weight module, simply an extremal weight crystal, appears as a subcrystal of that of the modified quantized enveloping algebra $\widetilde{U}_q(\mathfrak{g})$ [11]. Suppose that $\mathfrak{g}$ is an affine Kac–Moody algebra of finite rank. Then an extremal weight crystal of positive (resp. negative) level is isomorphic to the crystal base of an integrable highest (resp. lowest) weight module. In [1,13], a level zero extremal weight module has been studied in detail, and it was conjectured by Kashiwara [13] that the structure of a level zero extremal weight crystal can be described in terms of a tensor product of the crystal bases of level zero fundamental weight modules and Laurent Schur polynomials. In [3], Beck and Nakajima proved this conjecture (see also [2,24] for the case when $\mathfrak{g}$ is symmetric), and

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furthermore based on the study of extremal weight modules they proved the Kashiwara's conjecture on Peter–Weyl decomposition and the Lusztig's conjecture on two-sided cell structure of $\widetilde{U}_q(\mathfrak{g})$ in a purely algebraic way, though there is a geometric background related with quiver varieties.

A natural question arises whether there is also a nice description of extremal weight crystals and their tensor products when $\mathfrak{g}$ is an infinite rank affine Lie algebra. In [19] the author studied extremal weight crystals of type $A_{+\infty}$, and he showed that an extremal weight crystal is isomorphic to the tensor product of a lowest weight crystal and a highest weight crystal, and the Grothendieck ring generated by the isomorphism classes of extremal weight crystals is isomorphic to the Weyl algebra of infinite rank. A Littlewood–Richardson rule of extremal weight crystals is then described explicitly using the operators induced from the multiplication by Schur functions together with their adjoints.

The purpose of this paper is to study extremal weight crystals of type A_∞ (or extremal weight $\mathfrak{gl}_\infty$-crystals), where $\mathfrak{gl}_\infty$ denotes the infinite rank affine Lie algebra of type A_∞. In this case, it should be noted that (1) an extremal weight crystal is always connected (Proposition 4.1), (2) there are extremal weight crystals of non-zero level, which are not isomorphic to a highest weight or lowest weight crystal (Theorem 4.6), and (3) the tensor product of two extremal weight crystals of non-negative level is a disjoint union of extremal weight crystals of non-negative level (Theorem 5.1). These are important features of extremal weight $\mathfrak{gl}_\infty$-crystals, which do not necessarily hold in the affine types of finite rank. Also, as in the case of $A_{+\infty}$, we need certain non-commuting operators to describe the tensor product of extremal $\mathfrak{gl}_\infty$-crystals because of the non-existence of characters and the non-commutativity of tensor products.

Let us explain our results in detail. For an integral weight Λ of level $k \geqslant 0$, we denote by $\mathbf{B}(\Lambda)$ the crystal base of the extremal weight module over $U_q(\mathfrak{gl}_\infty)$ with extremal weight Λ. Let $\mathbf{B}$ be the crystal base of the natural representation of $U_q(\mathfrak{gl}_\infty)$. The connected components of $\mathbf{B}^{\otimes n}$ $(n \geqslant 1)$ are parameterized by partitions λ of n, say $\mathbf{B}_\lambda$. Note that the crystal $\mathbf{B}_\lambda$ is not isomorphic to a highest weight or lowest weight crystal. Then we show that $\mathbf{B}(\Lambda)$ is connected and there exist unique partitions μ, ν and the dominant integral weight Λ' of level k such that

$$\mathbf{B}(\Lambda) \simeq \mathbf{B}_\mu \otimes \mathbf{B}_\nu^\vee \otimes \mathbf{B}(\Lambda'),$$

where $\mathbf{B}_\nu^\vee$ is the dual crystal of $\mathbf{B}_\nu$ (Theorem 4.6). Note that $\mathbf{B}(\Lambda')$ is the crystal base of the integrable highest weight module with highest weight Λ'. An extremal weight crystal of non-positive level is also characterized from the above result by taking its dual.

Next, we consider a category $\mathcal{C}$ of $\mathfrak{gl}_\infty$-crystals, whose objects are disjoint unions of extremal weight crystals of non-negative level with certain finite conditions on the multiplicity of connected components (see Section 5.1). We show that $\mathcal{C}$ is a monoidal category under tensor product of crystals (Theorem 5.1). We remark that the tensor product in $\mathcal{C}$ is not necessarily commutative, that is, there is not always an isomorphism between $B \otimes B'$ and $B' \otimes B$ for any two objects B, B' in $\mathcal{C}$ (see for example, Proposition 4.9).

The Grothendieck group $\mathscr{K}$ of the category $\mathcal{C}$ admits a natural structure of a non-commutative associative $\mathbb{Z}$-algebra with 1 induced from the monoidal structure of $\mathcal{C}$. Let $z = \{z_k \mid k \in \mathbb{Z}\}$ be a set of formal commuting variables, and let $\mathscr{R}$ be the ring of formal power series in z with coefficients in $\mathbb{Z}$. Now, let $\mathscr{D}$ be an Ore extension of $\mathscr{R}_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} \mathscr{R}$ associated with a commuting family of derivations $\gamma_n^\pm = (-1)^{n-1} \sum_{k\in\mathbb{Z}} z_{k\mp n} \frac{\partial}{\partial z_k}$ $(n \in \mathbb{Z})$. We may view $\mathscr{D}$ as a non-commutative polynomial ring over $\mathscr{R}_\mathbb{Q}$ in $\gamma^\pm = \{\gamma_n^\pm \mid n \in \mathbb{Z}\}$. Then we show that there exists a $\mathbb{Q}$-algebra isomorphism

$$\mathscr{K}_\mathbb{Q} \xrightarrow{\sim} \mathscr{D}^{\mathrm{opp}}$$

(Theorem 5.7), where isomorphism classes of an integrable highest weight crystal and a level zero extremal weight crystal are mapped to polynomials in z and $\gamma^\pm$, respectively. Here $\mathscr{K}_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} \mathscr{K}$ and $\mathscr{D}^{\mathrm{opp}}$ denotes the opposite algebra of $\mathscr{D}$.

Based on the above results, we obtain a Littlewood–Richardson rule for extremal weight crystals of non-negative level (Theorem 5.14), which is given explicitly in terms of classical Littlewood–Richardson coefficients. In fact, the tensor product of level zero extremal weight crystals corresponds to the product of double symmetric functions, whose decomposition can be given by the classical Littlewood–Richardson rule due to a crystal $(\mathfrak{gl}_\infty, \mathfrak{gl}_n)$-duality on the n-th exterior algebra generated by the natural representation of $\mathfrak{gl}_\infty$ (Proposition 3.10), and the decomposition of the tensor product of integrable highest weight crystals is explained by using a crystal version of the $(\mathfrak{gl}_\infty, \mathfrak{gl}_n)$-duality on the level n fermionic Fock space due to Frenkel [5] (Proposition 3.11) following [30]. Hence the only non-trivial part is the decomposition of the tensor product $\mathbf{B}(\Lambda) \otimes \mathbf{B}_\mu \otimes \mathbf{B}_\nu^\vee$, where Λ is a dominant integral weight and μ, ν are partitions, and it is obtained by analyzing commutation relations for monomials in z and $\gamma^\pm$, equivalently Pieri rules for extremal weight crystals (Proposition 4.9).

Finally, we discuss some applications. Let $\mathcal{C}^\vee$ be the category of $\mathfrak{gl}_\infty$-crystals consisting of dual crystals $B^\vee$ for $B \in \mathcal{C}$ and let $\mathcal{C}^{\text{l.w.}}$ be a full subcategory of $\mathcal{C}^\vee$ whose objects are disjoint union of integrable lowest weight crystal. We denote by $\mathscr{K}^\vee$ and $\mathscr{K}^{\text{l.w.}}$ the corresponding Grothendieck groups. Then we consider a left $\mathscr{K}^\vee$-module structure on $\mathscr{K}^{\text{l.w.}}$, which is induced from the action of $\mathscr{D}$ on $\mathscr{R}_\mathbb{Q}$ as differential operators, or from the composite of following two functors;

$$\mathcal{C}^\vee \times \mathcal{C}^{\text{l.w.}} \xrightarrow{\otimes} \mathcal{C}^\vee \xrightarrow{\text{pr}} \mathcal{C}^{\text{l.w.}},$$

where pr is the natural projection functor. Using the Littlewood–Richardson rule, we obtain an explicit combinatorial description of the action of $\mathscr{K}^\vee$ on $\mathscr{K}^{\text{l.w.}}$. We observe that the action of level zero extremal weight crystals are transitive on the set of integrable lowest weight crystals of a fixed level. As an application, we obtain a new interpretation of a one-to-one correspondence between level n integrable highest (or lowest) weight $\mathfrak{gl}_\infty$-modules and finite-dimensional irreducible $\mathfrak{gl}_n$-modules, which comes from the $(\mathfrak{gl}_\infty, \mathfrak{gl}_n)$-duality on level n fermionic Fock space [5] (Theorem 6.8). As another application, we construct an action of the Hall–Littlewood vertex operators [8] on $\mathbb{Z}[q] \otimes_\mathbb{Z} \mathscr{K}^{\text{l.w.}}$ (Theorem 6.15), which naturally yields an A_∞-analogue of Hall–Littlewood function.

For a combinatorial realization, most crystals in this paper are embedded in a set of binary matrices of various shapes, equivalently an (infinite) abacus model. Also, two kinds of Kashiwara operators on binary matrices [4,18,21], which produces various dualities, play a crucial role in proving our main results, while the rational semistandard tableaux for $\mathfrak{gl}_n$ [27,29] were used to understand extremal weight crystals of type $A_{+\infty}$ [19].

We also expect a similar result for the other infinite rank affine Lie algebras, that is, roughly speaking, the Grothendieck ring generated by extremal weight crystals can be realized as a ring of differential operators acting on the character ring of integrable highest weight or lowest weight modules.

The paper is organized as follows. In Section 2, we review briefly the notion of crystals. In Section 3, we introduce a double crystal (or bicrystal) structure on binary matrices, which is our main method. In Section 4, we give a characterization of extremal weight crystals. In Section 5, we introduce the monoidal category $\mathcal{C}$, characterize its Grothendieck ring, and give a Littlewood–Richardson rule for extremal weight crystals. In Section 6, we study the action of $\mathscr{K}^\vee$ on $\mathscr{K}^{\text{l.w.}}$ and discuss its applications.

2. Crystals

2.1. Review on crystals

Let I be an index set. Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra associated with a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$. Denote the weight lattice of $\mathfrak{g}$ by P, the set of simple roots by $\Pi = \{\alpha_i \mid i \in I\} \subset P$, and the set of simple coroots by $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$ with $\langle \alpha_j, h_i \rangle = a_{ij}$.

Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra associated with $\mathfrak{g}$ introduced by Drinfeld and Jimbo. In [9], Kashiwara introduced the notion of crystal base of a $U_q(\mathfrak{g})$-module V, which can be viewed as a limit of V at $q = 0$. The crystal base is an I-colored oriented graph with important combinatorial information of V. The existence of the crystal bases of $U_q(\mathfrak{g})$-modules which are related with the work in this paper has been established in [9–11,15].

Based on the properties of crystal bases, one can define the notion of $\mathfrak{g}$-crystal (or crystal for short) as follows (see [12] for a general review and references therein).

A $\mathfrak{g}$-*crystal* is a set B together with the maps $\mathrm{wt}: B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{\mathbf{0}\}$ ($i \in I$) such that for $b \in B$ and $i \in I$

(1) $\varphi_i(b) = \langle \mathrm{wt}(b), h_i\rangle + \varepsilon_i(b)$,
(2) $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$, $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\tilde{e}_i b \neq \mathbf{0}$,
(3) $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$, $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\tilde{f}_i b \neq \mathbf{0}$,
(4) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in B$,
(5) $\tilde{e}_i b = \tilde{f}_i b = \mathbf{0}$ if $\varphi_i(b) = -\infty$,

where $\mathbf{0}$ is a formal symbol. Here we assume that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$. For example, a crystal base of an integrable $U_q(\mathfrak{g})$-module is a $\mathfrak{g}$-crystal.

Note that B is equipped with an I-colored oriented graph structure, where $b \xrightarrow{i} b'$ if and only if $b' = \tilde{f}_i b$ for $b, b' \in B$ and $i \in I$. We call B *connected* if it is connected as a graph.

The *dual crystal* $B^\vee$ *of* B is defined to be the set $\{b^\vee \mid b \in B\}$ with

$$\begin{gathered} \mathrm{wt}\big(b^\vee\big) = -\mathrm{wt}(b), \\ \varepsilon_i\big(b^\vee\big) = \varphi_i(b), \qquad \varphi_i\big(b^\vee\big) = \varepsilon_i(b), \\ \tilde{e}_i\big(b^\vee\big) = (\tilde{f}_i b)^\vee, \qquad \tilde{f}_i\big(b^\vee\big) = (\tilde{e}_i b)^\vee, \end{gathered} \tag{2.1}$$

for $b \in B$ and $i \in I$. We assume that $\mathbf{0}^\vee = \mathbf{0}$.

Let B_1 and B_2 be crystals. A *morphism* $\psi : B_1 \to B_2$ is a map from $B_1 \cup \{\mathbf{0}\}$ to $B_2 \cup \{\mathbf{0}\}$ such that

(1) $\psi(\mathbf{0}) = \mathbf{0}$,
(2) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$ if $\psi(b) \neq \mathbf{0}$,
(3) $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$ if $\psi(b) \neq \mathbf{0}$ and $\psi(\tilde{e}_i b) \neq \mathbf{0}$,
(4) $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$ if $\psi(b) \neq \mathbf{0}$ and $\psi(\tilde{f}_i b) \neq \mathbf{0}$,

for $b \in B_1$ and $i \in I$. We call ψ an *embedding* and B_1 a *subcrystal of* B_2 when ψ is injective, and call ψ *strict* if $\psi : B_1 \cup \{\mathbf{0}\} \to B_2 \cup \{\mathbf{0}\}$ commutes with $\tilde{e}_i$ and $\tilde{f}_i$ for all $i \in I$, where we assume that $\tilde{e}_i \mathbf{0} = \tilde{f}_i \mathbf{0} = \mathbf{0}$.

For $b_i \in B_i$ ($i = 1, 2$), we say that b_1 *is* ($\mathfrak{g}$-)*equivalent to* b_2, and write $b_1 \equiv b_2$ if there exists a crystal isomorphism $C(b_1) \to C(b_2)$ sending b_1 to b_2, where $C(b_i)$ denotes the connected component of B_i including b_i ($i = 1, 2$).

For a crystal B and a non-negative integer m, we denote by $B^{\oplus m}$ the disjoint union $B_1 \sqcup \cdots \sqcup B_m$ with $B_i \simeq B$, where $B^{\oplus 0}$ means the empty set.

We call a crystal B *normal* if $\varepsilon_i(b) = \max\{k \mid \tilde{e}_i^k b \neq \mathbf{0}\}$ and $\varphi_i(b) = \max\{k \mid \tilde{f}_i^k b \neq \mathbf{0}\}$ for $b \in B$ and $i \in I$, and put $\tilde{e}_i^{\max} b = \tilde{e}_i^{\varepsilon_i(b)} b$ and $\tilde{f}_i^{\max} b = \tilde{f}_i^{\varphi_i(b)} b$. We say that B is *regular* if B is as a $\mathfrak{g}_J$-crystal, isomorphic to the crystal base of an integrable $U_q(\mathfrak{g}_J)$-module for $J \subset I$ such that $\{\alpha_i \mid i \in J\}$ is of finite type, where $\mathfrak{g}_J$ is the Kac–Moody algebra associated with $A_J = (a_{ij})_{i,j \in J}$.[1] Note that if B is regular, then B is normal. Suppose that B_1, B_2 are regular crystals. If $\psi : B_1 \to B_2$ is an embedding, then ψ is strict. Hence $B_1 \simeq \psi(B_1)$ and $B_2 \simeq B_1 \sqcup (B_2 \setminus \psi(B_1))$. In particular, $b \equiv \psi(b)$ for $b \in B_1$.

[1] Erratum: the definition of regular crystal in [19, p. 1343], which was given as that of normal crystal, should be replaced with the one given here.

Let W be the Weyl group of $\mathfrak{g}$, that is, the subgroup of $GL(P)$ generated by r_i $(i \in I)$, where r_i is the simple reflection given by $r_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i$ for $\lambda \in P$. A regular crystal B admits an action of the Weyl group on B as follows; for $i \in I$ and $b \in B$

$$S_{r_i} b = \begin{cases} \tilde{f}_i^{\langle \mathrm{wt}(b), h_i \rangle} b & \text{if } \langle \mathrm{wt}(b), h_i \rangle \geqslant 0, \\ \tilde{e}_i^{-\langle \mathrm{wt}(b), h_i \rangle} b & \text{if } \langle \mathrm{wt}(b), h_i \rangle \leqslant 0, \end{cases} \tag{2.2}$$

and $S_w = S_{r_{i_1}} \cdots S_{r_{i_t}}$ for $w \in W$ with a reduced expression $w = r_{i_1} \cdots r_{i_t}$.

A *tensor product* $B_1 \otimes B_2$ of crystals B_1 and B_2 is defined to be $B_1 \times B_2$ as a set with elements denoted by $b_1 \otimes b_2$, where

$$\begin{aligned} \mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), h_i \rangle\}, \\ \varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_1) + \langle \mathrm{wt}(b_2), h_i \rangle, \varphi_i(b_2)\}, \\ \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2), \end{cases} \end{aligned} \tag{2.3}$$

for $i \in I$. Here we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$. Then $B_1 \otimes B_2$ is a crystal. Note that $(B_1 \otimes B_2)^\vee \simeq B_2^\vee \otimes B_1^\vee$, and $B_1 \otimes B_2$ is regular if B_1, B_2 are regular.

2.2. The Lie algebra $\mathfrak{gl}_\infty$

Let $\mathfrak{gl}_\infty$ denote the Lie algebra of complex matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ with finitely many non-zero entries.

Let E_{ij} be the elementary matrix with 1 at the i-th row and the j-th column and zero elsewhere. Let $\mathfrak{h} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C} E_{ii}$ be the Cartan subalgebra of $\mathfrak{gl}_\infty$ and $\langle \cdot, \cdot \rangle$ denote the natural pairing on $\mathfrak{h}^* \times \mathfrak{h}$. We denote by $\Pi^\vee = \{h_i = E_{ii} - E_{i+1 i+1} \mid i \in \mathbb{Z}\}$ the set of simple coroots, and $\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}\}$ the set of simple roots, where $\epsilon_i \in \mathfrak{h}^*$ is determined by $\langle \epsilon_i, E_{jj} \rangle = \delta_{ij}$.

Let $P = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}\epsilon_i \oplus \mathbb{Z}\Lambda_0 = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}\Lambda_i$ be the weight lattice of $\mathfrak{gl}_\infty$, where Λ_0 is defined by $\langle \Lambda_0, E_{-j+1 -j+1} \rangle = -\langle \Lambda_0, E_{jj} \rangle = \frac{1}{2}$ for $j \geqslant 1$, and Λ_i is given by $\Lambda_0 - \sum_{k=i+1}^{0} \epsilon_k$ (resp. $\Lambda_0 + \sum_{k=1}^{i} \epsilon_k$) for $i < 0$ (resp. $i > 0$). We call Λ_i the i-th fundamental weight. A partial ordering on P is defined as usual.

For $k \in \mathbb{Z}$, let $P_k = k\Lambda_0 + \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}\epsilon_i$ be the set of integral weights of level k. Let $P^+ = \{\Lambda \in P \mid \langle \Lambda, h_i \rangle \geqslant 0,\ i \in \mathbb{Z}\} = \sum_{i \in \mathbb{Z}} \mathbb{Z}_{\geqslant 0} \Lambda_i$ be the set of dominant integral weights. We also put $P_k^+ = P^+ \cap P_k$ for $k \geqslant 0$ (note that $P_0^+ = \{0\}$). Note that for $\Lambda = \sum_{i \in \mathbb{Z}} c_i \Lambda_i \in P$, the level of Λ is $\sum_{i \in \mathbb{Z}} c_i$ since $\epsilon_i = \Lambda_i - \Lambda_{i-1}$ for $i \in \mathbb{Z}$. If we put $\Lambda_\pm = \sum_{i; c_i \gtrless 0} |c_i| \Lambda_i$, then $\Lambda = \Lambda_+ - \Lambda_-$ with $\Lambda_\pm \in P^+$.

For $n \geqslant 1$, let $\mathbb{Z}_+^n = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{Z},\ \lambda_1 \geqslant \cdots \geqslant \lambda_n\}$ be the set of generalized partitions of length n. For $\lambda \in \mathbb{Z}_+^n$, we put

$$\Lambda_\lambda = \Lambda_{\lambda_1} + \cdots + \Lambda_{\lambda_n}. \tag{2.4}$$

Note that $\mathbb{Z}_+^n$ parameterizes P_n^+.

Let I be an interval in $\mathbb{Z}$. We denote by $\mathfrak{gl}_I$ the subalgebra of $\mathfrak{gl}_\infty$ spanned by E_{ij} for $i, j \in I$. For $p, q \in \mathbb{Z}$, we put $[p, q] = \{p, p+1, \ldots, q\}$ $(p < q)$, $[p, \infty) = \{p, p+1, \ldots\}$ and $(-\infty, q] = \{\ldots, q-1, q\}$. For $n \geqslant 1$, we denote $[1, n]$ by $[n]$ for simplicity.

For $\Lambda \in P^+$, we denote by $\mathbf{B}(\pm\Lambda)$ the crystal base of the irreducible $U_q(\mathfrak{gl}_\infty)$-module with highest (resp. lowest) weight vector $u_{\pm\Lambda}$ of weight $\pm\Lambda$, which is a connected regular $\mathfrak{gl}_\infty$-crystal. We denote by $\mathbf{B}$ and $\mathbf{B}^\vee$ the crystal base of the natural representation of $U_q(\mathfrak{gl}_\infty)$ and its dual respectively, which are also connected regular crystals. The associated colored oriented graphs are

$$\mathbf{B} : \cdots \xrightarrow{-3} -2 \xrightarrow{-2} -1 \xrightarrow{-1} 0 \xrightarrow{0} 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{3} \cdots,$$
$$\mathbf{B}^\vee : \cdots \xrightarrow{3} 3^\vee \xrightarrow{2} 2^\vee \xrightarrow{1} 1^\vee \xrightarrow{0} 0^\vee \xrightarrow{-1} -1^\vee \xrightarrow{-2} -2^\vee \xrightarrow{-3} \cdots,$$

where $\mathrm{wt}(i) = \epsilon_i$ for $i \in \mathbb{Z}$.

3. Double crystal structure on binary matrices

3.1. Crystal operators on binary matrices

For intervals I, J in $\mathbb{Z}$, let $\mathbf{M}_{I,J}$ be the set of $I \times J$ matrices $A = (a_{ij})$ with $a_{ij} \in \{0,1\}$. We denote by A_i the i-th row of A for $i \in I$. Let $I^\circ = \{i \mid i, i+1 \in I\}$ and $J^\circ = \{j \mid j, j+1 \in J\}$.

Let $A \in \mathbf{M}_{I,J}$ be given. For $i \in I$ and $j \in J^\circ$, let

$$\sigma_j(A_i) = \begin{cases} + & \text{if } (a_{ij}, a_{ij+1}) = (1,0), \\ - & \text{if } (a_{ij}, a_{ij+1}) = (0,1), \\ \cdot & \text{otherwise.} \end{cases}$$

We say that A is *row j-admissible* if there exist $M, N \in I$ $(M \leqslant N)$ such that $\sigma_j(A_i) \neq +$ for all $i < M$ and $\sigma_j(A_i) \neq -$ for all $i > N$. Note that if I is finite, then A is row j-admissible.

