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Family of multivariate generalized *t* distributions

Olcay Arslan

Department of Statistics, University of Cukurova Balcali 01330, Adana, Turkey

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Abstract

In this paper, we introduce a new family of multivariate distributions as the scale mixture of the multivariate power exponential distribution introduced by Gómez et al. (Comm. Statist. Theory Methods 27(3) (1998) 589) and the inverse generalized gamma distribution. Since the resulting family includes the multivariate *t* distribution and the multivariate generalization of the univariate GT distribution introduced by McDonald and Newey (Econometric Theory 18 (11) (1988) 4039) we call this family as the "multivariate generalized *t*-distributions family", or MGT for short. We show that this family of distributions belongs to the elliptically contoured distributions family, and investigate the properties. We give the stochastic representation of a random variable distributed as a multivariate generalized *t* distribution. We give the marginal distribution, the conditional distribution and the distribution of the quadratic forms. We also investigate the other properties, such as, asymmetry, kurtosis and the characteristic function.

AMS 2000 subject classifications: 60E05; 62H10

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1. Introduction

McDonald and Newey [11] introduced the univariate generalized *t* (GT) distributions family. This family includes the normal distribution, the power exponential distribution and the univariate *t* distribution as the special or limiting cases. Butler et al. [8] pointed out that the GT distribution can be obtained as a scale mixture of a power exponential and an inverse generalized gamma distributions.

E-mail address: oarslan@mail.cu.edu.tr.

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Since the univariate family of generalized t distributions was introduced by McDonald and Newey [11] it has been widely used as a robust alternative to the normal distribution for modeling the errors in regression. Some other references on the family of the univariate generalized t distributions are [4,5,9,10].

Arellano-Valle and Bolfarine [1] proposed a generalized multivariate t distribution family and studied the properties of the distributions included by this family. They obtained this family as a scale mixture of normal and inverse gamma distribution. This distribution family includes the multivariate t distribution as a special case.

In this paper, we define a new family of multivariate generalized distributions as a scale mixture of a multivariate power exponential distribution introduced by Gómez et al. [3] and an inverse generalized gamma distribution with a scale parameter. We study the properties of the distributions included by this family. We show that this family of distributions belongs to the family of elliptically contoured distributions, and the multivariate normal distribution, the multivariate t distribution and the generalized multivariate t distribution introduced by Arellano-Valle and Bolfarine [1] are the special or limiting cases of the newly proposed family of multivariate generalized distributions. We also show that the univariate GT distribution introduced by McDonald and Newey [11] is also a special case of this family. Further, we can define a multivariate generalization of the GT distribution as a subclass of this family. Since this family includes the multivariate t distribution and the multivariate generalization of the GT distribution we will call this family as the family of multivariate generalized t distributions, or MGT for short.

The paper is organized as follows. In next section, we will define the family of multivariate generalized t distributions. In Section 3, we will investigate some of the properties of the multivariate generalized t distributions.

2. Family of multivariate generalized t distributions

Let Z be a p-dimensional random variable distributed as a p-dimensional multivariate power exponential distribution defined by Gómez et al. [3] $(Z \sim PE_p(0, I_p, \beta))$ with the density function

$$f_z(z; 0, I_p, \beta) = k \exp\left\{-\tfrac{1}{2}(z^T z)^{\beta}\right\}, \tag{1}$$

where, $z \in \mathbf{R}^p$, $k = p\Gamma(p/2)/\pi^{p/2}\Gamma(1 + p/2\beta)2^{1+\frac{p}{2\beta}}$, $\Gamma(\cdot)$ is the gamma function, and $\beta > 0$ is a parameter and $p \geqslant 1$. Further, let V be a scale random variable distributed as an inverse generalized gamma distribution $(V \sim IGG(\lambda, \beta, q))$ with the density function

$$f_v(v; \beta, q, \lambda) = \frac{\beta}{\frac{\lambda}{2^{1/\beta}}\Gamma(q)} \left(\frac{\frac{\lambda}{2^{1/\beta}}}{v}\right)^{\beta q+1} \exp\left\{-\tfrac{1}{2}\left(\tfrac{\lambda}{v}\right)^{\beta}\right\}, \tag{2}$$

where $v > 0$, $q > 0$ and $\beta > 0$ are called the shape parameters and $\lambda > 0$ is called the scale parameter [4, vol.2, pp. 401].

