A question of Kaplansky

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Received 28 November 2002

Dedicated to Claudio Procesi on the occasion of his 60th birthday

Abstract

We show, using affinization, that over any field F there exists a primitive, algebraic affine F-algebra that is not locally finite, answering a question of Kaplansky in the affirmative. © 2002 Elsevier Science (USA). All rights reserved.

1. Introduction

In [4] Kurosh asked whether it was true that an algebraic algebra was necessarily locally finite. Golod and Shafarevich [1] constructed an infinite-dimensional affine nil ring, disproving the conjecture of Kurosh. Kaplansky later asked whether one could find both a primitive and a primitive, affine counter-example to the Kurosh conjecture (see [3, Problem 15]). Amitsur [unpublished] was able to answer the first half of this problem by decorating the Golod–Shafarevich example to obtain a primitive counter-example to the Kurosh conjecture. This example, however, is not affine; indeed, it is not even countably infinite-dimensional over its base field. We give an example of an algebraic, primitive affine ring that is not locally finite by using the technique of affinization. Our main result is the following.

Theorem 1. *For any field F there exists an infinite-dimensional primitive, algebraic affine F-algebra.*

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2. Construction

We now give our construction.

Theorem 2. *Let F be a field. There exists a primitive algebraic affine F-algebra B that is not locally finite.*

Proof. Let T be the subring of row-finite $\aleph_0 \times \aleph_0$ matrices over F given by

$$T = \big\{(t_{i,j})_{i,j \geqslant 0}\colon \text{ there exists an } N \text{ such that } t_{i,j} = t_{i+N,j+N},$$
$$t_{k,j+N} = 0,\ t_{i+N,k} = 0 \text{ for } i,\, j = 0,\ 0 \leqslant k < N\big\}. \quad (2.1)$$

We observe that T is a countably infinite-dimensional primitive, algebraic algebra. From [1] we know that there exists an affine F-algebra $A = F\{x, y\}$ such that

$$A \text{ is infinite dimensional over } F, \quad (2.2)$$

$$\text{the homogeneous maximal ideal } (x, y) \text{ is nil.} \quad (2.3)$$

Let R be the free product of A and the polynomial ring $F[z]$ and let

$$S = \begin{pmatrix} F + Rz & R \\ Rz & R \end{pmatrix}. \quad (2.4)$$

Note that S is an affine F-algebra, generated by

$$\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix},$$

where $c \in \{1, x, y, z\}$. Notice also that $F + Rz$ is a free F-algebra on the set of generators $\{v_i z\colon i \geqslant 0\}$, where $\{v_i\colon i \geqslant 0\}$ is an F-basis for A. We can therefore find a surjective F-algebra homomorphism

$$\Phi : F + Rz \to T.$$

Let

$$P = \ker(\Phi) \quad (2.5)$$

and for $1 \leqslant i,\, j \leqslant 2$ let $e_{i,j}$ denote the 2×2 matrix with a one in the i, j entry and zero in every other entry. Observe that

$$J = S \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} S = \begin{pmatrix} P & PR \\ RzP & RzPR \end{pmatrix}$$

is an ideal in S with

$$e_{1,1} J e_{1,1} = P e_{1,1}. \quad (2.6)$$

By Zorn's lemma we can extend J to an ideal Q which is maximal with respect to satisfying Eq. (2.6). By maximality, Q is a prime ideal. Finally, we let

$$B = S/Q. \quad (2.7)$$

Then B is an infinite-dimensional, affine prime algebra; moreover, by construction, $e_{1,1} B e_{1,1} \cong T$, and hence B is primitive by [5, Theorem 1]. We now show that B is an algebraic F-algebra. Let I be the image of

$$\begin{pmatrix} F + Rz & R \\ 0 & 0 \end{pmatrix}$$

in B. We have that I is a right ideal. Moreover, I is locally finite since T is locally finite. It follows from [2, X.12, Proposition 2] that the two-sided ideal generated by I is algebraic. Notice that BI is the image of

$$\mathcal{I} = \begin{pmatrix} F + Rz & R \\ Rz & R \end{pmatrix} \cdot \begin{pmatrix} F + Rz & R \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} F + Rz & R \\ Rz & RzR \end{pmatrix}$$

in B. Hence B/BI is a homomorphic image of

$$S/\mathcal{I} \cong \begin{pmatrix} 0 & 0 \\ 0 & R/RzR \end{pmatrix} \cong A.$$

In particular B/BI is algebraic. Since BI is algebraic and B/BI is algebraic, we conclude that B is algebraic. Finally, since

$$e_{1,1} B e_{1,1} \cong T,$$

we conclude that B is infinite dimensional. Thus we have constructed an affine, algebraic primitive algebra that is not locally finite. $\quad\square$

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