Let us define the operators $\tilde{e}_j, \tilde{f}_j$ for $j \in J^\circ$ on the set of row j-admissible matrices in $\mathbf{M}_{I,J}$. Suppose that $|I| = 1$ $(I = \{i\})$. For $A \in \mathbf{M}_{I,J}$ $(A = A_i)$, we define

$$\tilde{e}_j A = \begin{cases} A + E_j - E_{j+1} & \text{if } \sigma_j(A) = -, \\ \mathbf{0} & \text{otherwise,} \end{cases}$$
$$\tilde{f}_j A = \begin{cases} A - E_j + E_{j+1} & \text{if } \sigma_j(A) = +, \\ \mathbf{0} & \text{otherwise,} \end{cases} \tag{3.1}$$

where E_j is the $\{i\} \times J$ matrix with 1 in the j-th column and 0 elsewhere.

Suppose that $|I| \geqslant 2$. For a row j-admissible $A \in \mathbf{M}_{I,J}$, consider the sequence

$$\sigma_j(A) = \big(\sigma_j(A_i)\big)_{i \in I} = \big(\ldots, \sigma_j(A_{i-1}), \sigma_j(A_i), \sigma_j(A_{i+1}), \ldots\big).$$

Then we replace a pair $(\sigma_j(A_s), \sigma_j(A_{s'})) = (+,-)$ such that $s < s'$ and $\sigma_j(A_t) = \cdot$ for $s < t < s'$ by $(\cdot,\cdot)$ in $\sigma_j(A)$, and repeat this process as far as possible until we get a sequence $\tilde{\sigma}_j(A) = (\tilde{\sigma}_j(A_i))_{i \in I}$ with no $+$ placed to the left of $-$. It is not difficult to see that this procedure must end after a finite number of steps since A is row j-admissible.

Now we define $\tilde{e}_j A$ (resp. $\tilde{f}_j A$) to be the matrix in $\mathbf{M}_{I,J}$ given by applying $\tilde{e}_j$ (resp. $\tilde{f}_j$) to the row of A corresponding to the right-most $-$ (resp. the left-most $+$) in $\tilde{\sigma}_j(A)$, and $\tilde{e}_j A = \mathbf{0}$ (resp. $\tilde{f}_j A = \mathbf{0}$) if there is no such row. Note that $\tilde{e}_j A$ (resp. $\tilde{f}_j A$) is also row j-admissible if $\tilde{e}_j A \neq \mathbf{0}$ (resp. $\tilde{f}_j A \neq \mathbf{0}$).

Remark 3.1. We may regard each A_i as an element of a regular $\mathfrak{gl}_{\{j,j+1\}}$-crystal with weight $a_{ij}\epsilon_j + a_{ij+1}\epsilon_{j+1}$. Hence if I is a finite set, say $I = [n]$, then A can be viewed as $A_1 \otimes \cdots \otimes A_n$ and the actions of $\tilde{e}_j$ and $\tilde{f}_j$ on A defined here coincides with tensor product rule of crystal, which is also known as *signature rule* (cf. [15]). But, we do not always have a crystal structure on the set of row j-admissible matrices, since the weight of A with respect to $\mathfrak{gl}_{\{j,j+1\}}$ is not always defined naturally when I is an infinite set. For example, let $A = (a_{ij})$ be a $\mathbb{Z} \times [2]$-matrix where $(a_{i1}, a_{i2}) = (0, 1)$ for $i \leqslant 0$ and $(a_{i1}, a_{i2}) = (1, 0)$ for $i > 0$. Then A is row 1-admissible and $\tilde{e}_1 A = A + E_{01} - E_{02}$ and $\tilde{f}_1 A = A - E_{11} + E_{12}$, though the weight of A with respect to $\mathfrak{gl}_{[2]}$ is not well defined.

Let

$$\rho : \mathbf{M}_{I,J} \to \mathbf{M}_{-J,I}$$

be a bijection given by $\rho((a_{ij})) = (a'_{-ji}) \in \mathbf{M}_{-J,I}$ with $a'_{-ji} = a_{ij}$, where $-J = \{-j \mid j \in J\}$.

For $A \in \mathbf{M}_{I,J}$ and $i \in I^\circ$, we say that A is *column i-admissible* if $\rho(A)$ is row i-admissible. Then for a column i-admissible $A \in \mathbf{M}_{I,J}$, we define

$$\widetilde{E}_i(A) = \rho^{-1}\big(\tilde{e}_i \rho(A)\big), \qquad \widetilde{F}_i(A) = \rho^{-1}\big(\tilde{f}_i \rho(A)\big). \tag{3.2}$$

The following lemma follows from [16, Lemma 3.4] (see also [4,21], where essentially the same facts are stated in a slightly different way).

Lemma 3.2. *Let $A \in \mathbf{M}_{I,J}$ be given. If A is both row j-admissible and column i-admissible for $i \in I^\circ$ and $j \in J^\circ$, then*

$$\tilde{x}_j \widetilde{X}_i A = \widetilde{X}_i \tilde{x}_j A,$$

where $x = e, f$ and $X = E, F$.

For $A \in \mathbf{M}_{I,J}$, we say that *A is row admissible* (resp. *column admissible*) if A is row j-admissible (resp. column i-admissible) for all $j \in J^\circ$ (resp. $i \in I^\circ$). Note that if I (resp. J) is a finite set, then A is always row (resp. column) admissible. Lemma 3.2 implies the following immediately (cf. [16, Lemma 3.5]).

Lemma 3.3. *Let $A \in \mathbf{M}_{I,J}$ be given. Suppose that A is row admissible and column i-admissible for $i \in I^\circ$. If $\widetilde{X}_i A \neq \mathbf{0}$ ($X = E, F$), then*

$$\tilde{x}_{j_1} \cdots \tilde{x}_{j_r} A \neq \mathbf{0} \quad \Leftrightarrow \quad \tilde{x}_{j_1} \cdots \tilde{x}_{j_r} (\widetilde{X}_I A) \neq \mathbf{0}$$

for $r \geqslant 1$ and $j_1, \ldots, j_r \in J^\circ$, where $x = e$ or f for each j_k.

We have a similar statement when A is column admissible and row j-admissible for $j \in J^\circ$.

Suppose that I is finite, say $I = [p, q]$. For $A = (a_{ij}) \in \mathbf{M}_{I,J}$, put

$$A^\vee = \big(a^\vee_{ij}\big) \in \mathbf{M}_{I,J} \quad \big(a^\vee_{ij} = 1 - a_{p+q-i\,j}\big). \tag{3.3}$$

Then we have for $j \in J^\circ$

$$\tilde{f}_j A = \big(\tilde{e}_j A^\vee\big)^\vee, \qquad \tilde{e}_j A = \big(\tilde{f}_j A^\vee\big)^\vee. \tag{3.4}$$

We call $A^\vee$ the *dual of A* (with respect to row).

3.2. Crystals of semistandard tableaux

Let $\mathscr{P}$ denote the set of partitions. We identify a partition $\lambda = (\lambda_i)_{i\geqslant 1}$ with a Young diagram as usual (see [22]). The number of non-zero parts in λ is denoted by $\ell(\lambda)$. We denote by $\lambda' = (\lambda'_i)_{i\geqslant 1}$ the conjugate partition of λ. For a skew Young diagram λ/μ, $|\lambda/\mu|$ denotes the number of dots or boxes in the diagram. Let $\mathcal{A}$ be a linearly ordered set. A tableau T obtained by filling λ/μ with entries in $\mathcal{A}$ is called a *semistandard tableau of shape* λ/μ if the entries in each row are weakly increasing from left to right, and the entries in each column are strictly increasing from top to bottom. We denote by $SST_{\mathcal{A}}(\lambda/\mu)$ the set of all semistandard tableaux of shape λ/μ with entries in $\mathcal{A}$. For $T \in SST_{\mathcal{A}}(\lambda/\mu)$, let $w(T)_{\mathrm{col}}$ (resp. $w(T)_{\mathrm{row}}$) denote the word of T with respect to column (resp. row) reading, where we read the entries of T column by column (resp. row by row) from right to left (resp. top to bottom), and in each column (resp. row) from top to bottom (resp. right to left).

Let $\mathcal{A}$ denote one of the crystals $\mathbf{B}$ and $\mathbf{B}^\vee$, which are linearly ordered with respect to the partial ordering on P. Note that the set of all finite words with letters in $\mathcal{A}$ is a $\mathfrak{gl}_\infty$-crystal, where each word of length $r \geqslant 1$ is identified with an element in $\mathcal{A}^{\otimes r} = \mathcal{A} \otimes \cdots \otimes \mathcal{A}$ (r times). Given a skew Young diagram λ/μ, the injective image of $SST_{\mathcal{A}}(\lambda/\mu)$ in the set of finite words under the map $T \mapsto w(T)_{\mathrm{col}}$ (or $w(T)_{\mathrm{row}}$) together with $\{\mathbf{0}\}$ is invariant under $\tilde{e}_i, \tilde{f}_i$ ($i \in \mathbb{Z}$). Hence it is a regular $\mathfrak{gl}_\infty$-crystal [15]. In particular, for $\lambda \in \mathscr{P}$, we have

$$SST_{\mathbf{B}}(\lambda)^\vee \simeq SST_{\mathbf{B}^\vee}\big(\lambda^\vee\big),$$

where $\lambda^\vee$ is the skew Young diagram obtained from λ by 180°-rotation.

For $\mu \in \mathscr{P}$, we put

$$\mathbf{B}_\mu = SST_{\mathbf{B}}(\mu), \tag{3.5}$$

and we identify $\mathbf{B}_\mu^\vee$ with $SST_{\mathbf{B}^\vee}(\mu^\vee)$. Note that $\mathbf{B}_\mu$ does not have a highest weight or lowest weight element.

Proposition 3.4. *For $\mu, \nu \in \mathscr{P}$, $\mathbf{B}_\mu \otimes \mathbf{B}_\nu^\vee$ and $\mathbf{B}_\nu^\vee \otimes \mathbf{B}_\mu$ are connected.*

Proof. First, we claim that $\mathbf{B}_\mu$ is connected. Suppose that $S, T \in \mathbf{B}_\mu$ are given. Choose $p \in \mathbb{Z}$ such that all entries in S and T are greater than p. Then S is an element in $SST_{[p,\infty)}(\mu)$, which is a connected $\mathfrak{gl}_{[p,\infty)}$-crystal with a unique highest weight element, say $u_\mu^{[p,\infty)}$ (see [15]). This implies that S and T are contained in the same connected component, and hence $\mathbf{B}_\mu$ is connected.

Let $S \otimes T \in \mathbf{B}_\mu \otimes \mathbf{B}_\nu^\vee$ be given. Choose $p \in \mathbb{Z}$ such that $S \in SST_{[p,\infty)}(\mu)$. Then we have $\tilde{e}_{i_1} \cdots \tilde{e}_{i_r} S = u_\mu^{[p,\infty)}$ for some $i_1, \ldots, i_r \in [p,\infty)$. By tensor product rule of crystals, we also have

$$\tilde{e}_{i_1}^{m_1} \cdots \tilde{e}_{i_r}^{m_r}(S \otimes T) = u_\mu^{[p,\infty)} \otimes T'$$

for some $m_1, \ldots, m_r \geqslant 1$ and $T' \in \mathbf{B}_\nu^\vee$.

If p is sufficiently small, then we may assume that all the entries in T (and hence in T') are smaller than $(p + \ell(\mu))^\vee$. Choose q such that $T' \in SST_{(-\infty,q]^\vee}(\nu^\vee)$. Note that $SST_{(-\infty,q]^\vee}(\nu^\vee)$ is a $\mathfrak{gl}_{(-\infty,q]}$-crystal with a unique highest weight element $v_\nu^{(-\infty,q]}$. Hence $\tilde{e}_{j_1} \cdots \tilde{e}_{j_s} T' = v_\nu^{(-\infty,q]}$ for some $j_1, \ldots, j_s \in (-\infty, q-1]$. Since $\{j_1, \ldots, j_s\}$ does not intersect with the entries in $u_\mu^{[p,\infty)}$, we have

$$\tilde{e}_{j_1} \cdots \tilde{e}_{j_s}\big(u_\mu^{[p,\infty)} \otimes T'\big) = u_\mu^{[p,\infty)} \otimes v_\nu^{(-\infty,q]}.$$

Now, let $U \otimes V \in \mathbf{B}_\mu \otimes \mathbf{B}_\nu^\vee$ be given. Then if p is sufficiently small and q is sufficiently large, then it follows from the same argument that $U \otimes V$ is also connected to $u_\mu^{[p,\infty)} \otimes v_\nu^{(-\infty,q]}$. This implies that $\mathbf{B}_\mu \otimes \mathbf{B}_\nu^\vee$ is connected.

The proof of the connectedness of $\mathbf{B}_\nu^\vee \otimes \mathbf{B}_\mu$ is almost identical and is omitted. $\square$

Remark 3.5. In the proof of Proposition 3.4, we showed that for $S \otimes T \in \mathbf{B}_\mu \otimes \mathbf{B}_\nu^\vee$ there exist $k_1, \ldots, k_t \in \mathbb{Z}$ such that $\tilde{e}_{k_1} \cdots \tilde{e}_{k_t}(S \otimes T) = u_\mu^{[p,\infty)} \otimes v_\nu^{(-\infty,q]}$ for some $p < q$. By applying suitable $\tilde{e}_k$'s, we may assume that $p \ll 0 \ll q$.

Let $\mathcal{E}$ be the subset of $\mathbf{M}_{\{1\},\mathbb{Z}}$ consisting of $(a_{1j})_{j\in\mathbb{Z}}$ such that $\sum_{j\in\mathbb{Z}} a_{1j} < \infty$. If we define $\mathrm{wt}(A) = \sum_{j\in\mathbb{Z}} a_{1j}\epsilon_j$ for $A = (a_{1j})_{j\in\mathbb{Z}} \in \mathcal{E}$, then $\mathcal{E}$ is a regular $\mathfrak{gl}_\infty$-crystal with respect to $\tilde{e}_k$, $\tilde{f}_k$ ($k \in \mathbb{Z}$) and wt (see Section 3.1 for the definitions of $\tilde{e}_k$ and $\tilde{f}_k$ on $\mathbf{M}_{\{1\},\mathbb{Z}}$). For $a \geqslant 1$, define

$$\sigma_a : \mathbf{B}_{(1^a)} \to \mathcal{E} \tag{3.6}$$

by $\sigma_a(S) = A$ with $\mathrm{wt}(S) = \mathrm{wt}(A)$. It is easy to check that σ_a is a strict embedding and

$$\mathcal{E} \simeq \bigsqcup_{a\geqslant 0} \mathbf{B}_{(1^a)}. \tag{3.7}$$

Indeed, $\mathcal{E}$ is the crystal base of the q-deformed exterior algebra of the natural representation of $U_q(\mathfrak{gl}_\infty)$ [14].

Let $\mathcal{E}^\vee = \{A^\vee \mid A \in \mathcal{E}\}$, where $A^\vee$ denotes the dual matrix of A (see (3.3)). If we define $\mathrm{wt}(A^\vee) = \sum_{j\in\mathbb{Z}}(a_{1j} - 1)\epsilon_j$ for $A = (a_{1j})_{j\in\mathbb{Z}} \in \mathcal{E}$, then $\mathcal{E}^\vee$ is a regular $\mathfrak{gl}_\infty$-crystal with respect to $\tilde{e}_k$, $\tilde{f}_k$ ($k \in \mathbb{Z}$) and wt, which is isomorphic to the dual crystal of $\mathcal{E}$. Similarly, for $b \geqslant 1$ we define

$$\tau_b : \mathbf{B}^\vee_{(1^b)} \to \mathcal{E}^\vee \tag{3.8}$$

by $\tau_b(T) = A^\vee$ with $\mathrm{wt}(T) = \mathrm{wt}(A^\vee)$. Then τ_b is a strict embedding. For convenience, we assume that σ_0 is a map sending trivial crystal to zero matrix in $\mathbf{M}_{\{1\},\mathbb{Z}}$, and τ_0 is a map sending trivial crystal to the matrix with 1 at all positions.

Lemma 3.6. *For $a, b \geqslant 0$, we have*

$$\mathbf{B}_{(1^a)} \otimes \mathbf{B}^\vee_{(1^b)} \simeq \mathbf{B}^\vee_{(1^b)} \otimes \mathbf{B}_{(1^a)}.$$

Proof. Consider $u_{(1^a)}^{[p,\infty)} \otimes v_{(1^b)}^{(-\infty,q]}$ (for simplicity write $u_a^p \otimes v_b^q$) for some $p \ll 0 \ll q$. We claim that

$$u_a^p \otimes v_b^q \equiv v_b^q \otimes u_a^p,$$

which implies that $\mathbf{B}_{(1^a)} \otimes \mathbf{B}^\vee_{(1^b)} \simeq \mathbf{B}^\vee_{(1^b)} \otimes \mathbf{B}_{(1^a)}$ by Proposition 3.4.

Define

$$\begin{aligned} \sigma_a \times \tau_b &: \mathbf{B}_{(1^a)} \otimes \mathbf{B}^\vee_{(1^b)} \to \mathcal{E} \otimes \mathcal{E}^\vee, \\ \tau_b \times \sigma_a &: \mathbf{B}^\vee_{(1^b)} \otimes \mathbf{B}_{(1^a)} \to \mathcal{E}^\vee \otimes \mathcal{E} \end{aligned} \tag{3.9}$$

by $(\sigma_a \times \tau_b)(S \otimes T) = \sigma_a(S) \otimes \tau_b(T)$ and $(\tau_b \times \sigma_a)(T \otimes S) = \tau_b(T) \otimes \sigma_a(S)$ for $S \in \mathbf{B}_{(1^a)}$ and $T \in \mathbf{B}^\vee_{(1^b)}$. (Recall that a tensor product of crystals is as a set the Cartesian product of the given crystals. So we understand $\sigma_a \times \tau_b$ as the Cartesian product of the maps σ_a and τ_b on $\mathbf{B}_{(1^a)} \times \mathbf{B}^\vee_{(1^b)}$. Throughout the paper, a product of crystal morphisms is understood in this manner.) Then $\sigma_a \times \tau_b$ and $\tau_b \times \sigma_a$ are strict embeddings. Here we assume $\mathcal{E} \otimes \mathcal{E}^\vee$ and $\mathcal{E}^\vee \otimes \mathcal{E}$ as subsets of $\mathbf{M}_{[2],\mathbb{Z}}$, where $A_1 \otimes A_2$ is identified with a matrix $A \in \mathbf{M}_{[2],\mathbb{Z}}$ whose i-th row is A_i ($i = 1, 2$).

Let $A = (a_{ij}) = (\sigma_a \times \tau_b)(u_a^p \otimes v_b^q)$, where $a_{1j} = 1$ if and only if $p \leqslant j \leqslant p + a - 1$, and $a_{2j} = 0$ if and only if $q - b + 1 \leqslant j \leqslant q$, and let $B = (b_{ij}) = (\tau_b \times \sigma_a)(v_b^q \otimes u_a^p)$ where $b_{ij} = a_{3-ij}$ for all i, j.

Choose $r \ll p$ and $s \gg q$. Let $\pi_{[r,s]} : \mathbf{M}_{[2],\mathbb{Z}} \to \mathbf{M}_{[2],[r,s]}$ be the restriction map sending a matrix to its $[2] \times [r,s]$ submatrix. Then $\pi_{[r,s]}(A)$ is column 1-admissible and $\widetilde{E}_1^{\max}\pi_{[r,s]}(A) = \pi_{[r,s]}(B)$. By Lemma 3.3, we have

$$\tilde{x}_{i_1} \cdots \tilde{x}_{i_t} \pi_{[r,s]}(A) \neq \mathbf{0} \quad \Leftrightarrow \quad \tilde{x}_{i_1} \cdots \tilde{x}_{i_t} \pi_{[r,s]}(B) \neq \mathbf{0},$$

and hence

$$\tilde{x}_{i_1} \cdots \tilde{x}_{i_t} A \neq \mathbf{0} \quad \Leftrightarrow \quad \tilde{x}_{i_1} \cdots \tilde{x}_{i_t} B \neq \mathbf{0}$$

for $t \geqslant 1$ and $r \leqslant i_1, \ldots, i_t \leqslant s - 1$, where $x = e$ or f for each i_k. Since r and s are arbitrary and $\mathrm{wt}(A) = \mathrm{wt}(B)$, we have $A \equiv B$, which implies that $u_a^p \otimes v_b^q \equiv v_b^q \otimes u_a^p$. □

Remark 3.7. More generally, we can check by the same argument as in Lemma 3.6 that for $S \otimes T \in \mathbf{B}_{(1^a)} \otimes \mathbf{B}^\vee_{(1^b)}$, if $\mathrm{wt}(S) = \epsilon_{j_1} + \cdots + \epsilon_{j_a}$ and $\mathrm{wt}(T) = -\epsilon_{j'_1} - \cdots - \epsilon_{j'_b}$ with $j_1 < \cdots < j_a < j'_1 < \cdots < j'_b$, then $S \otimes T \equiv T \otimes S$.

For $n \geqslant 1$, let $\mathcal{E}^n$ be the subset of $\mathbf{M}_{[n],\mathbb{Z}}$ consisting of matrices A such that $A_i \in \mathcal{E} \subset \mathbf{M}_{\{i\},\mathbb{Z}}$ for all $i \in [n]$. Then $\mathcal{E}^n$ is row admissible and can be identified with $\mathcal{E}^{\otimes n}$ as a $\mathfrak{gl}_\infty$-crystal, where $A \in \mathcal{E}^n$ is identified with $A_1 \otimes \cdots \otimes A_n \in \mathcal{E}^{\otimes n}$. Also we may identify the dual crystal $(\mathcal{E}^n)^\vee$ with the set $\{A^\vee \mid A \in \mathcal{E}^n\}$, where $A^\vee$ denotes the dual matrix of A.

Let $\mu, \nu \in \mathscr{P}$ be given with $\ell(\mu') = m$ and $\ell(\nu') = n$. We may regard $\mathbf{B}_\mu \subset \mathbf{B}_{(1^{\mu'_m})} \otimes \cdots \otimes \mathbf{B}_{(1^{\mu'_1})}$, where the k-th column of $S \in \mathbf{B}_\mu$ (from the right) is an element in $\mathbf{B}_{(1^{\mu'_{m-k+1}})}$. Composing with (3.6), we have a strict embedding

$$\sigma_\mu = \sigma_{\mu'_m} \times \cdots \times \sigma_{\mu'_1} : \mathbf{B}_\mu \to \mathcal{E}^m. \tag{3.10}$$

Similarly, we may regard $\mathbf{B}^\vee_\nu \subset \mathbf{B}^\vee_{(1^{\nu'_1})} \otimes \cdots \otimes \mathbf{B}^\vee_{(1^{\nu'_n})}$, and have a strict embedding

$$\tau_\nu = \tau_{\nu'_1} \times \cdots \times \tau_{\nu'_n} : \mathbf{B}^\vee_\nu \to \left(\mathcal{E}^n\right)^\vee. \tag{3.11}$$

Proposition 3.8. *For $\mu, \nu \in \mathscr{P}$, we have*

$$\mathbf{B}_\mu \otimes \mathbf{B}^\vee_\nu \simeq \mathbf{B}^\vee_\nu \otimes \mathbf{B}_\mu.$$

Proof. Let $\ell(\mu') = m$ and $\ell(\nu') = n$. Consider the strict embedding

$$\sigma_\mu \times \tau_\nu : \mathbf{B}_\mu \otimes \mathbf{B}^\vee_\nu \to \mathcal{E}^m \otimes \left(\mathcal{E}^n\right)^\vee.$$

Here we assume $\mathcal{E}^m \otimes (\mathcal{E}^n)^\vee$ as a subset of $\mathbf{M}_{[m+n],\mathbb{Z}}$, consisting of the matrices whose first m rows form an element in $\mathcal{E}^m$ and the other n rows form an element in $(\mathcal{E}^n)^\vee$.