Proposition 1. *Let $Z \sim PE_p(0, I_p, \beta)$ and $V \sim IGG(\lambda; \beta, q)$ be two independent random variables, and let X be a new random variable defined as*

$$X = \mu + q^{1/2\beta}\Sigma^{1/2}V^{1/2}Z, \tag{3}$$

where $\mu \in \mathbf{R}^p$, Σ is a positive-definite symmetric matrix and $\Sigma^{1/2}$ is the positive-definite square root of Σ. Then, X is an elliptically contoured random variable with the density function

$$f(x; \mu, \Sigma, \lambda, \beta, q) = \frac{\beta\Gamma\left(\frac{p}{2}\right)\Gamma\left(q + \frac{p}{2\beta}\right)(q)^q}{(\pi\lambda)^{\frac{p}{2}}\Gamma(q)\Gamma\left(\frac{p}{2\beta}\right)}|\Sigma|^{-1/2}\frac{1}{\left\{q + \left(\frac{s}{\lambda}\right)^\beta\right\}^{q+\frac{p}{2\beta}}}, \tag{4}$$

where $s = (x-\mu)^T\Sigma^{-1}(x-\mu)$.

Proof. Since Z and V are independent, $Z = \Sigma^{-1/2}(X-\mu)V^{-1/2}q^{-1/2\beta}$, $Z^TZ = (X-\mu)^T\Sigma^{-1}(X-\mu)V^{-1}q^{-1/\beta}$, and the Jacobian of the transformation is $|\Sigma|^{-1/2}V^{-p/2}q^{-p/2\beta}$, the joint density function of X and V can be obtained as

$$f(x, v; \mu, \Sigma, \lambda, \beta, q)$$
$$= \frac{k|\Sigma|^{-1/2}\beta}{\left(\frac{\lambda}{2^{1/\beta}}\right)^{\frac{p}{2}+1}\Gamma(q)q^{p/\beta}}\left\{\left(\frac{\frac{\lambda}{2^{1/\beta}}}{v}\right)^\beta\right\}^{q+\frac{1}{\beta}+\frac{p}{2\beta}}\exp\left\{-\frac{1}{2q}\left(\frac{\lambda}{v}\right)^\beta\left\{q + \left(\frac{s}{\lambda}\right)^\beta\right\}\right\}.$$

Then, the density function of the random variable X is

$$f(x; \mu, \Sigma, \lambda, \beta, q)$$
$$= \frac{k|\Sigma|^{-1/2}\beta}{\left(\frac{\lambda}{2^{1/\beta}}\right)^{\frac{p}{2}+1}\Gamma(q)q^{p/2\beta}}\int_0^\infty\left\{\left(\frac{\frac{\lambda}{2^{1/\beta}}}{v}\right)^\beta\right\}^{q+\frac{1}{\beta}+\frac{p}{2\beta}}\exp\left\{-\frac{1}{q}\left(\frac{\frac{\lambda}{2^{1/\beta}}}{v}\right)^\beta\left\{q + \left(\frac{s}{\lambda}\right)^\beta\right\}\right\}dv.$$

Setting $\frac{1}{t} = \left(\frac{\frac{\lambda}{2^{1/\beta}}}{v}\right)\frac{\{q+(\frac{s}{\lambda})^\beta\}^{1/\beta}}{(q)^{1/\beta}}$ gives $dv = \frac{\frac{\lambda}{2^{1/\beta}}\{q+(\frac{s}{\lambda})^\beta\}^{1/\beta}}{(q)^{1/\beta}}dt$, and substituting these in the integral yields

$$f(x; \mu, \Sigma, \lambda, \beta, q)$$
$$= \frac{k|\Sigma|^{-1/2}}{\left(\frac{\lambda}{2^{1/\beta}}\right)^{\frac{p}{2}}\Gamma(q)q^{p/2\beta}}\frac{(q)^{q+p/2\beta}}{\left\{q + \left(\frac{s}{\lambda}\right)^\beta\right\}^{q+\frac{p}{2\beta}}}\beta\int_0^\infty\left(\frac{1}{t}\right)^{\beta(q+\frac{p}{2\beta})+1}\exp\left\{-(1/t)^\beta\right\}dt.$$