Consider $u_\mu^{[p,\infty)} \otimes v_\nu^{(-\infty,q]}$ (see the proof of Proposition 3.4 for its definition). We assume that $p \ll 0 \ll q$. Let $A = (\sigma_\mu \times \tau_\nu)(u_\mu^{[p,\infty)} \otimes v_\nu^{(-\infty,q]}) \in \mathbf{M}_{[m+n],\mathbb{Z}}$ and write $A = (A_1, \ldots, A_m, A_{m+1}, \ldots, A_{m+n})$. Note that A_i $(1 \leqslant i \leqslant m)$ corresponds to the i-th column of $u_\mu^{[p,\infty)}$ and A_{m+j} $(1 \leqslant j \leqslant n)$ corresponds to the j-th column of $v_\nu^{(-\infty,q]}$ from the right. By Lemma 3.6 (and Remark 3.7), we have

$$\begin{aligned}(A_1,\ldots,A_{m-1},A_m,A_{m+1},\ldots,A_{m+n}) &\equiv (A_1,\ldots,A_{m-1},A_{m+1},A_m,\ldots,A_{m+n})\\ &\equiv (A_1,\ldots,A_{m+1},A_{m-1},A_m,\ldots,A_{m+n})\\ &\ \ \vdots\\ &\equiv (A_{m+1},A_1,\ldots,A_m,A_{m+2},\ldots,A_{m+n}).\end{aligned}$$

Note that we can identify each matrix given above with an element in a $\mathfrak{gl}_\infty$-crystal (a mixed tensor product of $\mathcal{E}$ and $\mathcal{E}^\vee$'s), and hence consider $\mathfrak{gl}_\infty$-equivalence between them.

Repeating the above process, we conclude that

$$(A_1,\ldots,A_m,A_{m+1},\ldots,A_{m+n}) \equiv (A_{m+1},\ldots,A_{m+n},A_1,\ldots,A_m).$$

Since the right-hand side of the above equivalence is the image of $v_\nu^{(-\infty,q]}\otimes u_\mu^{[p,\infty)}$ under the strict embedding $\tau_\nu\times\sigma_\mu:\mathbf{B}_\nu^\vee\otimes\mathbf{B}_\mu\to(\mathcal{E}^n)^\vee\otimes\mathcal{E}^m\subset\mathbf{M}_{[m+n],\mathbb{Z}}$, we have

$$u_\mu^{[p,\infty)}\otimes v_\nu^{(-\infty,q]}\equiv v_\nu^{(-\infty,q]}\otimes u_\mu^{[p,\infty)}.$$

By Proposition 3.4, it follows that $\mathbf{B}_\mu\otimes\mathbf{B}_\nu^\vee\simeq\mathbf{B}_\nu^\vee\otimes\mathbf{B}_\mu$. □

From now on, we write $\mathbf{B}_{\mu,\nu}=\mathbf{B}_\mu\otimes\mathbf{B}_\nu^\vee$ for $\mu,\nu\in\mathscr{P}$.

Proposition 3.9. *For $\mu,\nu,\sigma,\tau\in\mathscr{P}$, $\mathbf{B}_{\mu,\nu}\simeq\mathbf{B}_{\sigma,\tau}$ if and only if $(\mu,\nu)=(\sigma,\tau)$.*

Proof. Suppose that $S\otimes T\equiv S'\otimes T'$ for $S\otimes T\in\mathbf{B}_\mu\otimes\mathbf{B}_\nu^\vee$ and $S'\otimes T'\in\mathbf{B}_\sigma\otimes\mathbf{B}_\tau^\vee$. By Remark 3.5, we may assume that there exist $s\ll 0$ and $t\gg 0$ such that the entries in S and S' are less than s, and the entries in T and T' are less than $t^\vee$. Then it follows that S is $\mathfrak{gl}_{(-\infty,s]}$-equivalent to S', which implies that $S=S'$ and $\mu=\sigma$. Similarly, we have $T=T'$ and $\nu=\tau$. □

We define $\mathrm{wt}_{[n]}(A)=\sum_{i\in[n]}(\sum_{j\in\mathbb{Z}}a_{ij})\epsilon_i$ for $A\in\mathcal{E}^n$. Then $\mathcal{E}^n$ $(n\geqslant 2)$ is column admissible, and it is a regular $\mathfrak{gl}_{[n]}$-crystal with respect to $\widetilde{E}_i,\widetilde{F}_i$ $(i\in[n]^\circ)$ and $\mathrm{wt}_{[n]}$. By Lemma 3.2, $\mathcal{E}^n$ is a $(\mathfrak{gl}_\infty,\mathfrak{gl}_{[n]})$-bicrystal, that is, the operators $\tilde{e}_j,\tilde{f}_j$ $(j\in\mathbb{Z})$ on $\mathcal{E}^n\cup\{\mathbf{0}\}$ commute with $\widetilde{E}_i,\widetilde{F}_i$ $(i\in[n]^\circ)$.

For $k\in[n]$ and $\lambda\in\mathbb{Z}_+^n$, we put

$$\begin{aligned}\omega_k&=\epsilon_1+\cdots+\epsilon_k,\\ \omega_\lambda&=\lambda_1\epsilon_1+\cdots+\lambda_n\epsilon_n.\end{aligned}\tag{3.12}$$

We denote by $\mathbf{B}_{[n]}(\pm\omega_\lambda)$ the crystal base of the irreducible $U_q(\mathfrak{gl}_{[n]})$-module with highest (resp. lowest) weight vector $u_{\pm\omega_\lambda}^{[n]}$ of weight $\pm\omega_\lambda$. When $\lambda\in\mathscr{P}$, $\mathbf{B}_{[n]}(\omega_\lambda)$ can be realized as $SST_{[n]}(\lambda)$ [15]. Here we assume that $\mathbf{B}_{[1]}(\pm\omega_\lambda)=\{u_{\pm\omega_\lambda}^{[1]}\}$ with $\mathrm{wt}_{[1]}(u_{\pm\omega_\lambda}^{[1]})=\pm\lambda\epsilon_1$ for $\lambda\in\mathbb{Z}_+^1=\mathbb{Z}$.

Note that for $A\in\mathcal{E}^n$ $(n\geqslant 2)$, the j-th column A^j of A $(j\in\mathbb{Z})$ is $\mathfrak{gl}_{[n]}$-equivalent to an element in the trivial crystal $\mathbf{B}_{[n]}(0)$ or $\mathbf{B}_{[n]}(\omega_k)$ for some $k\in[n]$, and it is non-trivial for only finitely many j's.

Proposition 3.10. *As a $(\mathfrak{gl}_\infty,\mathfrak{gl}_{[n]})$-bicrystal, we have*

$$\mathcal{E}^n\simeq\bigsqcup_{\substack{\mu\in\mathscr{P}\\ \mu_1\leqslant n}}\mathbf{B}_\mu\times\mathbf{B}_{[n]}(\omega_{\mu'}).$$

Proof. We may assume that $n \geqslant 2$. Let $\mu \in \mathscr{P}$ be given with $\mu_1 \leqslant n$. Let $A_\mu = \sigma_\mu(u_\mu^{[1,\infty)})$ (see (3.10)). Recall that $A_\mu = (a_{ij})$ is of the form;

$$a_{ij} = \begin{cases} 1 & \text{if } 1 \leqslant j \leqslant \mu'_{n-i+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is straightforward to see that A_μ is $\mathfrak{gl}_{[n]}$-equivalent to the lowest weight vector in $\mathbf{B}_{[n]}(\omega_{\mu'})$, that is, $S_{w_{[n]}} u^{[n]}_{\omega_{\mu'}}$, where $w_{[n]}$ is the longest element in the Weyl group of $\mathfrak{gl}_{[n]}$. This implies that $C(A_\mu)$ the connected component in $\mathcal{E}^n$ including A_μ is isomorphic to $\mathbf{B}_\mu \times \mathbf{B}_{[n]}(\omega_{\mu'})$ as a $(\mathfrak{gl}_\infty, \mathfrak{gl}_{[n]})$-bicrystal. Note that for $\mu, \nu \in \mathscr{P}$, $C(A_\mu) = C(A_\nu)$ if and only if $\mu = \nu$.

Suppose that $B \in \mathcal{E}^n$ is given. Choose an interval $[p, q]$ in $\mathbb{Z}$ such that all non-zero entries of B are placed in its $[n] \times [p, q]$ submatrix. Since $\mathbf{M}_{[n],[p,q]}$ is a $(\mathfrak{gl}_{[n]}, \mathfrak{gl}_{[p,q]})$-bicrystal, B is connected to $B' = (b'_{ij})$, which is of the following form;

$$b'_{ij} = \begin{cases} 1 & \text{if } p \leqslant j \leqslant \mu'_{n-i+1} + p - 1, \\ 0 & \text{otherwise,} \end{cases}$$

for some $\mu \in \mathscr{P}$ with $\mu_1 \leqslant n$ (see [16, Theorem 4.5]). Now, we have $S_w B' = A_\mu$ for $w \in W$ such that $w(\mathrm{wt}(B')) = \mathrm{wt}(A_\mu)$, where S_w is defined with respect to $\tilde{e}_k$ and $\tilde{f}_k$ ($k \in \mathbb{Z}$). Hence $C(B') = C(A_\mu)$. This completes the proof. $\square$

Considering the dual crystal of $\mathcal{E}^n$, we have

$$\left(\mathcal{E}^n\right)^\vee \simeq \bigsqcup_{\substack{\mu \in \mathscr{P} \\ \mu_1 \leqslant n}} \mathbf{B}_\mu^\vee \times \mathbf{B}_{[n]}(-\omega_{\mu'}). \tag{3.13}$$

3.3. *Highest weight crystals*

Let $\mathcal{F}$ be the subset of $\mathbf{M}_{\{1\},\mathbb{Z}}$ consisting of $(a_{1j})_{j \in \mathbb{Z}}$ such that $a_{1j} = 1$ for $j \ll 0$ and $a_{1j} = 0$ for $j \gg 0$. For $A = (a_{1j})_{j \in \mathbb{Z}} \in \mathcal{F}$, define

$$\mathrm{wt}(A) = \Lambda_0 + \sum_{j > 0} a_{1j}\epsilon_j + \sum_{j \leqslant 0} (a_{1j} - 1)\epsilon_j. \tag{3.14}$$

Then $\mathcal{F}$ is a regular $\mathfrak{gl}_\infty$-crystal with respect to $\tilde{e}_k, \tilde{f}_k$ ($k \in \mathbb{Z}$) and wt, and there exists a strict embedding

$$\iota_i : \mathbf{B}(\Lambda_i) \to \mathcal{F} \tag{3.15}$$

for $i \in \mathbb{Z}$, where the highest weight vector u_{Λ_i} of $\mathbf{B}(\Lambda_i)$ is mapped to the unique element of weight Λ_i, that is, $\iota_i(u_{\Lambda_i}) = (a_{1j})_{j \in \mathbb{Z}}$ with $a_{1j} = 1$ for $j \leqslant i$ and $a_{1j} = 0$ otherwise. Then, we have

$$\mathcal{F} \simeq \bigsqcup_{i \in \mathbb{Z}} \mathbf{B}(\Lambda_i). \tag{3.16}$$

Recall that $\mathcal{F}$ is the crystal base of the q-deformed Fock space representation, which can be realized as the space of semi-infinite wedge vectors [23,28].

For $n \geqslant 1$, let $\mathcal{F}^n$ be the set of matrices A in $\mathbf{M}_{[n],\mathbb{Z}}$ such that $A_i \in \mathcal{F} \subset \mathbf{M}_{\{i\},\mathbb{Z}}$ for $i \in [n]$. Then $\mathcal{F}^n$ is row admissible and can be identified with $\mathcal{F}^{\otimes n}$ as a $\mathfrak{gl}_\infty$-crystal, where $A \in \mathcal{F}^n$ is identified with

$A_1 \otimes \cdots \otimes A_n \in \mathcal{F}^{\otimes n}$. Also for $\lambda \in \mathbb{Z}_+^n$, we may regard $\mathbf{B}(\Lambda_\lambda) \subset \mathbf{B}(\Lambda_{\lambda_n}) \otimes \cdots \otimes \mathbf{B}(\Lambda_{\lambda_1})$ by identifying u_{Λ_λ} with $u_{\Lambda_{\lambda_n}} \otimes \cdots \otimes u_{\Lambda_{\lambda_1}}$. Composing with (3.15), we have a strict embedding

$$\iota_\lambda = \iota_{\lambda_n} \times \cdots \times \iota_{\lambda_1} : \mathbf{B}(\Lambda_\lambda) \to \mathcal{F}^n. \tag{3.17}$$

Taking dual crystals in (3.15) and (3.17), we also have embeddings $\iota_i^\vee$ and $\iota_\lambda^\vee$, respectively.

On the other hand, define

$$\mathrm{wt}_{[n]}(A) = \sum_{i \in [n]} \Big(\sum_{j>0} a_{ij} \Big) \epsilon_i + \sum_{i \in [n]} \Big(\sum_{j \leqslant 0} (a_{ij} - 1) \Big) \epsilon_i, \tag{3.18}$$

for $A \in \mathcal{F}^n$. Then $\mathcal{F}^n$ $(n \geqslant 2)$ is column admissible and it is a regular $\mathfrak{gl}_{[n]}$-crystal with respect to $\widetilde{E}_i, \widetilde{F}_i$ $(i \in [n]^\circ)$ and $\mathrm{wt}_{[n]}$. Hence, $\mathcal{F}^n$ is a $(\mathfrak{gl}_\infty, \mathfrak{gl}_{[n]})$-bicrystal by Lemma 3.2.

As in the case of $\mathcal{E}^n$, for $A \in \mathcal{F}^n$ $(n \geqslant 2)$, the j-th column A^j of A $(j \in \mathbb{Z})$ is $\mathfrak{gl}_{[n]}$-equivalent to an element in $\mathbf{B}_{[n]}(0)$ or $\mathbf{B}_{[n]}(\pm\omega_k)$ for some $1 \leqslant k \leqslant n$, and it is non-trivial for only finitely many j's.

The following theorem is a crystal version of the $(\mathfrak{gl}_\infty, \mathfrak{gl}_{[n]})$-duality on the level n fermionic Fock space [5].

Proposition 3.11. *(Cf. [19].) As a $(\mathfrak{gl}_\infty, \mathfrak{gl}_{[n]})$-bicrystal, we have*

$$\mathcal{F}^n \simeq \bigsqcup_{\lambda \in \mathbb{Z}_+^n} \mathbf{B}(\Lambda_\lambda) \times \mathbf{B}_{[n]}(\omega_\lambda).$$

If we consider $(\mathcal{F}^n)^\vee$, then we obtain

$$\big(\mathcal{F}^n\big)^\vee \simeq \bigsqcup_{\lambda \in \mathbb{Z}_+^n} \mathbf{B}(-\Lambda_\lambda) \times \mathbf{B}_{[n]}(-\omega_\lambda).$$

As in the case of $(\mathcal{E}^n)^\vee$, we may view $(\mathcal{F}^n)^\vee = \{A^\vee \mid A \in \mathcal{F}^n\}$.

3.4. Littlewood–Richardson coefficients

Let us recall some basic notions in symmetric functions [22]. Let $x = \{x_1, x_2, x_3, \ldots\}$ be the set of formal commuting variables. Let *Sym* be the ring of symmetric functions in x. For $k \geqslant 1$, denote by $e_k(x)$, $h_k(x)$ and $p_k(x)$ the k-th elementary, complete and power sum symmetric functions in x, respectively. It is well known that $\{e_k(x) \mid k \geqslant 1\}$ and $\{h_k(x) \mid k \geqslant 1\}$ are algebraically independent over $\mathbb{Z}$ in *Sym* and $\{p_k(x) \mid k \geqslant 1\}$ is algebraically independent over $\mathbb{Q}$ in $Sym_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} Sym$. For $\lambda \in \mathscr{P}$, let $s_\lambda(x)$ be the Schur function in x corresponding to λ. The *Littlewood–Richardson coefficients* $c^\lambda_{\mu\nu}$ for $\lambda, \mu, \nu \in \mathscr{P}$ are defined by

$$s_\mu(x) s_\nu(x) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(x). \tag{3.19}$$

For $n \geqslant 1$, let $x_{[n]} = \{x_1, \ldots, x_n\}$. For $\lambda \in \mathscr{P}$ with $\ell(\lambda) \leqslant n$, let $s_\lambda(x_{[n]})$ be the corresponding Schur polynomial in $x_{[n]}$. Put $\mathrm{ch}\,\mathbf{B}_{[n]}(\omega_\lambda) = \sum_{T \in \mathbf{B}_{[n]}(\omega_\lambda)} x_{[n]}^T$, where $x_{[n]}^T = \prod_i x_i^{m_i}$ for $T \in \mathbf{B}_{[n]}(\omega_\lambda)$ with $\mathrm{wt}_{[n]}(T) = \sum_i m_i \epsilon_i$. Then we have $\mathrm{ch}\,\mathbf{B}_{[n]}(\omega_\lambda) = s_\lambda(x_{[n]})$.

Proposition 3.12. *For $\mu, \nu \in \mathscr{P}$, we have*

$$\mathbf{B}_\mu \otimes \mathbf{B}_\nu \simeq \bigsqcup_{\lambda \in \mathscr{P}} \mathbf{B}_\lambda^{\oplus c^\lambda_{\mu\nu}}.$$

Proof. Let m, n be positive integers. Put $[n] + m = \{m+1, \ldots, m+n\}$. Then $\mathfrak{gl}_{[m]} \oplus \mathfrak{gl}_{[n]+m}$ is a subalgebra of $\mathfrak{gl}_{[m+n]}$. By Proposition 3.10, we have

$$\mathcal{E}^m \otimes \mathcal{E}^n \simeq \bigsqcup_{\substack{\mu,\nu \in \mathscr{P} \\ \mu_1 \leqslant m,\ \nu_1 \leqslant n}} (\mathbf{B}_\mu \otimes \mathbf{B}_\nu) \times \big(\mathbf{B}_{[m]}(\omega_{\mu'}) \times \mathbf{B}_{[n]+m}(\omega_{\nu'})\big), \tag{3.20}$$

as a $(\mathfrak{gl}_\infty, \mathfrak{gl}_{[m]} \oplus \mathfrak{gl}_{[n]+m})$-bicrystal. On the other hand, we have

$$\mathcal{E}^{m+n} \simeq \bigsqcup_{\substack{\lambda \in \mathscr{P} \\ \lambda_1 \leqslant m+n}} \mathbf{B}_\lambda \times \mathbf{B}_{[m+n]}(\omega_{\lambda'}), \tag{3.21}$$

as a $(\mathfrak{gl}_\infty, \mathfrak{gl}_{[m+n]})$-bicrystal. Since $s_{\lambda'}(x_{[m+n]}) = \sum_{\mu'\nu'} c^{\lambda'}_{\mu'\nu'} s_{\mu'}(x_{[m]}) s_{\nu'}(x_{[n]+m})$ and $c^{\lambda'}_{\mu'\nu'} = c^\lambda_{\mu\nu}$, we have as a $\mathfrak{gl}_{[m]} \oplus \mathfrak{gl}_{[n]+m}$-crystal,

$$\mathbf{B}_{[m+n]}(\omega_{\lambda'}) \simeq \bigsqcup_{\mu',\nu'} \mathbf{B}_{[m]}(\omega_{\mu'}) \times \mathbf{B}_{[n]+m}(\omega_{\nu'})^{\oplus c^\lambda_{\mu\nu}}.$$

Since $\mathcal{E}^{m+n} \simeq \mathcal{E}^m \otimes \mathcal{E}^n$ as a $(\mathfrak{gl}_\infty, \mathfrak{gl}_{[m]} \oplus \mathfrak{gl}_{[n]+m})$-bicrystal, we obtain the required decomposition of $\mathbf{B}_\mu \otimes \mathbf{B}_\nu$ by comparing (3.20) and (3.21). □

Let $m, n \geqslant 1$ be given. Recall that for $\mu \in \mathbb{Z}^m_+$, $\operatorname{ch} \mathbf{B}_{[m]}(\omega_\mu) = s_\mu(x_{[m]}) = (x_1 \cdots x_m)^{-p} s_{\mu+(p^m)}(x_{[m]})$ is the Laurent Schur polynomial corresponding to μ, where p is a non-negative integer such that $\mu + (p^m) \in \mathscr{P}$.

For $\lambda \in \mathbb{Z}^{m+n}_+$, $\mu \in \mathbb{Z}^m_+$ and $\nu \in \mathbb{Z}^n_+$, we define

$$c^\lambda_{\mu\nu} = c^{\lambda+(p^{m+n})}_{\mu+(p^m)\nu+(p^n)}, \tag{3.22}$$

where p is a non-negative integer such that $\lambda + (p^{m+n}), \mu + (p^m), \nu + (p^n) \in \mathscr{P}$. Note that

$$s_\lambda(x_{[m+n]}) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x_{[m]}) s_\nu(x_{[n]+m}) \tag{3.23}$$

and $c^\lambda_{\mu\nu}$ does not depend on p. Also, for $\lambda, \mu, \nu \in \mathbb{Z}^m_+$, we define $c^\lambda_{\mu\nu}$ to be the coefficient of $s_\lambda(x_{[m]})$ in $s_\mu(x_{[m]}) s_\nu(x_{[m]})$.

Proposition 3.13. *For $\mu \in \mathbb{Z}^m_+$, $\nu \in \mathbb{Z}^n_+$, we have*

$$\mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(\Lambda_\nu) \simeq \bigsqcup_{\lambda \in \mathbb{Z}^{m+n}_+} \mathbf{B}(\Lambda_\lambda)^{\oplus c^\lambda_{\mu\nu}}.$$

Proof. The proof is almost the same as that of Proposition 3.12. Here we compare the $(\mathfrak{gl}_\infty, \mathfrak{gl}_{[m]} \oplus \mathfrak{gl}_{[n]+m})$-bicrystal decompositions of $\mathcal{F}^m \otimes \mathcal{F}^n$ and $\mathcal{F}^{m+n}$. □

Remark 3.14. Note that there are infinitely many connected components in $\mathbf{B}(\Lambda_\mu) \otimes \mathbf{B}(\Lambda_\nu)$, but the multiplicity of each connected component is finite.

3.5. Tensor product

Let us end this section with introducing another family of regular connected $\mathfrak{gl}_\infty$-crystals.

Proposition 3.15. *For $\mu, \nu \in \mathscr{P}$ and $\Lambda \in P^+$, $\mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda)$ is connected.*

Proof. Suppose that $S \otimes T \otimes U \in \mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda)$ is given. By Remark 3.5, there exist $i_1, \ldots, i_r \in \mathbb{Z}$ such that $\tilde{e}_{i_1} \cdots \tilde{e}_{i_r}(S \otimes T) = u_\mu^{[p,\infty)} \otimes v_\nu^{(-\infty,q]}$ for some $p < q$. By tensor product rule of crystals, we have

$$\tilde{e}_{i_1}^{m_1} \cdots \tilde{e}_{i_r}^{m_r}(S \otimes T \otimes U) = u_\mu^{[p,\infty)} \otimes v_\nu^{(-\infty,q]} \otimes U'$$

for some $m_1, \ldots, m_r \geqslant 1$ and $U' \in \mathbf{B}(\Lambda)$. We may assume that $p \ll 0$ and $q \gg 0$ so that $\tilde{e}_{j_1} \cdots \tilde{e}_{j_s} U' = u_\Lambda$ for some $j_1, \ldots, j_s \in [p + \ell(\mu) + 1, q - \ell(\nu) - 1]$. Since $\tilde{x}_{j_t}(u_\mu^{[p,\infty)} \otimes v_\nu^{(-\infty,q]}) = \mathbf{0}$ for $1 \leqslant t \leqslant s$ and $x = e, f$, we get

$$\tilde{e}_{j_1} \cdots \tilde{e}_{j_s}\big(u_\mu^{[p,\infty)} \otimes v_\nu^{(-\infty,q]} \otimes U'\big) = u_\mu^{[p,\infty)} \otimes v_\nu^{(-\infty,q]} \otimes u_\Lambda.$$

Since p (resp. q) can be arbitrarily small (resp. large), we conclude that $\mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda)$ is connected. $\square$

Proposition 3.16. *For $\mu, \nu, \sigma, \tau \in \mathscr{P}$ and $\Lambda, \Lambda' \in P^+$, $\mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda) \simeq \mathbf{B}_{\sigma,\tau} \otimes \mathbf{B}(\Lambda')$ if and only if $(\mu, \nu, \Lambda) = (\sigma, \tau, \Lambda')$.*

Proof. Suppose that $\mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda) \simeq \mathbf{B}_{\sigma,\tau} \otimes \mathbf{B}(\Lambda')$. Let $S \otimes T \otimes U \in \mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda)$ be equivalent to $S' \otimes T' \otimes U' \in \mathbf{B}_{\sigma,\tau} \otimes \mathbf{B}(\Lambda')$. Applying suitable $\tilde{e}_k$'s, we assume that there exist $s \ll 0$ and $t \gg 0$ such that the entries in S and S' are less than s, and the entries in T and T' are less than $t^\vee$ (see Remark 3.5 and the proof of Proposition 3.15). We may further assume that $\tilde{x}_k U = \tilde{x}_k U' = \mathbf{0}$ for $k \notin [s,t]$. By similar arguments as in Proposition 3.9, we have $S = S'$ and $T = T'$. Also U is $\mathfrak{gl}_{[s,t]}$-equivalent to U', which implies that $U = U'$. $\square$

4. Realization of extremal weight crystals

4.1. Extremal weight crystals for $\mathfrak{gl}_\infty$

Let us briefly recall the crystal bases of the modified quantized enveloping algebra of $\mathfrak{gl}_\infty$ and an extremal weight module over $U_q(\mathfrak{gl}_\infty)$ (see [11,13] for more details). Let $\widetilde{U}_q(\mathfrak{gl}_\infty) = \bigoplus_{\Lambda \in P} U_q(\mathfrak{gl}_\infty) a_\Lambda$ be the modified quantized enveloping algebra of $\mathfrak{gl}_\infty$ and let

$$\mathbf{B}\big(\widetilde{U}_q(\mathfrak{gl}_\infty)\big) = \bigsqcup_{\Lambda \in P} \mathbf{B}\big(U_q(\mathfrak{gl}_\infty) a_\Lambda\big) \tag{4.1}$$

be its crystal base. It is known that $\mathbf{B}(\widetilde{U}_q(\mathfrak{gl}_\infty))$ is regular, and

$$\mathbf{B}\big(U_q(\mathfrak{gl}_\infty) a_\Lambda\big) \simeq \mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty) \tag{4.2}$$

for $\Lambda \in P$, where $\mathbf{B}(\infty)$ (resp. $\mathbf{B}(-\infty)$) is the crystal base of the negative (resp. positive) part of $U_q(\mathfrak{gl}_\infty)$, and $T_\Lambda = \{t_\Lambda\}$ is a $\mathfrak{gl}_\infty$-crystal with $\mathrm{wt}(t_\Lambda) = \Lambda$ and $\varepsilon_i(t_\Lambda) = \varphi_i(t_\Lambda) = -\infty$ $(i \in \mathbb{Z})$.

An element b of a regular $\mathfrak{gl}_\infty$-crystal B with $\mathrm{wt}(b) = \Lambda$ is called *extremal* if $\{S_w b \mid w \in W\}$ satisfies the following conditions; (1) $\tilde{e}_i S_w b = \mathbf{0}$ if $\langle w(\Lambda), h_i \rangle \geqslant 0$, (2) $\tilde{f}_i S_w(b) = \mathbf{0}$ if $\langle w(\Lambda), h_i \rangle \leqslant 0$.

For $\Lambda \in P$, let

$$\mathbf{B}(\Lambda) = \{ b \in \mathbf{B}(U_q(\mathfrak{gl}_\infty)a_\Lambda) \mid b^* \text{ is extremal} \}, \tag{4.3}$$

where $*$ is the star involution on $\mathbf{B}(\widetilde{U}_q(\mathfrak{gl}_\infty))$. Then $\mathbf{B}(\Lambda)$ is the crystal base of the $U_q(\mathfrak{gl}_\infty)$-module generated by an extremal weight vector u_Λ of weight Λ, which is called an extremal weight module. Note that (1) $\mathbf{B}(\Lambda) \simeq \mathbf{B}(w\Lambda)$ for $w \in W$, and (2) if $\Lambda \in P^+$, then $\mathbf{B}(\Lambda) \simeq \mathbf{B}(\Lambda_\lambda)$ for some $\lambda \in \mathbb{Z}_+^n$. From now on, we call $\mathbf{B}(\Lambda)$ simply an *extremal weight crystal*.

Proposition 4.1. *For $\Lambda \in P$, $\mathbf{B}(\Lambda)$ is connected.*

Proof. We regard $\mathbf{B}(\Lambda)$ as a subcrystal of $\mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$ and identify $u_\Lambda \in \mathbf{B}(\Lambda)$ with $u_\infty \otimes t_\Lambda \otimes u_{-\infty}$, where $u_{\pm\infty}$ is the highest (resp. lowest) weight element in $\mathbf{B}(\pm\infty)$. Let $b \in \mathbf{B}(\Lambda)$ be given. We may assume that b is extremal since any element in $\mathbf{B}(\Lambda)$ is connected to an extremal one. By the same argument as in [13, Theorem 5.1], b is connected to $b_1 \otimes t_\Lambda \otimes u_{-\infty}$, where $\langle \mathrm{wt}(b_1), h_i \rangle \geqslant 0$ for all $i \in \mathbb{Z}$. Since $\mathrm{wt}(b_1) = \sum_{i\in\mathbb{Z}} m_i\alpha_i = \sum_{i\in\mathbb{Z}} m_i(\epsilon_i - \epsilon_{i+1})$ with $m_i \in \mathbb{Z}_{\leqslant 0}$ and $P_0^+ = \{0\}$, we have $m_i = 0$ for all $i \in \mathbb{Z}$ and $b_1 = u_\infty$. Therefore, $\mathbf{B}(\Lambda)$ is connected. □

Corollary 4.2. *For $\Lambda \in P$, $\mathbf{B}(\Lambda)$ is isomorphic to the connected component in $\mathbf{B}(\Lambda_+) \otimes \mathbf{B}(-\Lambda_-)$ including $u_{\Lambda_+} \otimes u_{-\Lambda_-}$.*

Proof. Recall that there is a strict embedding of regular crystals

$$\mathbf{B}(\Lambda_+) \otimes \mathbf{B}(-\Lambda_-) \to \mathbf{B}(\infty) \otimes T_\Lambda \otimes \mathbf{B}(-\infty)$$

sending $u_{\Lambda_+} \otimes u_{-\Lambda_-}$ to $u_\infty \otimes t_\Lambda \otimes u_{-\infty}$. Since $\mathbf{B}(\Lambda) \simeq C(u_\infty \otimes t_\Lambda \otimes u_{-\infty})$ by Proposition 4.1, we have $\mathbf{B}(\Lambda) \simeq C(u_{\Lambda_+} \otimes u_{-\Lambda_-}) \subset \mathbf{B}(\Lambda_+) \otimes \mathbf{B}(-\Lambda_-)$. □

4.2. Realization of extremal weight crystals

Lemma 4.3. *For $i, j \in \mathbb{Z}$ ($i \leqslant j$), we have*

$$\mathbf{B}(\Lambda_i) \otimes \mathbf{B}(-\Lambda_j) \simeq \bigsqcup_{a\geqslant 0} \mathbf{B}_{(1^a),(1^{a+j-i})},$$

$$\mathbf{B}(\Lambda_j) \otimes \mathbf{B}(-\Lambda_i) \simeq \bigsqcup_{a\geqslant 0} \mathbf{B}_{(1^{a+j-i}),(1^a)}.$$

Proof. Let us prove the first isomorphism. The second one is obtained by considering the dual crystals of both sides in the first isomorphism.

Suppose that $S \otimes T \in \mathbf{B}(\Lambda_i) \otimes \mathbf{B}(-\Lambda_j)$ is given. Applying suitable $\tilde{e}_k$'s, we assume that $S = u_{\Lambda_i}$. Then applying $\tilde{f}_k$'s ($k \neq i$), $u_{\Lambda_i} \otimes T$ is connected to $u_{\Lambda_i} \otimes T'$ such that $\tilde{f}_k(u_{\Lambda_i} \otimes T') = \mathbf{0}$ for all $k \neq i$.

Let A be the image of $u_{\Lambda_i} \otimes T'$ under the strict embedding $\iota_i \times \iota_j^\vee : \mathbf{B}(\Lambda_i) \otimes \mathbf{B}(-\Lambda_j) \to \mathcal{F} \otimes \mathcal{F}^\vee \subset \mathbf{M}_{[2],\mathbb{Z}}$ (cf. (3.9)). Then there exists $a \geqslant 0$ such that

$$\begin{cases} a_{1k} = 1 & \text{if and only if } k \leqslant i, \\ a_{2k} = 1 & \text{if and only if } i - a + 1 \leqslant k \leqslant i \text{ or } j + a + 1 \leqslant k. \end{cases}$$

Let $B \in \mathcal{E} \otimes \mathcal{E}^\vee \subset \mathbf{M}_{[2],\mathbb{Z}}$ be such that

$$\begin{cases} b_{1k} = 1 & \text{if and only if } i - a + 1 \leqslant k \leqslant i, \\ b_{2k} = 0 & \text{if and only if } i + 1 \leqslant k \leqslant j + a. \end{cases}$$

Note that $C(B) \simeq \mathbf{B}_{(1^a),(1^{a+j-i})}$. For $p, q \in \mathbb{Z}$ $(p < q)$, let $\pi_{[p,q]} : \mathbf{M}_{[2],\mathbb{Z}} \to \mathbf{M}_{[2],[p,q]}$ be the map sending a matrix to its $[2] \times [p,q]$ submatrix. Assume that $p \ll 0 \ll q$. Then $\pi_{[p,q]}(A)$ is column admissible and $\widetilde{F}_1^{\max}\pi_{[p,q]}(A) = \pi_{[p,q]}(B)$. By Lemma 3.3, we have

$$\tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \pi_{[p,q]}(A) \neq \mathbf{0} \quad \Leftrightarrow \quad \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \pi_{[p,q]}(B) \neq \mathbf{0},$$

and hence

$$\tilde{x}_{i_1} \cdots \tilde{x}_{i_r} A \neq \mathbf{0} \quad \Leftrightarrow \quad \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} B \neq \mathbf{0}$$

for $r \geqslant 1$ and $p \leqslant i_1, \ldots, i_r \leqslant q - 1$, where $x = e$ or f for each i_k. Since p and q are arbitrary and $\mathrm{wt}(A) = \mathrm{wt}(B)$, we have $A \equiv B$ or $u_{\Lambda_i} \otimes T' \equiv B$, which implies that $C(u_{\Lambda_i} \otimes T') \simeq \mathbf{B}_{(1^a),(1^{a+j-i})}$.

Conversely, for each $a \geqslant 0$, let B be the matrix in $\mathcal{E} \otimes \mathcal{E}^\vee$ which is of the above form. Then there exists a unique $T \in \mathbf{B}(-\Lambda_j)$ such that $u_{\Lambda_i} \otimes T \equiv B$ since the construction of B is reversible. This completes the proof. □

Lemma 4.4. *For $i \in \mathbb{Z}$ and $k \geqslant 0$, we have*

$$\mathbf{B}(\Lambda_i) \otimes \mathbf{B}_{(1^k)} \simeq \bigsqcup_{a=0}^{k} \mathbf{B}_{(1^a)} \otimes \mathbf{B}(\Lambda_{i+k-a}),$$

$$\mathbf{B}(\Lambda_i) \otimes \mathbf{B}^\vee_{(1^k)} \simeq \bigsqcup_{a=0}^{k} \mathbf{B}^\vee_{(1^a)} \otimes \mathbf{B}(\Lambda_{i-k+a}).$$

Proof. The proof is similar to that of Lemma 4.3. Let us prove the first isomorphism. Suppose that $S \otimes T \in \mathbf{B}(\Lambda_i) \otimes \mathbf{B}_{(1^k)}$ is given. Applying $\tilde{e}_k$'s, we assume that $S = u_{\Lambda_i}$.

Let A be the image of $u_{\Lambda_i} \otimes T$ under the strict embedding $\iota_i \times \sigma_k : \mathbf{B}(\Lambda_i) \otimes \mathbf{B}_{(1^k)} \to \mathcal{F} \otimes \mathcal{E} \subset \mathbf{M}_{[2],\mathbb{Z}}$. Applying suitable $\tilde{x}_s$'s for $s \neq i$ and $x = e, f$, we may assume that $a_{2j} = 1$ if and only if $i - a + 1 \leqslant j \leqslant i - a + k$ for some $a \geqslant 0$. Let $B \in \mathcal{E} \otimes \mathcal{F} \subset \mathbf{M}_{[2],\mathbb{Z}}$ be such that

$$\begin{cases} b_{1j} = 1 & \text{if and only if } i - a + 1 \leqslant j \leqslant i, \\ b_{2j} = 1 & \text{if and only if } j \leqslant i - a + k. \end{cases}$$

Note that $C(B) \simeq \mathbf{B}_{(1^a)} \otimes \mathbf{B}(\Lambda_{i-a+k})$. Choose $p \ll 0 \ll q$. Then $\pi_{[p,q]}(A)$ is column admissible and $\widetilde{F}_1^{\max}\pi_{[p,q]}(A) = \pi_{[p,q]}(B)$. By Lemma 3.3, we have

$$\tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \pi_{[p,q]}(A) \neq \mathbf{0} \quad \Leftrightarrow \quad \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} \pi_{[p,q]}(B) \neq \mathbf{0},$$

and hence

$$\tilde{x}_{i_1} \cdots \tilde{x}_{i_r} A \neq \mathbf{0} \quad \Leftrightarrow \quad \tilde{x}_{i_1} \cdots \tilde{x}_{i_r} B \neq \mathbf{0}$$

for $r \geqslant 1$ and $p \leqslant i_1, \ldots, i_r \leqslant q - 1$, where $x = e$ or f for each i_s. Since p and q are arbitrary and $\mathrm{wt}(A) = \mathrm{wt}(B)$, we have $A \equiv B$ or $u_{\Lambda_i} \otimes T \equiv B$, which implies that $C(u_{\Lambda_i} \otimes T) \simeq \mathbf{B}_{(1^a)} \otimes \mathbf{B}(\Lambda_{i-a+k})$.

Conversely, for $a \geqslant 0$, let B be the matrix in $\mathcal{E} \otimes \mathcal{F}$ which is of the above form. Then we can find a unique $T \in \mathbf{B}_{(1^k)}$ such that $u_{\Lambda_i} \otimes T \equiv B$ since the construction of B is reversible. This establishes the first isomorphism.

The second isomorphism can be proved by modifying the above argument. $\square$

Proposition 4.5. *For $m \geqslant n \geqslant 0$, a connected component in $\mathcal{F}^m \otimes (\mathcal{F}^n)^\vee$ is isomorphic to $\mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda)$ for some $\mu, \nu \in \mathscr{P}$ and $\Lambda \in P^+_{m-n}$.*

Proof. We claim that each $A \in \mathcal{F}^m \otimes (\mathcal{F}^n)^\vee$ is equivalent to an element in $\mathcal{E}^n \otimes (\mathcal{E}^n)^\vee \otimes \mathcal{F}^{m-n}$. Then it follows from Propositions 3.10, 3.11 and 3.15 that $C(A) \simeq \mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda)$ for some $\mu, \nu \in \mathscr{P}$ and $\Lambda \in P^+_{m-n}$.

We use induction on $m+n$. Suppose that $m+n=2$. If $m=n=1$, then it is clear by Lemma 4.3. If $m=2$ and $n=0$, then it follows from Proposition 3.11. Suppose that $m+n \geqslant 3$. Let $A = A_1 \otimes \cdots \otimes A_{m+n} \in \mathcal{F}^m \otimes (\mathcal{F}^n)^\vee$ be given, where $A_i \in \mathcal{F}$ and $A_{m+j} \in \mathcal{F}^\vee$ for $i \in [m]$ and $j \in [n]$. Consider $A_m \otimes A_{m+1} \in \mathcal{F} \otimes \mathcal{F}^\vee$. By (3.16) and Lemma 4.3, $A_m \otimes A_{m+1}$ is equivalent to some $A'_m \otimes A'_{m+1} \in \mathcal{E} \otimes \mathcal{E}^\vee$. Applying Lemma 4.4 to $A_1 \otimes \cdots \otimes A_{m-1}$ and $A'_m \otimes A'_{m+1}$ repeatedly, we can say that A is equivalent to some $B = B_1 \otimes \cdots \otimes B_{m+n}$ in $\mathcal{E} \otimes \mathcal{E}^\vee \otimes \mathcal{F}^{m-1} \otimes (\mathcal{F}^{n-1})^\vee$. By induction hypothesis, $B_3 \otimes \cdots \otimes B_{m+n}$ is equivalent to some $B'_3 \otimes \cdots \otimes B'_{m+n}$ in $\mathcal{E}^{n-1} \otimes (\mathcal{E}^{n-1})^\vee \otimes \mathcal{F}^{m-n}$. Finally, by Lemma 3.6, $B_1 \otimes B_2 \otimes B'_3 \otimes \cdots \otimes B'_{2n} \in \mathcal{E} \otimes \mathcal{E}^\vee \otimes \mathcal{E}^{n-1} \otimes (\mathcal{E}^{n-1})^\vee$ is equivalent to an element in $\mathcal{E}^n \otimes (\mathcal{E}^n)^\vee$. Therefore A is equivalent to an element in $\mathcal{E}^n \otimes (\mathcal{E}^n)^\vee \otimes \mathcal{F}^{m-n}$. This completes the induction. $\square$

Theorem 4.6. *For $\Lambda \in P_\ell$ ($\ell \in \mathbb{Z}$), there exist unique $\mu, \nu \in \mathscr{P}$ and $\Lambda' \in P^+_{|\ell|}$ such that*

$$\mathbf{B}(\Lambda) \simeq \begin{cases} \mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda') & \text{if } \ell \geqslant 0, \\ \mathbf{B}(-\Lambda') \otimes \mathbf{B}_{\mu,\nu} & \text{if } \ell \leqslant 0. \end{cases}$$

Proof. The first isomorphism follows immediately from Proposition 3.11, Corollary 4.2 and Proposition 4.5. If $\ell \leqslant 0$, then $\mathbf{B}(\Lambda)$ is embedded into $\mathcal{F}^m \otimes (\mathcal{F}^n)^\vee$ for some m, n with $m-n=\ell$. Since $\mathbf{B}(\Lambda)^\vee$ is embedded into $\mathcal{F}^n \otimes (\mathcal{F}^m)^\vee$, it is isomorphic to $\mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda')$ for some $\Lambda' \in P^+_{|\ell|}$ and $\mu, \nu \in \mathscr{P}$. Hence, $\mathbf{B}(\Lambda) \simeq \mathbf{B}(-\Lambda') \otimes \mathbf{B}_{\nu,\mu}$. The uniqueness follows from Proposition 3.16. $\square$

Corollary 4.7. *For $\Lambda \in P_0$, there exist unique $\mu, \nu \in \mathscr{P}$ such that $\mathbf{B}(\Lambda) \simeq \mathbf{B}_{\mu,\nu}$.*

Remark 4.8. Combining with a tableaux description of $\mathbf{B}(\Lambda)$ ($\Lambda \in P^+$) (see for example [20]), we obtain a combinatorial realization of an extremal weight crystal.

4.3. Pieri rules of extremal weight crystals

We have the following generalization of Lemma 4.4.

Proposition 4.9. *For $\lambda \in \mathbb{Z}^n_+$ and $k \geqslant 1$, we have*

$$\mathbf{B}(\Lambda_\lambda) \otimes \mathbf{B}_{(1^k)} \simeq \bigsqcup_{a=0}^{k} \bigsqcup_{\substack{\mu \in \mathbb{Z}^n_+ \\ (\mu - (\lambda^n_n))/(\lambda - (\lambda^n_n)): \\ \text{a horizontal strip of length } k-a}} \mathbf{B}_{(1^a)} \otimes \mathbf{B}(\Lambda_\mu),$$

$$\mathbf{B}(\Lambda_\lambda) \otimes \mathbf{B}^\vee_{(1^k)} \simeq \bigsqcup_{a=0}^{k} \bigsqcup_{\substack{\nu \in \mathbb{Z}^n_+ \\ (\lambda - (\nu^n_n))/(\nu - (\nu^n_n)): \\ \text{a horizontal strip of length } k-a}} \mathbf{B}^\vee_{(1^a)} \otimes \mathbf{B}(\Lambda_\nu).$$

Proof. First, consider $\mathbf{B}(\Lambda_\lambda)\otimes\mathbf{B}_{(1^k)}$. Given $S\otimes T\in\mathbf{B}(\Lambda_\lambda)\otimes\mathbf{B}_{(1^k)}$, let A be the image of $S\otimes T$ under the strict embedding $\iota_\lambda\times\sigma_k:\mathbf{B}(\Lambda_\lambda)\otimes\mathbf{B}_{(1^k)}\to\mathcal{F}^n\otimes\mathcal{E}\subset\mathbf{M}_{[n+1],\mathbb{Z}}$. By applying suitable $\tilde{x}_s$'s for $x=e,f$ and $s\in\mathbb{Z}$, we may assume that A is of the following form;

(1) for $i\in[n]$, $a_{ij}=1$ if and only if $j\leqslant\lambda_{n-i+1}$, that is, $S=u_{\Lambda_\lambda}$,
(2) $a_{n+1\,j}=0$ for $j\leqslant\lambda_n-k$,
(3) $\sum_{i\in[n+1]}a_{ij}\geqslant\sum_{i\in[n+1]}a_{ij+1}$ for $j\in[\lambda_n-k+1,\infty)$.

Let $a=\sum_{j=\lambda_n-k+1}^{\lambda_n}a_{n+1\,j}$ and let $\mu\in\mathbb{Z}_+^n$ be given by

$$\mu_i=\begin{cases}\lambda_1+\sum_{j>\lambda_1}a_{n+1\,j} & \text{if } i=1,\\ \lambda_i+\sum_{j=\lambda_i+1}^{\lambda_{i-1}}a_{n+1\,j} & \text{if } 2\leqslant i\leqslant n \text{ and } \lambda_i<\lambda_{i-1},\\ \lambda_i & \text{if } 2\leqslant i\leqslant n \text{ and } \lambda_i=\lambda_{i-1}.\end{cases}$$

Note that μ is a well-defined generalized partition by (1) and (3), and $(\mu-(\lambda_n^n))/(\lambda-(\lambda_n^n))$ is a horizontal strip of length $k-a$. We denote the matrix of the above form by $A_{a,\mu}$. Let $B_{a,\mu}$ be the image of $u_{(1^a)}^{[\lambda_n-k+1,\infty)}\otimes u_{\Lambda_\mu}$ under the strict embedding $\sigma_a\times\iota_\mu:\mathbf{B}_{(1^a)}\otimes\mathbf{B}(\Lambda_\mu)\to\mathcal{E}\otimes\mathcal{F}^n\subset\mathbf{M}_{[n+1],\mathbb{Z}}$. Choose $p\ll 0\ll q$. Then $\pi_{[p,q]}(A_{a,\mu})$ is column admissible and

$$\widetilde{F}_1^{\max}\cdots\widetilde{F}_n^{\max}\pi_{[p,q]}(A_{a,\mu})=\pi_{[p,q]}(B_{a,\mu}).$$

By the same arguments as in Lemmas 4.3 and 4.4, we have $A_{a,\mu}\equiv B_{a,\mu}$, which implies that $C(S\otimes T)\simeq C(A_{a,\mu})\simeq C(B_{a,\mu})\simeq\mathbf{B}_{(1^a)}\otimes\mathbf{B}(\Lambda_\mu)$. Conversely, suppose that $0\leqslant a\leqslant k$ and $\mu\in\mathbb{Z}_+^n$ are given where $(\mu-(\lambda_n^n))/(\lambda-(\lambda_n^n))$ is a horizontal strip of length $k-a$. Let $B_{a,\mu}$ be as above. Then we can check that there exists a unique $T\in\mathbf{B}_{(1^k)}$ such that $u_{\Lambda_\lambda}\otimes T\equiv B_{a,\mu}$ since the construction of $B_{a,\mu}$ is reversible. This proves the first isomorphism.