Since

$$\beta\int_0^\infty\left(\frac{1}{t}\right)^{\beta(q+\frac{p}{2\beta})+1}\exp\left\{-(1/t)^\beta\right\}dt = \Gamma\left(q + \frac{p}{2\beta}\right)$$

we get

$$f(x;\mu,\Sigma,\lambda,\beta,q)=\frac{k|\Sigma|^{-1/2}\Gamma\left(q+\frac{p}{2\beta}\right)}{\lambda^{\frac{p}{2}}\Gamma(q)q^{p/\beta}}\frac{(q)^{q+p/2\beta}}{\left\{q+\left(\frac{s}{\lambda}\right)^{\beta}\right\}^{q+\frac{p}{2\beta}}}$$

and substituting $k=p\Gamma(p/2)/\pi^{p/2}\Gamma(1+p/2\beta)2^{1+\frac{p}{2\beta}}$ in the above identity we obtain the density function given in (4).

The function $f(x;\mu,\Sigma,\lambda,\beta,q)$ is actually the density function of an elliptically contoured random variable with

$$g(t)=\{q+t^{\beta}\}^{-(q+\frac{p}{2\beta})}$$

since the function $g(t)$ satisfies the condition

$$\int_0^{\infty}t^{p/2-1}g(t)\,dt=(q)^{-q}\beta^{-1}\frac{\Gamma(q)\Gamma\left(\frac{p}{2\beta}\right)}{\Gamma\left(q+\frac{p}{2\beta}\right)}<\infty$$

(see [2, p. 59]). Thus, X has an elliptically contoured distribution $(X\sim EC_p(\mu,\Sigma,g))$. □

When $\beta=1$, $\lambda=1$, $\Sigma=2\Sigma_1$ and $v=2q$ we obtain the standard multivariate t distribution with the location and scatter parameters μ and Σ_1, and the degrees of freedom v. The case $\beta=1$, $\lambda=1$, and $q\to\infty$ gives the normal distribution. The multivariate power exponential distribution can be obtained as $q\to\infty$. Setting $\beta=1$, $\lambda q=\alpha$, and $2q=v$ yields the generalized version of the t distributions family defined by Arellano-Valle and Bolfarine [1] with the parameters μ, Σ, α and v.

When $\lambda=1$ and $\beta=\tau/2$, $\tau>0$, (4) gives a multivariate generalization of the univariate GT distributions introduced by McDonald and Newey [11]. The density function of the multivariate generalization of the GT distribution can be obtained as

$$f(x;\mu,\Sigma,\tau,q)=C|\Sigma|^{-1/2}\frac{1}{\{q+|d|^{\tau}\}^{q+\frac{p}{\tau}}},$$

where $d=\sqrt{s}=\sqrt{(x-\mu)^T\Sigma^{-1}(x-\mu)}$ and $C=\frac{\tau\Gamma\left(\frac{p}{2}\right)(q)^q}{2\pi^{\frac{p}{2}}B\left(q,\frac{p}{\tau}\right)}$. Note that, here p is the dimension of the random variable, τ and q are the shape parameters, and μ and Σ are the location and the scatter parameters.

Finally, when $\lambda=1$, and β tends to infinity, (4) tends to the density function of the uniform distribution on the ellipse $s<1$.

Now, we can give the definition of the multivariate generalized t distributions.

Definition. A random variable $X=(X_1,X_2,\dots,X_p)^T\in\mathbf{R}^p$, with $p\geqslant 1$, has a p dimensional multivariate generalized t distribution with μ,Σ,λ,β and q parameters, where $\mu\in\mathbf{R}^p$, $\Sigma_{p\times p}$ is a positive-definite symmetric matrix, and $\lambda,\beta,q>0$, if its density function is

$$f(x;\mu,\Sigma,\lambda,\beta,q)=C\lambda^{-p/2}|\Sigma|^{-1/2}\frac{1}{\left\{q+\left(\frac{s}{\lambda}\right)^{\beta}\right\}^{q+\frac{p}{2\beta}}},\tag{5}$$

where

$$C=\frac{\beta\Gamma\left(\frac{p}{2}\right)\Gamma\left(q+\frac{p}{2\beta}\right)(q)^{q}}{\pi^{\frac{p}{2}}\Gamma(q)\Gamma\left(\frac{p}{2\beta}\right)}=\frac{\beta\Gamma\left(\frac{p}{2}\right)(q)^{q}}{\pi^{\frac{p}{2}}B\left(q,\frac{p}{2\beta}\right)}\tag{6}$$

is the normalizing constant, $s=(x-\mu)^{T}\Sigma^{-1}(x-\mu)$, and $B\left(q,\frac{p}{2\beta}\right)$ denotes the beta function. Here μ and Σ are the location and scatter parameters, λ is the scale parameter, and β and q are the shape parameters.