Next, consider $\mathbf{B}(\Lambda_\lambda)\otimes\mathbf{B}^\vee_{(1^k)}$. Let $\iota^*_\lambda:\mathbf{B}(\Lambda_\lambda)\to\mathcal{F}^n$ be the strict embedding which sends u_{Λ_λ} to $u_{\Lambda_{\lambda_1}}\otimes\cdots\otimes u_{\Lambda_{\lambda_n}}$ (cf. (3.17)). Given $S\otimes T\in\mathbf{B}(\Lambda_\lambda)\otimes\mathbf{B}^\vee_{(1^k)}$, let A be the image of $S\otimes T$ under the strict embedding $\iota^*_\lambda\times\tau_k:\mathbf{B}(\Lambda_\lambda)\otimes\mathbf{B}^\vee_{(1^k)}\to\mathcal{F}^n\otimes\mathcal{E}^\vee\subset\mathbf{M}_{[n+1],\mathbb{Z}}$. By applying suitable $\tilde{x}_s$'s for $x=e,f$ and $s\in\mathbb{Z}$, we may assume that A is of the following form;

(1*) for $i\in[n]$, $a_{ij}=1$ if and only if $j\leqslant\lambda_i$, that is, $S=u_{\Lambda_\lambda}$,
(2*) $a_{n+1\,j}=1$ for $j\geqslant\lambda_1+k+1$,
(3*) $\sum_{i\in[n+1]}a_{ij}\leqslant\sum_{i\in[n+1]}a_{ij-1}$ for $j\in(-\infty,\lambda_1+k]$.

Let $a=\sum_{j=\lambda_1+1}^{\lambda_1+k}(1-a_{n+1\,j})$ and let $\nu\in\mathbb{Z}_+^n$ be given by

$$\nu_i=\begin{cases}\lambda_i-\sum_{j=\lambda_{i+1}+1}^{\lambda_i}(1-a_{n+1\,j}) & \text{if } 1\leqslant i\leqslant n-1 \text{ and } \lambda_{i+1}<\lambda_i,\\ \lambda_i & \text{if } 1\leqslant i\leqslant n-1 \text{ and } \lambda_{i+1}=\lambda_i,\\ \lambda_n-\sum_{j\leqslant\lambda_n}(1-a_{n+1\,j}) & \text{if } i=n.\end{cases}$$

Note that ν is a well-defined generalized partition by (1*) and (3*), and $(\lambda-(\nu_n^n))/(\nu-(\nu_n^n))$ is a horizontal strip of length $k-a$. We denote the matrix of the above form by $A^*_{a,\nu}$. Let $B^*_{a,\nu}$ be the image of $v_{(1^a)}^{(-\infty,\lambda_1+k]}\otimes u_{\Lambda_\nu}$ under the strict embedding $\tau_a\times\iota^*_\nu:\mathbf{B}^\vee_{(1^a)}\otimes\mathbf{B}(\Lambda_\nu)\to\mathcal{E}\otimes\mathcal{F}^n\subset\mathbf{M}_{[n+1],\mathbb{Z}}$. Choose $p\ll 0\ll q$. Then $\pi_{[p,q]}(A^*_{a,\nu})$ is column admissible and

$$\widetilde{E}_1^{\max}\cdots\widetilde{E}_n^{\max}\pi_{[p,q]}\big(A^*_{a,\nu}\big)=\pi_{[p,q]}\big(B^*_{a,\nu}\big).$$

As in Lemmas 4.3 and 4.4, we have $A^*_{a,\nu} \equiv B^*_{a,\nu}$, which implies that $C(S \otimes T) \simeq C(A^*_{a,\nu}) \simeq C(B^*_{a,\nu}) \simeq \mathbf{B}^\vee_{(1^a)} \otimes \mathbf{B}(\Lambda_\nu)$. Conversely, suppose that $0 \leqslant a \leqslant k$ and $\nu \in \mathbb{Z}^n_+$ are given where $(\lambda - (\nu^n_n))/(\nu - (\nu^n_n))$ is a horizontal strip of length $k - a$. Let $B^*_{a,\nu}$ be as above. Then there exists a unique $T \in \mathbf{B}^\vee_{(1^k)}$ such that $u_{\Lambda_\lambda} \otimes T \equiv B^*_{a,\nu}$ since the construction of $B^*_{a,\nu}$ is also reversible. This proves the second isomorphism. $\square$

Corollary 4.10. *Let $\lambda \in \mathbb{Z}^n_+$ and $\mu \in \mathscr{P}$ be given. Then*

(1) *$\mathbf{B}(\Lambda_\lambda) \otimes \mathbf{B}_\mu$ is a finite disjoint union of $\mathbf{B}_\nu \otimes \mathbf{B}(\Lambda_\eta)$'s for some $\nu \in \mathscr{P}$ and $\eta \in \mathbb{Z}^n_+$ such that $|\nu| = a \leqslant |\mu|$ and $(\eta - (\lambda^n_n))/(\lambda - (\lambda^n_n))$ is a skew Young diagram of size $|\mu| - a$,*
(2) *$\mathbf{B}(\Lambda_\lambda) \otimes \mathbf{B}^\vee_\mu$ is a finite disjoint union of $\mathbf{B}^\vee_\nu \otimes \mathbf{B}(\Lambda_\eta)$'s for some $\nu \in \mathscr{P}$ and $\eta \in \mathbb{Z}^n_+$ such that $|\nu| = a \leqslant |\mu|$ and $(\lambda - (\eta^n_n))/(\eta - (\eta^n_n))$ is a skew Young diagram of size $|\mu| - a$.*

Proof. It follows immediately from Proposition 3.10 and Proposition 4.9. $\square$

5. Tensor product of extremal weight crystals

5.1. A monoidal category of $\mathfrak{gl}_\infty$-crystals

Let $\mathcal{C}$ be the category of $\mathfrak{gl}_\infty$-crystals, where each object B in $\mathcal{C}$ satisfies the following conditions;

($C1$) there exists a finite subset $S \subset \mathscr{P} \times \mathscr{P}$ such that each connected component in B is isomorphic to $\mathbf{B}_{\mu,\nu}$ or $\mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda_\lambda)$ for some $(\mu,\nu) \in S$ and $\lambda \in \mathbb{Z}^n_+$,
($C2$) for each $\Lambda \in P_n$ ($n \geqslant 0$), the number of connected components in B isomorphic to $\mathbf{B}(\Lambda)$ is finite,

and a morphism is a crystal morphism.

Theorem 5.1. *$\mathcal{C}$ is a monoidal category under tensor product of crystals.*

Proof. It is enough to show that $B \otimes B' \in \mathcal{C}$ for B, B' in $\mathcal{C}$ since the map sending $(b_1 \otimes b_2) \otimes b_3$ to $b_1 \otimes (b_2 \otimes b_3)$ is an isomorphism of crystals for $B_i \in \mathcal{C}$ and $b_i \in B_i$ ($i = 1, 2, 3$). By ($C1$), it suffices to prove the case when

$$B = \bigsqcup_{m \geqslant 1} \bigsqcup_{\lambda \in \mathbb{Z}^m_+} \mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda_\lambda)^{\oplus c_{\lambda\mu\nu}} = \mathbf{B}_{\mu,\nu} \otimes \left(\bigsqcup_{m \geqslant 1} \bigsqcup_{\lambda \in \mathbb{Z}^m_+} \mathbf{B}(\Lambda_\lambda)^{\oplus c_{\lambda\mu\nu}} \right),$$

$$B' = \bigsqcup_{n \geqslant 1} \bigsqcup_{\eta \in \mathbb{Z}^n_+} \mathbf{B}_{\sigma,\tau} \otimes \mathbf{B}(\Lambda_\eta)^{\oplus c_{\eta\sigma\tau}} = \mathbf{B}_{\sigma,\tau} \otimes \left(\bigsqcup_{n \geqslant 1} \bigsqcup_{\eta \in \mathbb{Z}^n_+} \mathbf{B}(\Lambda_\eta)^{\oplus c_{\eta\sigma\tau}} \right)$$

for $\mu, \nu, \sigma, \tau \in \mathscr{P}$ and $c_{\lambda\mu\nu}, c_{\eta\sigma\tau} \in \mathbb{Z}_{\geqslant 0}$.

Step 1. Suppose that $\mu, \nu, \sigma, \tau = \emptyset$. Then for $\zeta \in \mathbb{Z}^l_+$, the multiplicity of $\mathbf{B}(\Lambda_\zeta)$ in $B \otimes B'$ is equal to $\sum_{\lambda,\eta} c_{\lambda\emptyset\emptyset} c_{\eta\emptyset\emptyset} c^\zeta_{\lambda\eta}$ by Proposition 3.13. Since there are only finitely many $\lambda \in \mathbb{Z}^m_+$ and $\eta \in \mathbb{Z}^n_+$ such that $m + n = l$ and $c^\zeta_{\lambda\eta} \neq 0$ (see (3.23)), it is a well-defined integer.

Step 2. Let $B'' = (\bigsqcup_{m \geqslant 1} \bigsqcup_{\lambda \in \mathbb{Z}^m_+} \mathbf{B}(\Lambda_\lambda)^{\oplus c_{\lambda\mu\nu}}) \otimes \mathbf{B}_{\sigma,\tau}$. By Proposition 3.8 and Corollary 4.10, we have

$$B'' \simeq \bigsqcup_{(\alpha,\beta) \in S} \mathbf{B}_{\alpha,\beta} \otimes \left(\bigsqcup_{m \geqslant 1} \bigsqcup_{\gamma \in \mathbb{Z}^m_+} \mathbf{B}(\Lambda_\gamma)^{\oplus d_{\gamma\alpha\beta}} \right)$$

for some finite subset S of $\mathscr{P}\times\mathscr{P}$ and $d_{\gamma\alpha\beta}\in\mathbb{Z}_{\geqslant 0}$. This implies that $B''\in\mathcal{C}$. By Step 1, we have $B'''=B''\otimes(\bigsqcup_{n\geqslant 1}\bigsqcup_{\eta\in\mathbb{Z}_+^n}\mathbf{B}(\Lambda_\eta)^{\oplus c_{\eta\sigma\tau}})\in\mathcal{C}$. Finally, by Proposition 3.12, $B\otimes B'=\mathbf{B}_{\mu,\nu}\otimes B'''\in\mathcal{C}$. □

Remark 5.2.

(1) A connected component of the tensor product $\mathbf{B}(\Lambda)\otimes\mathbf{B}(\Lambda')$ with $\Lambda\in P_{-m}$ and $\Lambda'\in P_n$ for $m,n>0$ is not necessarily isomorphic to an extremal weight crystal. For example, consider $B=\mathbf{B}(-\Lambda_0)\otimes\mathbf{B}(\Lambda_0)$. We can check that any element in B is connected to $u_{-\Lambda_0}\otimes u_{\Lambda_0}$, and hence B is connected. Suppose that B is isomorphic to an extremal weight crystal. By Corollary 4.7, $B\simeq\mathbf{B}_{\mu,\nu}$ for some $\mu,\nu\in\mathscr{P}$. By Remark 3.5, for given $S\otimes T\in\mathbf{B}_{\mu,\nu}$ and $p,q\in\mathbb{Z}$ $(p<q)$, there exist $i_1,\ldots,i_r\in\mathbb{Z}$ such that $\tilde{x}_k(\tilde{e}_{i_1}\cdots\tilde{e}_{i_r}(S\otimes T))=\mathbf{0}$ for $x=e,f$ and $k\in[p,q]$. Suppose that $S\otimes T$ is equivalent to $u_{-\Lambda_0}\otimes u_{\Lambda_0}$. Then we have

$$\tilde{e}_{i_1}\cdots\tilde{e}_{i_r}(u_{-\Lambda_0}\otimes u_{\Lambda_0})=(\tilde{e}_{i_1}\cdots\tilde{e}_{i_r}u_{-\Lambda_0})\otimes u_{\Lambda_0}.$$

On the other hand, we have

$$\tilde{f}_0\big(\tilde{e}_{i_1}\cdots\tilde{e}_{i_r}(u_{-\Lambda_0}\otimes u_{\Lambda_0})\big)=\begin{cases}(\tilde{e}_{i_1}\cdots\tilde{e}_{i_r}u_{-\Lambda_0})\otimes\tilde{f}_0u_{\Lambda_0} & \text{if } \varphi_0(\tilde{e}_{i_1}\cdots\tilde{e}_{i_r}u_{-\Lambda_0})=0,\\ (\tilde{f}_0\tilde{e}_{i_1}\cdots\tilde{e}_{i_r}u_{-\Lambda_0})\otimes u_{\Lambda_0} & \text{otherwise,}\end{cases}$$

which is not $\mathbf{0}$ in any case. This is a contradiction. Hence B is not isomorphic to an extremal weight crystal. In general, for $\Lambda,\Lambda'\in P^+$, we can check by similar arguments that $\mathbf{B}(-\Lambda)\otimes\mathbf{B}(\Lambda')$ is a connected regular crystal but not isomorphic to an extremal weight crystal.

(2) When we consider a tensor product of arbitrary extremal weight crystals, the multiplicity of each connected component is not necessarily finite. For example, the multiplicity of $\mathbf{B}(\Lambda_0)$ in $\mathbf{B}(\Lambda_0)\otimes\mathbf{B}(\Lambda_0)\otimes\mathbf{B}(-\Lambda_0)$ is infinite.

5.2. Grothendieck ring

Let $\mathscr{K}$ be the additive abelian group generated by the symbol $[B]$ $(B\in\mathcal{C})$ subject to the relations; $[B]=[B']$ if $B\simeq B'$ and $[B\sqcup B']=[B]+[B']$ for $B,B'\in\mathcal{C}$.

Proposition 5.3. *$\mathscr{K}$ is an associative $\mathbb{Z}$-algebra with $1=[\mathbf{B}(0)]$ under the multiplication $[B]\cdot[B']=[B\otimes B']$ for $B,B'\in\mathcal{C}$.*

Let $\mathcal{C}_n$, $\mathcal{C}^{\text{h.w.}}$ and $\mathcal{C}_n^{\text{h.w.}}$ $(n\geqslant 0)$ be the full subcategories of $\mathcal{C}$ consisting of objects whose connected components are isomorphic to $\mathbf{B}(\Lambda)$ for Λ in P_n, P^+ and P_n^+, respectively. We denote by $\mathscr{K}_n$, $\mathscr{K}^{\text{h.w.}}$ and $\mathscr{K}_n^{\text{h.w.}}$ the corresponding subgroups of $\mathscr{K}$, respectively. Note that $\mathscr{K}_0$ is a subalgebra of $\mathscr{K}$. By Proposition 3.16, $[B]=[B']$ if and only if the multiplicities of each $\mathbf{B}_{\mu,\nu}\otimes\mathbf{B}(\Lambda_\lambda)$ in B and B' are equal for $B,B'\in\mathcal{C}$. Hence, as $\mathbb{Z}$-modules, we have

$$\begin{aligned}\mathscr{K}_n&\simeq\mathscr{K}_0\otimes_{\mathbb{Z}}\mathscr{K}_n^{\text{h.w.}},\\ \mathscr{K}&\simeq\mathscr{K}_0\otimes_{\mathbb{Z}}\mathscr{K}^{\text{h.w.}},\end{aligned}\tag{5.1}$$

and $\mathscr{K}_n$, $\mathscr{K}$ are free $\mathscr{K}_0$-modules.

Let $\mathcal{C}^\vee$ be the category of $\mathfrak{gl}_\infty$-crystals with objects $B^\vee$ for $B\in\mathcal{C}$, and let $\mathcal{C}_{-n}$, $\mathcal{C}^{\text{l.w.}}$ and $\mathcal{C}_{-n}^{\text{l.w.}}$ $(n\geqslant 0)$ be its subcategories whose objects are $B^\vee$ for B in $\mathcal{C}_n$, $\mathcal{C}^{\text{h.w.}}$ and $\mathcal{C}_n^{\text{h.w.}}$, respectively. We denote by $\mathscr{K}^\vee$, $\mathscr{K}_{-n}$, $\mathscr{K}^{\text{l.w.}}$ and $\mathscr{K}_{-n}^{\text{l.w.}}$ the corresponding groups, respectively. Then $\mathscr{K}^\vee$ is a $\mathbb{Z}$-algebra under tensor product of crystals and isomorphic to $\mathscr{K}^{\text{opp}}$, the opposite $\mathbb{Z}$-algebra of $\mathscr{K}$. We have $\mathscr{K}_{-n}\simeq\mathscr{K}_{-n}^{\text{l.w.}}\otimes_{\mathbb{Z}}\mathscr{K}_0$ and $\mathscr{K}^\vee\simeq\mathscr{K}^{\text{l.w.}}\otimes_{\mathbb{Z}}\mathscr{K}_0$ as $\mathbb{Z}$-modules.

We denote by Sym_x and Sym_y the ring of symmetric functions in x and y, respectively, where $x = \{x_1, x_2, \ldots\}$ and $y = \{y_1, y_2, \ldots\}$ are two sets of formal commuting variables. Let $Sym_x \otimes_{\mathbb{Z}} Sym_y$ be the tensor product of $\mathbb{Z}$-algebras with $\mathbb{Z}$-basis $\{s_\mu(x)s_\nu(y) \mid \mu, \nu \in \mathscr{P}\}$.

Proposition 5.4. *The assignment $s_\mu(x)s_\nu(y) \mapsto [\mathbf{B}_{\mu,\nu}]$ $(\mu, \nu \in \mathscr{P})$ defines an isomorphism of $\mathbb{Z}$-algebras $\Phi : Sym_x \otimes_{\mathbb{Z}} Sym_y \to \mathscr{K}_0$.*

Proof. Note that $\{[\mathbf{B}_{\mu,\nu}] \mid \mu, \nu \in \mathscr{P}\}$ is a $\mathbb{Z}$-basis of $\mathscr{K}_0$ by Proposition 3.9 and Corollary 4.7. Let $\Phi : Sym_x \otimes_{\mathbb{Z}} Sym_y \to \mathscr{K}_0$ be a linear isomorphism sending $s_\mu(x)s_\nu(y)$ to $[\mathbf{B}_{\mu,\nu}]$ $(\mu, \nu \in \mathscr{P})$. It follows immediately from Propositions 3.8 and 3.12 that Φ is a homomorphism of algebras, and hence an isomorphism. □

Let $\mathbb{Q}[P]$ be the group algebra of P over $\mathbb{Q}$ with basis $\{e^\Lambda \mid \Lambda \in P\}$. For $\Lambda \in P^+$, let $\mathrm{ch}\,\mathbf{B}(\Lambda) = \sum_{b \in \mathbf{B}(\Lambda)} e^{\mathrm{wt}(b)}$ be the character of $\mathbf{B}(\Lambda)$, which is a formal power series in $\{e^{\Lambda_0}, e^{\pm\epsilon_i}\ (i \in \mathbb{Z})\}$ and equal to the character of the irreducible highest weight $U_q(\mathfrak{gl}_\infty)$-module with highest weight Λ. Then $\{\mathrm{ch}\,\mathbf{B}(\Lambda) \mid \Lambda \in P^+\}$ is linearly independent, and $\mathrm{ch}\,B$ is also well defined for $B \in \mathfrak{C}^{\text{h.w.}}$ (use the formula for $e^{-n\Lambda_0}\,\mathrm{ch}\,\mathbf{B}(\Lambda_\lambda)$ $(\lambda \in \mathbb{Z}_+^n)$ in [17, Theorem 5.5] and then apply [17, Proposition 3.18] for the well-definedness of $\mathrm{ch}\,B$ for $B \in \mathfrak{C}^{\text{h.w.}}$). Let R be the $\mathbb{Z}$-algebra spanned by $\{\mathrm{ch}\,B \mid B \in \mathfrak{C}^{\text{h.w.}}\}$. Then the map $\psi : \mathscr{K}^{\text{h.w.}} \to R$ given by $\psi([B]) = \mathrm{ch}\,B$ is an algebra isomorphism.

Let $z = \{z_k \mid k \in \mathbb{Z}\}$ be another set of formal commuting variables, and let $\mathscr{R}$ be the ring of formal power series in z with coefficients in $\mathbb{Z}$. Then we have an A_∞-analogue of the fundamental theorem on symmetric functions as follows.

Proposition 5.5. *The assignment $z_k \mapsto [\mathbf{B}(\pm\Lambda_k)]$ $(k \in \mathbb{Z})$ defines isomorphisms of $\mathbb{Z}$-algebras*

$$\Psi_+ : \mathscr{R} \to \mathscr{K}^{\text{h.w.}},$$
$$\Psi_- : \mathscr{R} \to \mathscr{K}^{\text{l.w.}},$$

respectively.

Proof. Let us identify $\mathscr{K}^{\text{h.w.}}$ with R. Put $H_k = \mathrm{ch}\,\mathbf{B}(\Lambda_k)$ for $k \in \mathbb{Z}$.

For $\lambda, \mu \in \mathbb{Z}_+^n$, we define $\lambda > \mu$ if and only if there exists $i \geqslant 1$ such that $\lambda_k = \mu_k$ for $1 \leqslant k < i$ and $\lambda_i > \mu_i$. Then $>$ is a linear ordering on $\mathbb{Z}_+^n$. Put $H_\mu = \prod_{i=1}^n H_{\mu_i}$. Then we have

$$H_\mu = \sum_{\lambda \in \mathbb{Z}_+^n} K_{\lambda\mu}\,\mathrm{ch}\,\mathbf{B}(\Lambda_\lambda) \tag{5.2}$$

for some $K_{\lambda\mu} \in \mathbb{Z}_{\geqslant 0}$, where $K_{\lambda\mu} = 0$ unless $\lambda \geqslant \mu$ and $\sum_i \lambda_i = \sum_i \mu_i$ (see (4.1) and (4.2) in [17]). Note that we have a Jacobi–Trudi formula

$$\mathrm{ch}\,\mathbf{B}(\Lambda_\lambda) = \det(H_{\lambda_i - i + j})_{1 \leqslant i, j \leqslant n} \tag{5.3}$$

[6] (see also [17, Theorem 4.5]).

Define an algebra homomorphism $\Psi_+ : \mathscr{R} \to R$ by $\Psi_+(z_k) = H_k$ for $k \in \mathbb{Z}$. Suppose that $f(z) = c + \sum_{n \geqslant 1} \sum_{\mu \in \mathbb{Z}_+^n} c_\mu z_\mu$ is given (not necessarily a finite sum), where $z_\mu = \prod_{i=1}^n z_{\mu_i}$. Note that for $\lambda \in \mathbb{Z}_+^n$, there are only finitely many $\mu \in \mathbb{Z}_+^n$ such that $\lambda \geqslant \mu$ and $\sum_i \lambda_i = \sum_i \mu_i$. Then it follows from (5.2) that $\sum_{\mu \in \mathbb{Z}_+^n} c_\mu K_{\lambda\mu}$ is a well-defined integer, and

$$\Psi_+\big(f(z)\big) = c + \sum_{n \geqslant 1} \sum_{\mu \in \mathbb{Z}_+^n} c_\mu H_\mu = c + \sum_{n \geqslant 1} \sum_{\lambda \in \mathbb{Z}_+^n} \left(\sum_{\mu \in \mathbb{Z}_+^n} c_\mu K_{\lambda\mu} \right) \mathrm{ch}\,\mathbf{B}(\Lambda_\lambda) \in R. \tag{5.4}$$

Hence Ψ_+ is well defined.

Also by (5.3) there exists M such that the coefficient of H_μ in $\mathrm{ch}\,\mathbf{B}(\Lambda_\lambda)$ is non-zero only if $\sum_{i=1}^{n}|\lambda_i-\mu_i|\leqslant M$. Hence, for $c+\sum_{n\geqslant 1}\sum_{\lambda\in\mathbb{Z}_+^n}c'_\lambda\,\mathrm{ch}\,\mathbf{B}(\Lambda_\lambda)\in R$, we have by (5.3)

$$c+\sum_{n\geqslant 1}\sum_{\lambda\in\mathbb{Z}_+^n}c'_\lambda\,\mathrm{ch}\,\mathbf{B}(\Lambda_\lambda)=c+\sum_{n\geqslant 1}\sum_{\mu\in\mathbb{Z}_+^n}c_\mu H_\mu$$

for some $c_\mu\in\mathbb{Z}$. This implies that Ψ_+ is surjective.

Finally, Ψ_+ is injective since $\{H_\mu \mid \mu\in\mathbb{Z}_+^n,\ n\geqslant 1\}$ is linearly independent by (5.2). Hence Ψ_+ is an isomorphism. The proof for Ψ_- is almost the same. □

Let $t^\pm=\{t_1^\pm,t_2^\pm,t_3^\pm,\ldots\}$ be two sets of mutually commuting formal variables. Consider

$$\mathscr{A}=\mathscr{R}\big[t^+,t^-\big]$$

the free $\mathscr{R}$-module of the polynomials in $t^\pm$ over $\mathscr{R}$. For $n\geqslant 1$, let $\mathscr{A}_n=\mathscr{R}[t_1^\pm,\ldots,t_n^\pm]$. Note that as an $\mathscr{R}$-module

$$\mathscr{A}_{n-1}\subset\mathscr{A}_n=\mathscr{A}_{n-1}\big[t_n^+,t_n^-\big],\qquad \mathscr{A}=\sum_{n\geqslant 0}\mathscr{A}_n,$$

where $\mathscr{A}_0=\mathscr{R}$. Now, let us define a $\mathbb{Z}$-algebra structure on $\mathscr{A}$ inductively as follows;

(1) $\mathscr{A}_0=\mathscr{R}$ with the usual multiplication.
(2) Suppose that a $\mathbb{Z}$-algebra structure on $\mathscr{A}_{n-1}$ is defined. Then we define a multiplication on $\mathscr{A}_n$ by

$$t_n^\pm a=at_n^\pm+\delta_n^\pm(a)\quad (a\in\mathscr{A}_{n-1}),\tag{5.5}$$

where $\delta_n^\pm$ is the derivations on $\mathscr{A}_{n-1}$ given by

$$\begin{aligned}\delta_n^\pm\big(t_k^\pm\big)&=0\quad (1\leqslant k\leqslant n-1),\\ \delta_n^\pm(z_k)&=z_{k\mp 1}t_{n-1}^\pm+z_{k\mp 2}t_{n-2}^\pm+\cdots+z_{k\mp n}\quad (k\in\mathbb{Z})\end{aligned}\tag{5.6}$$

(that is, $\mathscr{A}_n$ is an Ore extension [25] of $\mathscr{A}_{n-1}$ associated with derivations $\delta_n^\pm$).

Note that by induction hypothesis, we have for $\mu\in\mathbb{Z}_+^m$,

$$\delta_n^\pm(z_\mu)=\sum_{k=1}^{n}\Bigg(\sum_{\lambda\in\mathbb{Z}^m}c_\lambda z_\lambda\Bigg)t_{n-k}^\pm$$

for some $c_\lambda\in\mathbb{Z}_{\geqslant 0}$, where $c_\lambda=0$ unless $\sum_{i=1}^{m}|\lambda_i-\mu_i|=k$. (We assume that $t_0^\pm=1$.) Hence, $\delta_n^\pm$ is well defined on $\mathscr{A}_{n-1}$.

Proposition 5.6. *The assignment $z_k\mapsto[\mathbf{B}(-\Lambda_k)]$, $t_n^+\mapsto[\mathbf{B}_{(1^n)}]$ and $t_n^-\mapsto[\mathbf{B}_{(1^n)}^\vee]$ $(k\in\mathbb{Z},\ n\geqslant 1)$ defines an isomorphism of $\mathbb{Z}$-algebras $\Psi:\mathscr{A}\to\mathscr{K}^\vee$.*

Proof. Since the map sending $t_k^+ t_l^-$ to $e_k(x)e_l(y)$ $(k, l \geqslant 1)$ gives a $\mathbb{Z}$-algebra isomorphism from $\mathbb{Z}[t^+, t^-]$ to $Sym_x \otimes_{\mathbb{Z}} Sym_y$, composing it with Φ in Proposition 5.4 we have a $\mathbb{Z}$-algebra isomorphism

$$\Psi_0 : \mathbb{Z}\big[t^+, t^-\big] \to \mathscr{K}_0$$

given by $\Psi_0(t_n^+) = [\mathbf{B}_{(1^n)}]$ and $\Psi_0(t_n^-) = [\mathbf{B}^\vee_{(1^n)}]$ for $n \geqslant 1$.

Since $\mathscr{A} \simeq \mathscr{R} \otimes_{\mathbb{Z}} \mathbb{Z}[t^+, t^-]$ and $\mathscr{K}^\vee \simeq \mathscr{K}^{\mathrm{l.w.}} \otimes_{\mathbb{Z}} \mathscr{K}_0$ as $\mathbb{Z}$-modules, we have a $\mathbb{Z}$-linear isomorphism

$$\Psi = \Psi_- \otimes \Psi_0 : \mathscr{A} \to \mathscr{K}^\vee.$$

Next, we claim that Ψ is an algebra homomorphism. For $n \geqslant 0$, let $\mathscr{R}_n = \Psi(\mathscr{A}_n)$. We use induction on n to show that $\Psi|_{\mathscr{A}_n} : \mathscr{A}_n \to \mathscr{R}_n$ is an algebra homomorphism. If $n = 0$, then $\Psi|_{\mathscr{A}_0} = \Psi_- : \mathscr{R} \to \mathscr{K}^{\mathrm{l.w.}}$ is an algebra isomorphism by Proposition 5.5. Suppose that $\Psi|_{\mathscr{A}_n}$ is an algebra homomorphism. Note that $\mathscr{A}_{n+1} = \mathscr{A}_n[t_{n+1}^+, t_{n+1}^-]$. By Lemmas 3.6 and 4.4, we have

$$\Psi|_{\mathscr{A}_{n+1}}\big(t_{n+1}^\pm a - a t_{n+1}^\pm - \delta_{n+1}^\pm(a)\big) = 0, \tag{5.7}$$

when $a = z_k$ $(k \in \mathbb{Z})$ and $a = t_k^\pm$ $(1 \leqslant k \leqslant n)$, which implies that (5.7) holds for all $a \in \mathscr{A}_n$. Hence $\Psi|_{\mathscr{A}_{n+1}}$ preserves the multiplication, and it is an algebra homomorphism. This completes the induction. □

Let $s^\pm = \{s_1^\pm, s_2^\pm, s_3^\pm, \ldots\}$ be two sets of mutually commuting formal variables. Consider

$$\mathscr{D} = \mathscr{R}_{\mathbb{Q}}\big[s^+, s^-\big]$$

the free $\mathscr{R}_{\mathbb{Q}}$-module of the polynomials in $s_\pm$ over $\mathscr{R}_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{R}$. Then we define a $\mathbb{Q}$-algebra structure on $\mathscr{D}$ by

$$s_n^\pm z_k = z_k s_n^\pm + (-1)^{n-1} z_{k\mp n} \quad (n \geqslant 1,\ k \in \mathbb{Z}). \tag{5.8}$$

One may regard $\mathscr{D}$ as an Ore extension of $\mathscr{R}_{\mathbb{Q}}$ associated with derivations $\gamma_n^\pm = (-1)^{n-1} \sum_{k\in\mathbb{Z}} z_{k\mp n} \frac{\partial}{\partial z_k}$ $(n \geqslant 1)$.

Let u be a formal variable. Put

$$F(u) = \sum_{k\in\mathbb{Z}} z_k u^k, \qquad E^\pm(u) = \exp\left(\sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n} s_n^\pm u^n\right) \in \mathscr{D}[\![u]\!]. \tag{5.9}$$

Now, we obtain the following characterization of $\mathscr{K}_{\mathbb{Q}}^\vee = \mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{K}^\vee$ (hence $\mathscr{K}_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{K}$), which is the main result in this section.

Theorem 5.7. *There exists a $\mathbb{Q}$-algebra isomorphism $\Theta : \mathscr{D} \to \mathscr{K}_{\mathbb{Q}}^\vee$ such that*

$$\Theta\big(F(u)\big) = \sum_{k\in\mathbb{Z}} \big[\mathbf{B}(-\Lambda_k)\big] u^k,$$

$$\Theta\big(E^+(u)\big) = \sum_{n\geqslant 0} [\mathbf{B}_{(1^n)}] u^n, \qquad \Theta\big(E^-(u)\big) = \sum_{n\geqslant 0} \big[\mathbf{B}^\vee_{(1^n)}\big] u^n.$$

Proof. For $n \geqslant 1$, set

$$\hat{s}_n^+ = \Psi_0^{-1} \circ \Phi\big(p_n(x)\big), \qquad \hat{s}_n^- = \Psi_0^{-1} \circ \Phi\big(p_n(y)\big).$$

Note that $\hat{s}^\pm = \{\hat{s}_n^\pm \mid n \geqslant 1\}$ is algebraically independent over $\mathbb{Q}$, and $\mathscr{A}_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} \mathscr{A} = \mathscr{R}_\mathbb{Q}[\hat{s}^+, \hat{s}^-]$.

First, we use induction on n to show that for $k \in \mathbb{Z}$

$$\hat{s}_n^\pm z_k = z_k \hat{s}_n^\pm + (-1)^{n-1} z_{k\mp n}. \tag{5.10}$$

It is clear when $n = 1$. Recall that

$$(-1)^{n-1}\hat{s}_n^\pm = nt_n^\pm - \sum_{r=1}^{n-1} (-1)^{r-1}\hat{s}_r^\pm t_{n-r}^\pm. \tag{5.11}$$

By induction hypothesis, we have for $k \geqslant 1$,

$$\begin{aligned}(-1)^{n-1}\hat{s}_n^\pm z_k &= nt_n^\pm z_k - \sum_{r=1}^{n-1}(-1)^{r-1}\hat{s}_r^\pm t_{n-r}^\pm z_k \\ &= n\big(z_k t_n^\pm + z_{k\mp 1} t_{n-1}^\pm + \cdots + z_{k\mp n}\big) \\ &\quad - \sum_{r=1}^{n-1}(-1)^{r-1} \sum_{j=0}^{n-r} \big(z_{k\mp j}\hat{s}_r^\pm + (-1)^{r-1} z_{k\mp r\mp j}\big) t_{n-r-j}^\pm.\end{aligned}$$

Using (5.11), it is straightforward to check that the (right) coefficient of $z_{k\mp i}$ in the last equation is

$$\begin{cases} (-1)^{n-1}\hat{s}_n^\pm & \text{if } i = 0, \\ 1 & \text{if } i = n, \\ 0 & \text{otherwise}. \end{cases}$$

This proves (5.10) and completes the induction.

By (5.8) and (5.10), we obtain a $\mathbb{Q}$-algebra isomorphism $\theta : \mathscr{D} \to \mathscr{A}_\mathbb{Q}$ such that $\theta(z_k) = z_k$ and $\theta(s_n^\pm) = \hat{s}_n^\pm$ for $k \in \mathbb{Z}$ and $n \geqslant 1$. Since

$$\exp\bigg(\sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n} \hat{s}_n^\pm u^n\bigg) = \sum_{n\geqslant 0} t_n^\pm u^n$$

(cf. [22]), we obtain the required isomorphism $\Theta = \Psi \circ \theta$ by Proposition 5.6. □

Corollary 5.8. *$\mathscr{K}_\mathbb{Q}$ is isomorphic to $\mathscr{D}^{\mathrm{opp}}$ as a $\mathbb{Q}$-algebra, where $\mathscr{D}^{\mathrm{opp}}$ denotes the opposite algebra of $\mathscr{D}$.*

Note that there exists an involution $\omega : \mathscr{D} \to \mathscr{D}$ determined by

$$\omega(z_k) = z_{-k}, \qquad \omega\big(s_n^\pm\big) = s_n^\mp \tag{5.12}$$

for $k \in \mathbb{Z}$ and $n \geqslant 1$, which is well defined by (5.8). Then ω induces an involution on $\mathscr{K}_\mathbb{Q}^\vee$ under Θ, which we also denote by ω by abuse of notation.

Proposition 5.9. *For $\lambda \in \mathbb{Z}_+^n$ and $\mu, \nu \in \mathscr{P}$, we have*

$$\omega\big(\big[\mathbf{B}(-\Lambda_\lambda) \otimes \mathbf{B}_{\mu,\nu}\big]\big) = \big[\mathbf{B}(-\Lambda_{\lambda^*}) \otimes \mathbf{B}_{\nu,\mu}\big],$$

where $\lambda^* = (-\lambda_n, \ldots, -\lambda_1)$.

Proof. It follows directly from (5.3) that $\omega[\mathbf{B}(-\Lambda_\lambda)] = [\mathbf{B}(-\Lambda_{\lambda^*})]$. Since $\Theta^{-1} \circ \Phi(p_m(x)) = s_m^+$ and $\Theta^{-1} \circ \Phi(p_m(y)) = s_m^-$ for $m \geqslant 1$, we have

$$\big(\Phi^{-1} \circ \omega \circ \Phi\big)\big(s_\mu(x)s_\nu(y)\big) = s_\nu(x)s_\mu(y),$$

which implies $\omega([\mathbf{B}_{\mu,\nu}]) = [\mathbf{B}_{\nu,\mu}]$. Since ω is an algebra homomorphism, we obtain the above identity. $\square$

5.3. Littlewood–Richardson rule

Let u be a formal variable. Put

$$\mathscr{F}(u) = \sum_{k\in\mathbb{Z}} \big[\mathbf{B}(\Lambda_k)\big]u^k, \qquad \mathscr{E}(u) = \sum_{n\geqslant 0} [\mathbf{B}_{(1^n)}]u^n, \qquad \mathscr{E}^\vee(u) = \sum_{n\geqslant 0} \big[\mathbf{B}^\vee_{(1^n)}\big]u^n. \tag{5.13}$$

Lemma 5.10. *For $n \geqslant 1$, we have*

$$\mathscr{F}(x_{[n]}) = \sum_{\lambda\in\mathbb{Z}_+^n} \big[\mathbf{B}(\Lambda_\lambda)\big]s_\lambda(x_{[n]}),$$

$$\mathscr{E}(x_{[n]}) = \sum_{\substack{\mu\in\mathscr{P}\\ \mu_1\leqslant n}} [\mathbf{B}_\mu]s_{\mu'}(x_{[n]}), \qquad \mathscr{E}^\vee(x_{[n]}) = \sum_{\substack{\nu\in\mathscr{P}\\ \nu_1\leqslant n}} \big[\mathbf{B}^\vee_\nu\big]s_{\nu'}(x_{[n]}),$$

where $\mathscr{F}(x_{[n]}) = \prod_{k=1}^n \mathscr{F}(x_k)$, $\mathscr{E}(x_{[n]}) = \prod_{k=1}^n \mathscr{E}(x_k)$ and $\mathscr{E}^\vee(x_{[n]}) = \prod_{k=1}^n \mathscr{E}^\vee(x_k)$.

Proof. Consider the $(\mathfrak{gl}_\infty, \mathfrak{gl}_n)$-character associated with the decomposition in Proposition 3.11. Then we obtain the first identity by replacing $\mathrm{ch}\,\mathbf{B}(\Lambda_\lambda)$ with $[\mathbf{B}(\Lambda_\lambda)]$. The other two identities are obtained by considering the $(\mathfrak{gl}_\infty, \mathfrak{gl}_n)$-character associated with the decomposition in Proposition 3.10 and using the isomorphism in Proposition 5.4. $\square$

Let u, v be commuting formal variables. By Lemma 4.4, we have

$$\mathscr{F}(u)\mathscr{E}(v) = \mathscr{E}(v)\mathscr{F}(u)\frac{1}{(1-u^{-1}v)},$$
$$\mathscr{F}(u)\mathscr{E}^\vee(v) = \mathscr{E}^\vee(v)\mathscr{F}(u)\frac{1}{(1-uv)}. \tag{5.14}$$

Applying (5.14) successively, we obtain the following identities.

Lemma 5.11. *For $m, n \geqslant 1$, we have*

$$\mathscr{F}(x_{[m]})\mathscr{E}(y_{[n]}) = \mathscr{E}(y_{[n]})\mathscr{F}(x_{[m]})\frac{1}{\prod_{i\in[m],\, j\in[n]}(1 - x_i^{-1}y_j)},$$

$$\mathscr{F}(x_{[m]})\mathscr{E}^\vee(y_{[n]}) = \mathscr{E}^\vee(y_{[n]})\mathscr{F}(x_{[m]})\frac{1}{\prod_{i\in[m],\, j\in[n]}(1 - x_i y_j)}.$$

Lemma 5.12. *Let $\lambda, \eta \in \mathbb{Z}_+^m$ and $\mu, \nu \in \mathscr{P}$ be given.*

(1) *The multiplicity of $\mathbf{B}_\nu \otimes \mathbf{B}(\Lambda_\eta)$ in $\mathbf{B}(\Lambda_\lambda) \otimes \mathbf{B}_\mu$ is*

$$\sum_{\substack{\gamma\in\mathscr{P}\\ \ell(\gamma)\leqslant m}} c^\lambda_{\eta\gamma^*} c^{\mu'}_{\nu'\gamma},$$

where $\gamma^* = (\dots, -\gamma_2, -\gamma_1) \in \mathbb{Z}_+^m$.

(2) *The multiplicity of $\mathbf{B}^\vee_\nu \otimes \mathbf{B}(\Lambda_\eta)$ in $\mathbf{B}(\Lambda_\lambda) \otimes \mathbf{B}^\vee_\mu$ is*

$$\sum_{\substack{\gamma\in\mathscr{P}\\ \ell(\gamma)\leqslant m}} c^\lambda_{\eta\gamma} c^{\mu'}_{\nu'\gamma}.$$

Proof. (1) Choose $n \geqslant 1$ such that $\ell(\mu') \leqslant n$. The left-hand side of the first identity in Lemma 5.11 is given by

$$\sum_{\lambda\in\mathbb{Z}_+^m}\sum_{\substack{\mu\in\mathscr{P}\\ \mu_1\leqslant n}} \big[\mathbf{B}(\Lambda_\lambda) \otimes \mathbf{B}_\mu\big] s_\lambda(x_{[m]}) s_{\mu'}(y_{[n]}). \tag{5.15}$$

On the other hand, the right-hand side is

$$\begin{aligned}
&\sum_{\eta\in\mathbb{Z}_+^m}\sum_{\substack{\nu\in\mathscr{P}\\ \nu_1\leqslant n}} \big[\mathbf{B}_\nu \otimes \mathbf{B}(\Lambda_\eta)\big] s_\eta(x_{[m]}) s_{\nu'}(y_{[n]}) \sum_{\substack{\gamma\in\mathscr{P}\\ \ell(\gamma)\leqslant m,n}} s_\gamma\big(x_{[m]}^{-1}\big) s_\gamma(y_{[n]})\\
&\quad= \sum_{\lambda\in\mathbb{Z}_+^m}\sum_{\substack{\mu\in\mathscr{P}\\ \mu'\leqslant n}}\sum_{\eta,\nu,\gamma} c^\lambda_{\eta\gamma^*} c^{\mu'}_{\nu'\gamma} \big[\mathbf{B}_\nu \otimes \mathbf{B}(\Lambda_\eta)\big] s_\lambda(x_{[m]}) s_{\mu'}(y_{[n]}).
\end{aligned} \tag{5.16}$$

Since $\{s_\lambda(x_{[m]}) s_{\mu'}(y_{[n]}) \mid \lambda \in \mathbb{Z}_+^m,\ \mu \in \mathscr{P} \text{ with } \ell(\mu') \leqslant n\}$ is linearly independent, we have

$$\big[\mathbf{B}(\Lambda_\lambda) \otimes \mathbf{B}_\mu\big] = \sum_{\eta,\nu}\Big(\sum_\gamma c^\lambda_{\eta\gamma^*} c^{\mu'}_{\nu'\gamma}\Big)\big[\mathbf{B}_\nu \otimes \mathbf{B}(\Lambda_\eta)\big] \in \mathscr{K}$$

by comparing (5.15) and (5.16). Hence we obtain the required multiplicity.

(2) The proof is almost the same as in (1). We leave the details to the reader. □

Combining Lemma 5.12 (1) and (2), we have the following decomposition of the tensor product of a highest weight crystal and a level zero extremal weight crystal.

Proposition 5.13. *For $\lambda \in \mathbb{Z}_+^m$ and $\mu, \nu \in \mathscr{P}$, we have*

$$\mathbf{B}(\Lambda_\lambda) \otimes \mathbf{B}_{\mu,\nu} \simeq \bigsqcup_{\substack{\rho \in \mathbb{Z}_+^m \\ \sigma,\tau \in \mathscr{P}}} \mathbf{B}_{\sigma,\tau} \otimes \mathbf{B}(\Lambda_\rho)^{\oplus c^{(\lambda,\mu,\nu)}_{(\rho,\sigma,\tau)}},$$

where

$$c^{(\lambda,\mu,\nu)}_{(\rho,\sigma,\tau)} = \sum_{\eta \in \mathbb{Z}_+^m} \sum_{\substack{\alpha,\beta \in \mathscr{P} \\ \ell(\alpha),\ell(\beta) \leqslant m}} c^{\lambda}_{\eta\alpha^*} c^{\mu'}_{\sigma'\alpha} c^{\eta}_{\rho\beta} c^{\nu'}_{\tau'\beta}.$$

Now, we can describe the Littlewood–Richardson rule of extremal weight crystals of non-negative level, which is the main result in this paper.

Theorem 5.14. *For $\lambda \in \mathbb{Z}_+^m$, $\rho \in \mathbb{Z}_+^n$, and $\mu, \nu, \sigma, \tau \in \mathscr{P}$, we have*

$$\big(\mathbf{B}_{\mu,\nu} \otimes \mathbf{B}(\Lambda_\lambda)\big) \otimes \big(\mathbf{B}_{\sigma,\tau} \otimes \mathbf{B}(\Lambda_\rho)\big) \simeq \bigsqcup_{\substack{\zeta \in \mathbb{Z}_+^{m+n} \\ \eta,\theta \in \mathscr{P}}} \mathbf{B}_{\eta,\theta} \otimes \mathbf{B}(\Lambda_\zeta)^{\oplus c^{(\zeta,\eta,\theta)}_{(\lambda,\mu,\nu),(\rho,\sigma,\tau)}},$$

where

$$c^{(\zeta,\eta,\theta)}_{(\lambda,\mu,\nu),(\rho,\sigma,\tau)} = \sum_{\alpha \in \mathbb{Z}_+^m} \sum_{\beta,\gamma \in \mathscr{P}} c^{\zeta}_{\alpha\rho} c^{\eta}_{\beta\mu} c^{\theta}_{\gamma\nu} c^{(\lambda,\sigma,\tau)}_{(\alpha,\beta,\gamma)}$$

and $c^{(\lambda,\sigma,\tau)}_{(\alpha,\beta,\gamma)}$ is defined in Proposition 5.13.

Proof. It follows from Propositions 3.8, 3.12, 3.13, and 5.13. $\square$

Remark 5.15. We have the same Littlewood–Richardson rule for the crystals in $\mathcal{C}^\vee$ by taking the dual of the decomposition in Theorem 5.14.

6. Differential operators on lowest weight character ring

6.1. A twisted action of $\mathscr{K}^\vee$ on $\mathscr{K}^{\mathrm{l.w.}}$

We define for $B \in \mathcal{C}^\vee$

$$\mathrm{pr}(B) = \{b \in B \mid \exists r \geqslant 1 \text{ such that } \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} b = \mathbf{0} \text{ for all } i_1, \ldots, i_r \in \mathbb{Z}\}.$$

We can check that $\mathrm{pr}(B)$ is a union of connected components B' of B such that $B' \in \mathcal{C}^{\mathrm{l.w.}}$. Hence pr is a functor from $\mathcal{C}^\vee$ to $\mathcal{C}^{\mathrm{l.w.}}$, and by definition $\mathrm{pr}(B \sqcup B') \simeq \mathrm{pr}(B) \sqcup \mathrm{pr}(B')$ for $B, B' \in \mathcal{C}^\vee$. Consider a composite of the following two functors

$$\begin{aligned} &\mathcal{C}^\vee \times \mathcal{C}^{\mathrm{l.w.}} \xrightarrow{\otimes} \mathcal{C}^\vee \xrightarrow{\mathrm{pr}} \mathcal{C}^{\mathrm{l.w.}}, \\ &\big(B, B'\big) \mapsto B \otimes B' \mapsto \mathrm{pr}\big(B \otimes B'\big). \end{aligned} \tag{6.1}$$

Then it is straightforward to check that (6.1) induces a $\mathbb{Z}$-algebra homomorphism (or $\mathscr{K}^\vee$-module structure on $\mathscr{K}^{\text{l.w.}}$)

$$\rho : \mathscr{K}^\vee \to \operatorname{End}_{\mathbb{Z}}\big(\mathscr{K}^{\text{l.w.}}\big), \tag{6.2}$$

where $\rho([B])([B']) = [\operatorname{pr}(B \otimes B')]$ for $[B] \in \mathscr{K}^\vee$ and $[B'] \in \mathscr{K}^{\text{l.w.}}$. Hence $\mathscr{K}^{\text{l.w.}}$ is a left $\mathscr{K}^\vee$-module. Moreover, when restricted to $\mathscr{K}_0$, each $\mathscr{K}^{\text{l.w.}}_{-n}$ ($n \geqslant 0$) is a $\mathscr{K}_0$-submodule of $\mathscr{K}^{\text{l.w.}}$. Since the action of $\mathscr{K}^{\text{l.w.}}$ on $\mathscr{K}^{\text{l.w.}}$ is nothing but the left multiplication in $\mathscr{K}^\vee$, we will focus on the $\mathscr{K}_0$-module structure on $\mathscr{K}^{\text{l.w.}}$.

Recall that by Proposition 5.6, we may identify $\mathscr{K}^\vee$ with $\mathscr{A} = \mathscr{R}[t^+, t^-]$, while $\mathscr{K}^{\text{l.w.}}$ and $\mathscr{K}_0$ are identified with $\mathscr{R}$ and $\mathbb{Z}[t^+, t^-]$, respectively. Let $\mathscr{R}_n$ be the subspace of $\mathscr{R}$ consisting of formal power series in z of degree n. Then $\mathscr{K}^{\text{l.w.}}_{-n}$ corresponds to $\mathscr{R}_n$.

With these identification, the left $\mathscr{K}^\vee$-module $\mathscr{K}^{\text{l.w.}}$ corresponds to a left $\mathscr{A}$-module

$$\mathscr{A} \Big/ \sum_{m \geqslant 1} \big(\mathscr{A} t_m^+ + \mathscr{A} t_m^-\big),$$

which can be identified with $\mathscr{R}$ as a $\mathbb{Z}$-module. We still denote this $\mathscr{A}$-module structure on $\mathscr{R}$ by $\rho : \mathscr{A} \to \operatorname{End}_{\mathbb{Z}}(\mathscr{R})$. Note that the action of $\mathscr{R} \subset \mathscr{A}$ is the usual multiplication on $\mathscr{R}$, and $\mathscr{R}_n$ is a $\mathbb{Z}[t^+, t^-]$-submodule of $\mathscr{R}$.

6.2. *The action of $\mathscr{K}_0$ on $\mathscr{K}^{\text{l.w.}}$*

Let us describe the action of $\mathbb{Z}[t^+, t^-]$ on $\mathscr{R}$ more explicitly. Recall the following correspondences (see Propositions 5.5 and 5.6);

$$\begin{gathered} Sym_x \otimes Sym_y \xrightarrow{\Phi} \mathscr{K}_0 \xleftarrow{\Psi_0} \mathbb{Z}[t_+, t_-], \\ s_\mu(x) s_\nu(y) \leftrightarrow [\mathbf{B}_{\mu,\nu}] \leftrightarrow t^+_{\{\mu\}} t^-_{\{\nu\}}, \\ e_m(x) e_n(y) \leftrightarrow [\mathbf{B}_{(1^m),(1^n)}] \leftrightarrow t_m^+ t_n^-. \end{gathered} \tag{6.3}$$

Here $t^\pm_{\{\mu\}} = \det(t^\pm_{\mu'_i - i + j})_{1 \leqslant i,j \leqslant \ell(\mu')}$, where we assume $t^\pm_0 = 1$ and $t^\pm_k = 0$ for $k < 0$.

For $n \geqslant 1$, let

$$\mathsf{p}_n^\pm = \rho\big(\hat{s}_n^\pm\big), \tag{6.4}$$

where $\hat{s}_n^+ = \Psi_0^{-1} \circ \Phi(p_n(x))$ and $\hat{s}_n^- = \Psi_0^{-1} \circ \Phi(p_n(y))$. The following is an immediate consequence of Theorem 5.7 (see (5.10)).

Proposition 6.1. *For $n \geqslant 1$, $\mathsf{p}_n^\pm = \gamma_n^\pm = (-1)^{n-1} \sum_{k \in \mathbb{Z}} z_{k \mp n} \frac{\partial}{\partial z_k}$.*

For $\lambda, \mu \in \mathbb{Z}_+^n$, put

$$z_{\{\lambda/\mu\}} = \det(z_{\lambda_i - \mu_j - i + j})_{1 \leqslant i,j \leqslant n}. \tag{6.5}$$

Note that each $f(z)$ in $\mathscr{R}$ can be written uniquely as $f(z) = c_0 + \sum_{n \geqslant 1} \sum_{\lambda \in \mathbb{Z}_+^n} c_\lambda z_{\{\lambda\}}$ for some $c_0, c_\lambda \in \mathbb{Z}$ by Proposition 5.5 and (5.3).

Lemma 6.2. *For $\lambda, \mu \in \mathbb{Z}^n_+$, we have*

$$z_{\{\lambda/\mu\}} = \sum_{\nu \in \mathbb{Z}^n_+} c^{\lambda}_{\mu\nu} z_{\{\nu\}}.$$

Proof. For $p \ll \min\{0, \lambda_n\}$, let

$$\mathbf{B}(\Lambda_\lambda)_{>p} = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_{\Lambda_\lambda} \;\middle|\; r \geqslant 0,\ i_1, \ldots, i_r \in [p+1, \infty) \right\} \subset \mathbf{B}(\Lambda_\lambda).$$

Note that $\mathbf{B}(\Lambda_\lambda)_{>p}$ is the connected $\mathfrak{gl}_{[p+1,\infty)}$-subcrystal of $\mathbf{B}(\Lambda_\lambda)$ including u_{Λ_λ}. Put $D_{p,n} = \prod_{p+1 \leqslant i \leqslant 0} x_i^{-n}$. Then it is easy to see that

$$e^{-n\Lambda_0} \operatorname{ch} \mathbf{B}(\Lambda_\lambda)_{>p} = D_{p,n} s_{(\lambda-(p^n))'}(x_{[p+1,\infty)}),$$

where $x_i = e^{\epsilon_i}$ for $i \in [p+1, \infty)$, $x_{[p+1,\infty)} = \{x_{p+1}, x_{p+2}, \ldots\}$ and $s_{(\lambda-(p^n))'}(x_{[p+1,\infty)})$ is the Schur function in $x_{[p+1,\infty)}$. For $k \in [p+1, \infty)$, put

$$\hat{e}_k(x_{[p+1,\infty)}) = D_{p,1} e_{k-p}(x_{[p+1,\infty)}).$$

Then $e^{-\Lambda_0} \operatorname{ch} \mathbf{B}(\Lambda_k)_{>p} = \hat{e}_k(x_{[p+1,\infty)})$ and $\hat{e}_k(x_{[p+1,\infty)})$ has a well-defined limit when $p \to -\infty$, which is equal to $e^{-\Lambda_0} \operatorname{ch} \mathbf{B}(\Lambda_k) = e^{-\Lambda_0} H_k$ (cf. [18, Section 3.3]).

We have

$$\begin{aligned}
\det\left(\hat{e}_{\lambda_i-\mu_j-i+j}(x_{[p+1,\infty)})\right)_{1 \leqslant i,j \leqslant n} &= D_{p,n} \det\left(e_{\lambda_i-\mu_j-i+j-p}(x_{[p+1,\infty)})\right)_{1 \leqslant i,j \leqslant n} \\
&= D_{p,n} s_{(\lambda-(p^n)-(\mu_n^n))'/(\mu-(\mu_n^n))'}(x_{[p+1,\infty)}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
s_{(\lambda-(p^n)-(\mu_n^n))'/(\mu-(\mu_n^n))'}(x_{[p+1,\infty)})) &= \sum_{\nu \in \mathbb{Z}^n_+} c^{(\lambda-(p^n)-(\mu_n^n))'}_{(\mu-(\mu_n^n))'(\nu-(p^n))'} s_{(\nu-(p^n))'}(x_{[p+1,\infty)}) \\
&= \sum_{\nu \in \mathbb{Z}^n_+} c^{\lambda-(p^n)-(\mu_n^n)}_{\mu-(\mu_n^n)\nu-(p^n)} s_{(\nu-(p^n))'}(x_{[p+1,\infty)}) \\
&= \sum_{\nu \in \mathbb{Z}^n_+} c^{\lambda}_{\mu\nu} s_{(\nu-(p^n))'}(x_{[p+1,\infty)}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\det\left(\hat{e}_{\lambda_i-\mu_j-i+j}(x_{[p+1,\infty)})\right)_{1 \leqslant i,j \leqslant n} &= D_{p,n} \sum_{\nu \in \mathbb{Z}^n_+} c^{\lambda}_{\mu\nu} s_{(\nu-(p^n))'}(x_{[p+1,\infty)}) \\
&= D_{p,n} \sum_{\nu \in \mathbb{Z}^n_+} c^{\lambda}_{\mu\nu} \det\left(e_{\nu_i-i+j-p}(x_{[p+1,\infty)})\right)_{1 \leqslant i,j \leqslant n} \\
&= \sum_{\nu \in \mathbb{Z}^n_+} c^{\lambda}_{\mu\nu} \det\left(\hat{e}_{\nu_i-i+j}(x_{[p+1,\infty)})\right)_{1 \leqslant i,j \leqslant n}. \qquad (6.6)
\end{aligned}$$

Taking $p \to -\infty$ and then multiplying $e^{n\Lambda_0}$ on both sides of (6.6), we have

$$\det(H_{\lambda_i-\mu_j-i+j})_{1\leqslant i,j\leqslant n} = \sum_{\nu\in\mathbb{Z}^n_+} c^{\lambda}_{\mu\nu} \det(H_{\nu_i-i+j})_{1\leqslant i,j\leqslant n}.$$

By Proposition 5.5, we have $z_{\{\lambda/\mu\}} = \sum_{\nu\in\mathbb{Z}^n_+} c^{\lambda}_{\mu\nu} z_{\{\nu\}}$. □

Remark 6.3. We may realize $z_{\{\lambda/\mu\}}$ or $\Psi_{\pm}(z_{\{\lambda/\mu\}})$ as a weight generating function for certain pairs of semistandard tableaux [17, Definition 4.9]. A bijective proof of Lemma 6.2 using this realization is given in [17, Theorem 4.11].

For $\mu \in \mathscr{P}$, we put

$$\mathsf{s}^{\pm}_{\mu} = \rho\big(t^{\pm}_{\{\mu\}}\big). \tag{6.7}$$

Theorem 6.4. *For $\lambda \in \mathbb{Z}^n_+$ and $\mu \in \mathscr{P}$, we have*

$$\mathsf{s}^{+}_{\mu'}(z_{\{\lambda\}}) = \begin{cases} z_{\{\lambda/\mu\}} & \text{if } \ell(\mu) \leqslant n, \\ 0 & \text{otherwise}, \end{cases} \qquad \mathsf{s}^{-}_{\mu'}(z_{\{\lambda\}}) = \begin{cases} z_{\{\lambda/\mu^*\}} & \text{if } \ell(\mu) \leqslant n, \\ 0 & \text{otherwise}, \end{cases}$$

where $\mu^* = (\ldots, -\mu_2, -\mu_1) \in \mathbb{Z}^n_+$.

Proof. Note that the coefficient of $z_{\{\nu\}}$ for $\nu \in \mathbb{Z}^n_+$ in $\mathsf{s}^{+}_{\mu'}(z_{\{\lambda\}})$ is equal to the multiplicity of $\mathbf{B}(-\Lambda_\nu)$ in $\mathbf{B}_{\mu'} \otimes \mathbf{B}(-\Lambda_\lambda)$, or that of $\mathbf{B}(\Lambda_\nu)$ in $\mathbf{B}(\Lambda_\lambda) \otimes \mathbf{B}^{\vee}_{\mu'}$. By Lemma 5.12, it is equal to

$$\sum_{\substack{\gamma\in\mathscr{P} \\ \ell(\gamma)\leqslant n}} c^{\lambda}_{\nu\gamma} c^{\mu}_{\emptyset\gamma}. \tag{6.8}$$

Since

$$c^{\mu}_{\emptyset\gamma} = \begin{cases} 1 & \text{if } \mu = \gamma, \\ 0 & \text{otherwise}, \end{cases}$$

the multiplicity (6.8) is

$$\begin{cases} c^{\lambda}_{\nu\mu} & \text{if } \ell(\mu) \leqslant n, \\ 0 & \text{otherwise}. \end{cases}$$

Since $c^{\lambda}_{\nu\mu} = c^{\lambda}_{\mu\nu}$, we have $\mathsf{s}^{+}_{\mu'}(z_{\{\lambda\}}) = z_{\{\lambda/\mu\}}$ when $\ell(\mu) \leqslant n$ and 0 otherwise, by Lemma 6.2. The proof for $\mathsf{s}^{-}_{\mu'}(z_{\{\lambda\}})$ is similar. □

Corollary 6.5. *For $\mu \in \mathscr{P}$ with $\ell(\mu) \leqslant n$, we have*

$$\mathsf{s}^{+}_{\mu'}(z_{\{(0^n)\}}) = z_{\{\mu^*\}}, \qquad \mathsf{s}^{-}_{\mu'}(z_{\{(0^n)\}}) = z_{\{\mu\}}.$$

For $n \geqslant 1$, we put

$$\mathsf{h}^{\pm}_{n} = \mathsf{s}^{\pm}_{(n)} = \rho\big(t^{\pm}_{\{(n)\}}\big). \tag{6.9}$$

Proposition 6.6. *Let $m \geqslant 0$ and $\lambda \in \mathbb{Z}_+^n$ be given.*

(1) *If $m > n$, then $\mathsf{h}_m^{\pm}$ acts as identically zero on $\mathscr{R}_n$.*
(2) *If $m \leqslant n$, then we have*

$$\mathsf{h}_m^{+}(z_{\{\lambda\}}) = \sum_{\substack{\mu \in \mathbb{Z}_+^n \\ (\lambda - (\mu_n^n))/(\mu - (\mu_n^n)): \\ \text{a vertical strip of length } m}} z_{\{\mu\}},$$

$$\mathsf{h}_m^{-}(z_{\{\lambda\}}) = \sum_{\substack{\mu \in \mathbb{Z}_+^n \\ (\mu - (\lambda_n^n))/(\lambda - (\lambda_n^n)): \\ \text{a vertical strip of length } m}} z_{\{\mu\}}.$$

In particular, we have $\mathsf{h}_n^{\pm}(z_{\{\lambda\}}) = z_{\{\lambda \mp (1^n)\}}$.

Proof. (1) It follows directly from Theorem 6.4.

(2) By Lemma 6.2 and Theorem 6.4, we have

$$\mathsf{h}_m^{-}(z_{\{\lambda\}}) = z_{\{\lambda / -(0^{n-m}, 1^m)\}} = \sum_{\mu \in \mathbb{Z}_+^n} c^{\lambda}_{-(0^{n-m},1^m)\mu} z_{\{\mu\}}.$$

Since $c^{\lambda}_{-(0^{n-m},1^m)\mu} = c^{\lambda + (1^n)}_{(1^{n-m},0^m)\mu} = c^{\lambda + (1^n) - (\mu_n^n)}_{(1^{n-m},0^m)\mu - (\mu_n^n)}$, we have

$$c^{\lambda}_{-(0^{n-m},1^m)\mu} = \begin{cases} 1 & \text{if } \lambda + (1^n) - (\mu_n^n) \text{ is a partition and} \\ & (\lambda + (1^n) - (\mu_n^n))/(\mu - (\mu_n^n)) \text{ is a vertical strip of length } n - m, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $c^{\lambda}_{-(0^{n-m},1^m)\mu} = 1$ if and only if $(\mu - (\lambda_n^n))/(\lambda - (\lambda_n^n))$ is a vertical strip of length m. The proof for $\mathsf{h}_m^{+}(z_{\{\lambda\}})$ is similar. □

As a corollary, we have the following.

Corollary 6.7. *As operators on $\mathscr{R}_n$ $(n \geqslant 1)$, we have for $0 \leqslant i \leqslant n$*

$$\mathsf{h}_n^{+} \mathsf{h}_i^{-} = \mathsf{h}_{n-i}^{+}.$$

Here, we assume $\mathsf{h}_0^{\pm} = \mathrm{id}_{\mathscr{R}_n}$.

6.3. The action of $\mathscr{K}_0$ on $\mathscr{K}_{-n}^{\mathrm{l.w.}}$ and the $(\mathfrak{gl}_\infty, \mathfrak{gl}_n)$-duality

For $n \geqslant 1$, let $R^\circ(GL_n(\mathbb{C}))$ be the character ring of finite-dimensional polynomial representations of $GL_n(\mathbb{C})$, which is isomorphic to the ring of symmetric polynomials in $x_{[n]}$. Also, it is free commutative algebra generated by $e_k(x_{[n]})$ for $1 \leqslant k \leqslant n$. Let $R(GL_n(\mathbb{C}))$ be the character ring of finite-dimensional representations of $GL_n(\mathbb{C})$. Then $R(GL_n(\mathbb{C}))$ is the ring of symmetric Laurent polynomials in $x_{[n]}$, and it is the localization of $R^\circ(GL_n(\mathbb{C}))$ with respect to the multiplicative subset $\{e_n(x_{[n]})^m \mid m \geqslant 1\}$.

Theorem 6.8. *For $n \geqslant 1$, let $\mathscr{M}_n = \mathscr{K}_0 \cdot [\mathbf{B}(-n\Lambda_0)]$ be the $\mathscr{K}_0$-submodule of $\mathscr{K}_{-n}^{\mathrm{l.w.}}$ generated by $[\mathbf{B}(-n\Lambda_0)]$.*

(1) *We have*

$$\mathscr{M}_n = \bigoplus_{\lambda \in \mathbb{Z}_+^n} \mathbb{Z}\big[\mathbf{B}(-\Lambda_\lambda)\big].$$

In particular, $\mathscr{K}_{-n}^{\text{l.w.}}$ *is the completion of* $\mathscr{M}_n$.

(2) *There exists a* $\mathbb{Z}$*-algebra isomorphism*

$$R\big(GL_n(\mathbb{C})\big) \to \mathscr{K}_0/\mathrm{ann}_{\mathscr{K}_0}(\mathscr{M}_n),$$

where $s_\lambda(x_{[n]})$ *is mapped to* $\overline{[\mathbf{B}_{(n)}^{\otimes \ell} \otimes \mathbf{B}_{(\lambda+(\ell^n))'}^{\vee}]}$ $(\lambda \in \mathbb{Z}_+^n)$ *with* $\ell = \max\{-\lambda_n, 0\}$.

(3) *The ideal* $\mathrm{ann}_{\mathscr{K}_0}(\mathscr{M}_n)$ *is generated by*

$$[\mathbf{B}_{(k),\emptyset}], [\mathbf{B}_{\emptyset,(k)}] \quad (k > n), \qquad [\mathbf{B}_{(n),(i)}] - [\mathbf{B}_{(n-i),\emptyset}] \quad (0 \leqslant i \leqslant n).$$

Proof. (1) Since we may identify $\mathscr{K}_0$ with $\mathbb{Z}[t^+, t^-]$, and $\mathscr{M}_n$ with $\mathbb{Z}[t^+, t^-]z_{\{(0^n)\}}$ in $\mathscr{R}_n$, it is equivalent to show that $\mathscr{M}_n = \bigoplus_{\lambda \in \mathbb{Z}_+^n} \mathbb{Z}z_{\{\lambda\}}$. By Theorem 6.4, we have $\mathscr{M}_n \subset \bigoplus_{\lambda \in \mathbb{Z}_+^n} \mathbb{Z}z_{\{\lambda\}}$. Conversely, given $\lambda \in \mathbb{Z}_+^n$, put $\mu = \lambda + (\ell^n) \in \mathscr{P}$, where $\ell = \max\{-\lambda_n, 0\}$. By Theorem 6.4 and Proposition 6.6, we have

$$z_{\{\lambda\}} = \big(\big(\mathsf{h}_n^+\big)^{\ell} \circ \mathsf{s}_{\mu'}^-\big)(z_{\{(0^n)\}}) = \big(t_{\{(n)\}}^+\big)^{\ell} \cdot \big(t_{\{\mu'\}}^- \cdot z_{\{(0^n)\}}\big), \tag{6.10}$$

and hence $\bigoplus_{\lambda \in \mathbb{Z}_+^n} \mathbb{Z}z_{\{\lambda\}} \subset \mathscr{M}_n$.

(2) Note that the map $g : \mathscr{K}_0/\mathrm{ann}_{\mathscr{K}_0}(\mathscr{M}_n) \to \mathscr{M}_n$ given by $g(\bar{a}) = a \cdot z_{\{(0^n)\}}$ is a $\mathbb{Z}$-linear isomorphism. Define a ring homomorphism $f : R(GL_n(\mathbb{C})) \to \mathscr{K}_0/\mathrm{ann}_{\mathscr{K}_0}(\mathscr{M}_n)$ by $f(e_k(x_{[n]})) = \overline{t_{\{(k)\}}^-}$ $(1 \leqslant k \leqslant n)$ and $f(e_n(x_{[n]})^{-1}) = \overline{t_{\{(n)\}}^+}$. Since $\overline{t_{\{(n)\}}^+ t_{\{(n)\}}^-} = 1$ in $\mathscr{K}_0/\mathrm{ann}_{\mathscr{K}_0}(\mathscr{M}_n)$ by Corollary 6.7, f is well defined. For $\lambda \in \mathbb{Z}_+^n$, we have

$$f\big(s_\lambda(x_{[n]})\big) = f\big(e_n(x_{[n]})^{-\ell} s_{\lambda+(\ell^n)}(x_{[n]})\big) = \overline{\big(t_{\{(n)\}}^+\big)^{\ell} t_{\{(\lambda+(\ell^n))'\}}^-},$$

where $\ell = \max\{-\lambda_n, 0\}$. By (6.10), we have $g \circ f(s_\lambda(x_{[n]})) = z_{\{\lambda\}}$, and hence obtain a $\mathbb{Z}$-linear isomorphism

$$g \circ f : R\big(GL_n(\mathbb{C})\big) \to \mathscr{M}_n.$$

This implies that f is an isomorphism.

(3) Let $I_n = \langle t_{\{(m)\}}^{\pm}\ (m > n),\ t_{\{(n)\}}^+ t_{\{(i)\}}^- - t_{\{(n-i)\}}^+\ (0 \leqslant i \leqslant n)\rangle$, which is an ideal in $\mathbb{Z}[t^+, t^-]$. By Proposition 6.6 and Corollary 6.7, $I_n \subset \mathrm{ann}_{\mathscr{K}_0}(\mathscr{M}_n)$. Hence we have an algebra homomorphism $\iota : \mathscr{K}_0/I_n \to \mathscr{K}_0/\mathrm{ann}_{\mathscr{K}_0}(\mathscr{M}_n)$. Similarly, we have a surjective algebra homomorphism $h : R(GL_n(\mathbb{C})) \to \mathscr{K}_0/I_n$ such that $h(e_k(x_{[n]})) = \overline{t_{\{(k)\}}^-}$ $(1 \leqslant k \leqslant n)$ and $h(e_n(x_{[n]})^{-1}) = \overline{t_{\{(n)\}}^+}$. Since $\iota \circ h = f$ and f is an isomorphism by (2), it follows that h is an isomorphism and hence so is ι. This implies that $I_n = \mathrm{ann}_{\mathscr{K}_0}(\mathscr{M}_n)$. $\square$

By Theorem 6.8 (2), $\mathscr{M}_n$ is naturally equipped with an $R(GL_n(\mathbb{C}))$-module structure, while $R(GL_n(\mathbb{C}))$ is itself an $R(GL_n(\mathbb{C}))$-module by left multiplication. Hence, we obtain the following.

Corollary 6.9. *The map sending* $s_\lambda(x_{[n]})$ *to* $[\mathbf{B}(-\Lambda_\lambda)]$ $(\lambda \in \mathbb{Z}_+^n)$ *extends to an* $R(GL_n(\mathbb{C}))$*-module isomorphism* $R(GL_n(\mathbb{C})) \to \mathscr{M}_n$.

Remark 6.10. Consider a graded $\mathbb{Z}$-module $R(GL)=\bigoplus_{n\geqslant 0}R(GL_n(\mathbb{C}))$, where $R(GL_0(\mathbb{C}))=\mathbb{Z}$. Then $R(GL)$ is a graded coalgebra with a graded comultiplication Δ given by

$$\Delta(\chi)=\sum_{p+q=n}\operatorname{Res}^{GL_n(\mathbb{C})}_{GL_p(\mathbb{C})\times GL_q(\mathbb{C})}(\chi)$$

for $\chi\in R(GL_n(\mathbb{C}))$. Then by the $(\mathfrak{gl}_\infty,\mathfrak{gl}_n)$-duality (see Proposition 3.11), we have as $\mathbb{Z}$-algebras

$$\bigoplus_{n\geqslant 0}\mathscr{K}^{\mathrm{l.w.}}_{-n}\simeq R(GL)^\circ,$$

where $R(GL)^\circ$ is the restricted dual of $R(GL)$ [30]. Equivalently, the branching rule in $R(GL)$ corresponds to the tensor product rule of integrable lowest weight $U_q(\mathfrak{gl}_\infty)$-modules (or $\mathfrak{gl}_\infty$-modules). On the other hand, Theorem 6.8 explains this duality from a different point of view. That is, we can recover $R(GL)$ via the action of $\mathscr{K}_0$ on $\mathscr{K}^{\mathrm{l.w.}}$,

$$R(GL)=\mathbb{Z}\oplus\left(\bigoplus_{n\geqslant 1}\mathscr{K}_0/\operatorname{ann}_{\mathscr{K}_0}\big(\big[\mathbf{B}(-n\Lambda_0)\big]\big)\right).$$

This explains that the multiplication in $R(GL_n(\mathbb{C}))$, which is not a graded multiplication in $R(GL)$, corresponds to the composite of operators on $\mathscr{K}^{\mathrm{l.w.}}_{-n}$ associated with level zero extremal weight crystals.

6.4. Hall–Littlewood vertex operators on $\mathscr{K}^{\mathrm{l.w.}}$

Fix an indeterminate q. Let $\mathscr{B}^q$ be the associative $\mathbb{Z}[q]$-algebra generated by $\{B^q_n \mid n\in\mathbb{Z}\}$ subject to the relation;

$$B^q_mB^q_n=qB^q_nB^q_m+qB^q_{m+1}B^q_{n-1}-B^q_{n-1}B^q_{m+1}\quad (m,n\in\mathbb{Z}).\tag{6.11}$$

The above defining relations can be written in terms of generating functions as follows;

$$(u-qv)B^q(u)B^q(v)=(qu-v)B^q(v)B^q(u),\tag{6.12}$$

where u,v are formal commuting variables and $B^q(u)=\sum_{k\in\mathbb{Z}}B^q_ku^k$. Note that $\mathscr{B}^q$ is isomorphic to the "half" of the algebra generated by the Hall–Littlewood vertex operators on $\mathbb{Z}[q]\otimes_{\mathbb{Z}}\mathit{Sym}$ introduced by Jing [8] (see also [7,26]).

Our main claim in this section is that there is a natural embedding of $\mathscr{B}^q$ into $\operatorname{End}_{\mathbb{Z}}(\mathscr{K}^{\mathrm{l.w.}})[\![q]\!]$ and hence we can define a Hall–Littlewood vertex operator for $\mathfrak{gl}_\infty$. Put

$$\mathtt{F}^\vee(u)=\sum_{k\in\mathbb{Z}}\rho\big(\big[\mathbf{B}(-\Lambda_k)\big]\big)u^k,$$

$$\mathtt{S}(v)=\sum_{n\geqslant 0}\rho\big([\mathbf{B}_{(n)}]\big)v^n,\qquad \mathtt{E}(v)=\sum_{n\geqslant 0}\rho\big([\mathbf{B}_{(1^n)}]\big)v^n,\tag{6.13}$$

where $\rho:\mathscr{K}^\vee\to\operatorname{End}(\mathscr{K}^{\mathrm{l.w.}})$ is given in (6.2). We define

$$\mathtt{B}^q(u)=\mathtt{F}^\vee(u)\mathtt{S}\big(-u^{-1}\big)\mathtt{E}\big(qu^{-1}\big)=\sum_{k\in\mathbb{Z}}\mathtt{B}^q_ku^k\in\big(\operatorname{End}\big(\mathscr{K}^\vee\big)[\![q]\!]\big)\big[\!\big[u,u^{-1}\big]\!\big].\tag{6.14}$$

Equivalently, for $k \in \mathbb{Z}$

$$\mathtt{B}_k^q = \sum_{i \geqslant 0} \sum_{j=0}^{i} (-1)^{i-j} q^j \rho\big(\big[\mathbf{B}(-\Lambda_{i+k}) \otimes \mathbf{B}_{(i-j)} \otimes \mathbf{B}_{(1^j)}\big]\big)$$
$$= \sum_{j \geqslant 0} \bigg(\sum_{i \geqslant 0} (-1)^i \rho\big(\big[\mathbf{B}(-\Lambda_{i+j+k}) \otimes \mathbf{B}_{(i)} \otimes \mathbf{B}_{(1^j)}\big]\big)\bigg) q^j. \tag{6.15}$$

Note that the coefficient of q^j ($j \geqslant 0$) in $\mathtt{B}_k^q$ is a well-defined $\mathbb{Z}$-linear map from $\mathscr{K}_{-n}^{\text{l.w.}}$ to $\mathscr{K}_{-n-1}^{\text{l.w.}}$ for $n \geqslant 0$ by Proposition 6.6 (1), and hence a $\mathbb{Z}$-linear operator on $\mathscr{K}^{\text{l.w.}}$.

Lemma 6.11. *We have*

$$\mathtt{S}(v)\mathtt{F}^\vee(u) = \mathtt{F}^\vee(u)\mathtt{S}(v)(1 + uv).$$

Proof. By Lemma 5.12, we have for $k \in \mathbb{Z}$ and $n \geqslant 0$,

$$\mathbf{B}(\Lambda_k) \otimes \mathbf{B}_{(n)}^\vee \simeq \big(\mathbf{B}_{(n)}^\vee \otimes \mathbf{B}(\Lambda_k)\big) \sqcup \big(\mathbf{B}_{(n-1)}^\vee \otimes \mathbf{B}(\Lambda_{k-1})\big).$$

By taking its dual, we obtain the identity. □

Lemma 6.12. *We have*

$$(u - qv)\mathtt{B}^q(u)\mathtt{B}^q(v) = (qu - v)\mathtt{B}^q(v)\mathtt{B}^q(u).$$

Proof. It follows directly from the dual identity of (5.14) and Lemma 6.11. □

By Lemma 6.12, $\mathtt{B}_k^q$'s satisfy the defining relation (6.11) for $\mathscr{B}^q$, and we have a $\mathbb{Z}[q]$-algebra homomorphism

$$\varpi : \mathscr{B}^q \to \operatorname{End}_{\mathbb{Z}}\big(\mathscr{K}^{\text{l.w.}}\big)[[q]] \tag{6.16}$$

given by $\varpi(B_k^q) = \mathtt{B}_k^q$ for $k \in \mathbb{Z}$.

We define for $n \geqslant 1$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$

$$B_\alpha^q = \prod_{1 \leqslant i < j \leqslant n} (1 - qR_{ij}) B_{\alpha_1}^q \cdots B_{\alpha_n}^q, \tag{6.17}$$

where R_{ij} is a raising operator, i.e.

$$R_{ij}\big(B_{\alpha_1}^q \cdots B_{\alpha_n}^q\big) = B_{\alpha_1}^q \cdots B_{\alpha_i+1}^q \cdots B_{\alpha_j-1}^q \cdots B_{\alpha_n}^q.$$

We should remark that when B_k^q's are replaced by the Hall–Littlewood vertex operators on $\mathbb{Z}[q] \otimes_Z Sym$, B_λ^q ($\lambda \in \mathbb{Z}_+^n$) gives the operator introduced by Shimozono and Zabrocki [26]. By (6.11), we can check that $B_\alpha^q = -B_{(\alpha_1,\ldots,\alpha_{i+1}-1,\alpha_i+1,\ldots,\alpha_n)}^q$. More generally, for each permutation w in S_n, we have

$$B_\alpha^q = (-1)^{\ell(w)} B_{w(\alpha+\rho_n)-\rho_n}^q, \tag{6.18}$$

where $\ell(w)$ is the length of w and $\rho_n = (n-1, n-2, \ldots, 0)$. Now for $\lambda \in \mathbb{Z}_+^n$, we put

$$\mathtt{B}_\lambda^q = \varpi\big(B_\lambda^q\big). \tag{6.19}$$

Let

$$\mathtt{F}^\vee(x_{[n]}) = \prod_{k\in[n]} \mathtt{F}^\vee(x_k), \qquad \mathtt{S}\big(-x_{[n]}^{-1}\big) = \prod_{k\in[n]} \mathtt{S}\big(-x_k^{-1}\big), \qquad \mathtt{E}\big(qx_{[n]}^{-1}\big) = \prod_{k\in[n]} \mathtt{E}\big(qx_k^{-1}\big).$$

Then we have the following generating function for $\mathtt{B}_\lambda^q$.

Proposition 6.13. *For $n \geqslant 1$, we have*

$$\sum_{\lambda\in\mathbb{Z}_+^n} \mathtt{B}_\lambda^q s_\lambda(x_{[n]}) = \mathtt{F}^\vee(x_{[n]})\mathtt{S}\big(-x_{[n]}^{-1}\big)\mathtt{E}\big(qx_{[n]}^{-1}\big).$$

Proof. By (5.14) and Lemma 6.11, we have

$$\mathtt{B}^q(x_1)\cdots\mathtt{B}^q(x_n) = \prod_{1\leqslant i<j\leqslant n} \frac{1-x_i^{-1}x_j}{1-qx_i^{-1}x_j} \mathtt{F}^\vee(x_{[n]})\mathtt{S}\big(-x_{[n]}^{-1}\big)\mathtt{E}\big(qx_{[n]}^{-1}\big). \tag{6.20}$$

Since

$$\prod_{1\leqslant i<j\leqslant n} \big(1-qx_i^{-1}x_j\big)\mathtt{B}^q(x_1)\cdots\mathtt{B}^q(x_n) = \sum_{\lambda\in\mathbb{Z}_+^n} \mathtt{B}_\lambda^q \left(\sum_{w\in S_n} (-1)^{\ell(w)} x_{[n]}^{w(\lambda+\rho_n)-\rho_n}\right)$$

by (6.18) and (6.19), we have

$$\prod_{1\leqslant i<j\leqslant n} \frac{1-qx_i^{-1}x_j}{1-x_i^{-1}x_j} \mathtt{B}^q(x_1)\cdots\mathtt{B}^q(x_n) = \sum_{\lambda\in\mathbb{Z}_+^n} \mathtt{B}_\lambda^q s_\lambda(x_{[n]}). \qquad \square$$

Corollary 6.14. *For $\lambda \in \mathbb{Z}_+^n$, we have*

$$\mathtt{B}_\lambda^q = \sum_{\eta\in\mathbb{Z}_+^n} \sum_{\substack{\sigma,\mu,\nu\in\mathscr{P} \\ \ell(\sigma),\ell(\mu),\ell(\nu)\leqslant n}} (-1)^{|\mu|} q^{|\nu|} c_{\eta\sigma^*}^{\lambda} c_{\mu\nu}^{\sigma} \rho\big([\mathbf{B}(-\Lambda_\eta)\otimes\mathbf{B}_\mu\otimes\mathbf{B}_{\nu'}]\big).$$

Proof. Note that

$$\mathtt{S}(x_{[n]}) = \sum_{\mu\in\mathscr{P}} \rho\big([\mathbf{B}_\mu]\big) s_\mu(x_{[n]}),$$

which can be verified by modifying the arguments in Proposition 3.10 using matrices with non-negative integral entries (see [4,21]). By Lemma 5.10, we have

$$\mathtt{F}^{\vee}(x_{[n]})\mathtt{S}\big(-x_{[n]}^{-1}\big)\mathtt{E}\big(qx_{[n]}^{-1}\big)$$
$$=\sum_{\eta\in\mathbb{Z}_+^n}\sum_{\substack{\mu,\nu\in\mathscr{P}\\ \ell(\mu),\ell(\nu)\leqslant n}}(-1)^{|\mu|}q^{|\nu|}\rho\big(\big[\mathbf{B}(-\Lambda_\eta)\otimes\mathbf{B}_\mu\otimes\mathbf{B}_{\nu'}\big]\big)s_\eta(x_{[n]})s_\mu\big(x_{[n]}^{-1}\big)s_\nu\big(x_{[n]}^{-1}\big).$$

Note that $s_\zeta(x_{[n]}^{-1})=s_{\zeta^*}(x_{[n]})$ and $c^{\sigma^*}_{\mu^*\nu^*}=c^\sigma_{\mu\nu}$ for $\zeta,\sigma,\mu,\nu\in\mathbb{Z}_+^n$. Comparing the coefficient of $s_\lambda(x_{[n]})$ on both sides, we obtain the identity. □

Proposition 6.15. *The map $\varpi:\mathscr{B}^q\to\mathrm{End}_{\mathbb{Z}}(\mathscr{K}^{\mathrm{l.w.}})[[q]]$ in (6.16) is injective.*

Proof. By (6.17), we have for $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n$

$$B^q_{\alpha_1}\cdots B^q_{\alpha_n}=\prod_{1\leqslant i<j\leqslant n}(1-qR_{ij})^{-1}B^q_\alpha,$$

where R_{ij} acts on B^q_α by $R_{ij}(B^q_\alpha)=B^q_{(\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_j-1,\ldots,\alpha_n)}$. By the same arguments as in [22, Ex. III.6.4], we have

$$(-1)^{\ell(w)}B^q_{\alpha_1}\cdots B^q_{\alpha_n}=\sum_{\lambda\in\mathbb{Z}_+^n}K_{\lambda\mu}(q)B^q_\lambda,\tag{6.21}$$

where $w\in S_n$ and $\mu\in\mathbb{Z}_+^n$ are given by $w(\alpha+\rho_n)-\rho_n=\mu$, and $K_{\lambda\mu}(q)$ is the coefficient of $x_1^{\mu_1}\cdots x_n^{\mu_n}$ in $\det(x_i^{\lambda_i-i+j})/\prod_{1\leqslant i<j\leqslant n}(1-qx_i^{-1}x_j)$, or the Kostka–Foulkes polynomial associated with $\lambda,\mu\in\mathbb{Z}_+^n$. Note that the sum on the right-hand side is not necessarily finite.

Suppose that $b\in\mathscr{B}^q$ is given such that $\varpi(b)=0$. By (6.21), we may write

$$b=\sum_{n=1}^{N}\sum_{\lambda\in\mathbb{Z}_+^n}c_\lambda(q)B^q_\lambda$$

for some $N\geqslant 1$ and $c_\lambda(q)\in\mathbb{Z}[q]$, and

$$\varpi(b)=\sum_{n=1}^{N}\sum_{\lambda\in\mathbb{Z}_+^n}c_\lambda(q)\mathtt{B}^q_\lambda.$$

We first assume that $c_\mu(0)\neq 0$ for some $\mu\in\mathbb{Z}_+^n$. Then by Corollary 6.14, we have

$$\varpi(b)\cdot 1|_{q=0}=\sum_{n=1}^{N}\sum_{\lambda\in\mathbb{Z}_+^n}c_\lambda(0)\mathtt{B}^0_\lambda\cdot 1=\sum_{n=1}^{N}\sum_{\lambda\in\mathbb{Z}_+^n}c_\lambda(0)\big[\mathbf{B}(-\Lambda_\lambda)\big],$$

since $[\mathbf{B}_{(k)}]\cdot 1=[\mathbf{B}_{(1^k)}]\cdot 1=0$ for $k>0$. (Here $1=[\mathbf{B}(0)]\in\mathscr{K}^{\mathrm{l.w.}}$.) Since $\{[\mathbf{B}(-\Lambda_\lambda)]\mid\lambda\in\mathbb{Z}_+^n,\ n\geqslant 1\}$ is linearly independent, we have $c_\lambda(0)=0$ for all λ, which is a contradiction. Now suppose that $c_\lambda(0)=0$ for all λ. If $b\neq 0$, then we have $b=q^m b'$ for some $m\geqslant 1$ and $b'\in\mathscr{B}^q$, where $c_\mu(q)q^{-m}\in\mathbb{Z}[q]$ has a non-zero constant term for some μ. Since $\mathrm{End}(\mathscr{K}^{\mathrm{l.w.}})[[q]]$ is free over $\mathbb{Z}[q]$, we have $\varpi(b')=0$. By the same argument, we conclude that $b'=0$, and hence $b=0$, which is also a contradiction. Therefore, ϖ is injective. □

Recall that $[\mathbf{B}(-\Lambda_\mu)] = \Psi_-(z_{\{\mu\}})$ for $\mu \in \mathbb{Z}_+^n$. Then we define

$$R_{ij}\big[\mathbf{B}(-\Lambda_\mu)\big] = \Psi_-(z_{\{(\ldots,\mu_i+1,\ldots,\mu_j-1,\ldots)\}}) \quad (1 \leqslant i < j \leqslant n).$$

Proposition 6.16. *For $\mu \in \mathbb{Z}_+^n$, we have*

$$\mathrm{B}_{\mu_1}^q \cdots \mathrm{B}_{\mu_n}^q \cdot 1 = \frac{1}{\prod_{1\leqslant i<j\leqslant n}(1-qR_{ij})}\big[\mathbf{B}(-\Lambda_\mu)\big] = \sum_{\lambda\in\mathbb{Z}_+^n} K_{\lambda\mu}(q)\big[\mathbf{B}(-\Lambda_\mu)\big].$$

Proof. By (6.20), we have

$$\mathrm{B}^q(x_1)\cdots\mathrm{B}^q(x_n) = \prod_{1\leqslant i<j\leqslant n}\frac{1-x_i^{-1}x_j}{1-qx_i^{-1}x_j}\mathrm{F}^\vee(x_{[n]})\mathrm{S}\big(-x_{[n]}^{-1}\big)\mathrm{E}\big(qx_{[n]}^{-1}\big).$$

If we apply both sides to $1 = [\mathbf{B}(0)] \in \mathscr{K}^{\mathrm{l.w.}}$, then

$$\mathrm{B}^q(x_1)\cdots\mathrm{B}^q(x_n)\cdot 1 = \prod_{1\leqslant i<j\leqslant n}\frac{1-x_i^{-1}x_j}{1-qx_i^{-1}x_j}\mathrm{F}^\vee(x_{[n]})\cdot 1,$$

since $[\mathbf{B}_{(k)}]\cdot 1 = [\mathbf{B}_{(1^k)}]\cdot 1 = 0$ for $k>0$. Given $\mu \in \mathbb{Z}_+^n$, equating the coefficients of $x_1^{\mu_1}\cdots x_n^{\mu_n}$ on both sides, we obtain the first identity since $z_{\{\mu\}} = \prod_{1\leqslant i<j\leqslant n}(1-R_{ij})z_\mu$. The second identity follows from the same arguments as in (6.21). □

Remark 6.17. If $q=0$, then $\mathscr{B}^0$ is the algebra of Bernstein operators [22] and $\mathrm{B}_{\lambda_1}^0\cdots\mathrm{B}_{\lambda_n}^0\cdot 1 = [\mathbf{B}(-\Lambda_\lambda)]$ ($\lambda \in \mathbb{Z}_+^n$), which is a Rodrigues type formula. Also, when $q=1$, we have $\mathrm{B}_{\lambda_1}^1\cdots\mathrm{B}_{\lambda_n}^1\cdot 1 = [\mathbf{B}(-\Lambda_{\lambda_1})\otimes\cdots\otimes\mathbf{B}(-\Lambda_{\lambda_n})]$. Hence, B_k^q may be viewed as a Hall–Littlewood vertex operator for $\mathfrak{gl}_\infty$.

One may define another Hall–Littlewood vertex operator for $\mathfrak{gl}_\infty$ using the involution ω on $\mathscr{D}$ (5.12). Put

$$\bar{\mathrm{B}}^q(u) = \sum_{k\in\mathbb{Z}}\bar{\mathrm{B}}_k^q u^k = \mathrm{F}^\vee\big(u^{-1}\big)\mathrm{S}^\vee\big(-u^{-1}\big)\mathrm{E}^\vee\big(qu^{-1}\big), \tag{6.22}$$

where

$$\mathrm{S}^\vee(v) = \sum_{n\geqslant 0}\rho\big(\big[\mathbf{B}_{(n)}^\vee\big]\big)v^n, \qquad \mathrm{E}^\vee(v) = \sum_{n\geqslant 0}\rho\big(\big[\mathbf{B}_{(1^n)}^\vee\big]\big)v^n.$$

Note that for $k\in\mathbb{Z}$

$$\begin{aligned}\bar{\mathrm{B}}_k^q &= \sum_{j\geqslant 0}\bigg(\sum_{i\geqslant 0}(-1)^i\rho\big(\big[\mathbf{B}(-\Lambda_{-i-j-k})\otimes\mathbf{B}_{(i)}^\vee\otimes\mathbf{B}_{(1^j)}^\vee\big]\big)\bigg)q^j\\ &= \sum_{j\geqslant 0}\bigg(\sum_{i\geqslant 0}(-1)^i\rho\big(\omega\big[\mathbf{B}(-\Lambda_{i+j+k})\otimes\mathbf{B}_{(i)}\otimes\mathbf{B}_{(1^j)}\big]\big)\bigg)q^j. \end{aligned}\tag{6.23}$$

By (6.15), we also have a $\mathbb{Z}[q]$-algebra homomorphism $\overline{\varpi}:\mathscr{B}^q \to \mathrm{End}_{\mathbb{Z}}(\mathscr{K}^{\mathrm{l.w.}})[[q]]$ given by $\overline{\varpi}(B_k^q) = \bar{\mathrm{B}}_k^q$ for $k\in\mathbb{Z}$.

Proposition 6.18. *For $m, n \in \mathbb{Z}$, we have*

$$\overline{\mathrm{B}}_m^q \mathrm{B}_n^q = \mathrm{B}_n^q \overline{\mathrm{B}}_m^q.$$

Proof. It is straightforward to check that

$$\begin{aligned}\mathrm{B}^q(u)\overline{\mathrm{B}}^q(v) &= \mathrm{F}^\vee(u)\mathrm{S}\big(-u^{-1}\big)\mathrm{E}\big(qu^{-1}\big)\mathrm{F}^\vee\big(v^{-1}\big)\mathrm{S}^\vee\big(-v^{-1}\big)\mathrm{E}^\vee\big(qv^{-1}\big) \\ &= \mathrm{F}^\vee(u)\mathrm{F}^\vee\big(v^{-1}\big)\mathrm{S}\big(-u^{-1}\big)\mathrm{E}\big(qu^{-1}\big)\mathrm{S}^\vee\big(-v^{-1}\big)\mathrm{E}^\vee\big(qv^{-1}\big)\frac{(1-u^{-1}v^{-1})}{(1-qu^{-1}v^{-1})} \\ &= \overline{\mathrm{B}}^q(v)\mathrm{B}^q(u),\end{aligned}$$

which implies that $\overline{\mathrm{B}}_m^q \mathrm{B}_n^q = \mathrm{B}_n^q \overline{\mathrm{B}}_m^q$ for $m, n \in \mathbb{Z}$. □

Remark 6.19. For $\lambda \in \mathbb{Z}_+^n$, we have $\overline{\mathrm{B}}_{\lambda_1}^0 \cdots \overline{\mathrm{B}}_{\lambda_n}^0 \cdot 1 = z_{\{\lambda^*\}} = [\mathbf{B}(-\Lambda_{\lambda^*})]$.

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