Note that we will use the notation $MGT_p(\mu,\Sigma,\lambda,\beta,q)$ for the family of multivariate generalized t distributions.

When $p=2$ we can give graphics of the density function. In Fig. 1, we present the graphs for some values of β. We set $\Sigma=I_2,\ \mu=(0,0)^{T},\ \lambda=1$ and $q=2$. Figs. 1 (a–d) shows the cases $\beta=0.25,1,2$ and 10, respectively. We can observe that for the small values of β the shape is very sharp. When β increases the sharpness of the density diminishes.

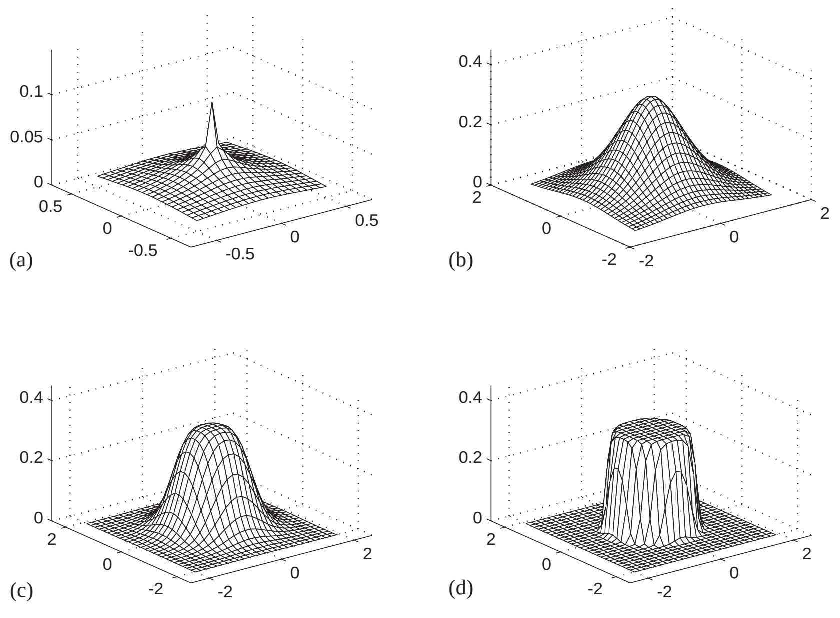

Fig. 1. Plots of the density functions of the $MGT_2(0,I_2,1,\beta,2)$ distributions for some values of β.

3. Some properties of the MGT distributions

Since the multivariate generalized t distributions family $MGT_p(\mu, \Sigma, \lambda, \beta, q)$ belongs to the family of elliptically contoured distributions $(EC_p(\mu, \Sigma, g))$, they can be restated with the stochastic representation (e.g. see [2]).

Proposition 2. (i) *Let A be a symmetric nonsingular matrix with $A^T A = \lambda\Sigma$. If the random variable X is distributed as $MGT_p(\mu, \Sigma, \lambda, \beta, q)$, then X has the same distribution as*

$$\mu + RA^T U^{(p)}, \tag{7}$$

where $U^{(p)}$ is a random variable uniformly distributed on the unit sphere in $\mathbf{R}^p$, R is an absolutely continues random variable, independent from $U^{(p)}$, and R has the density function

$$h_R(r) = \frac{2\beta}{B\left(q, \frac{p}{2\beta}\right)(q)^{p/2\beta}} r^{p-1}\left\{1 + \frac{r^{2\beta}}{q}\right\}^{-q-\frac{p}{2\beta}}, \quad r > 0. \tag{8}$$

(ii) *$R \sim GB2(r; 2\beta, q^{1/2\beta}, p/2\beta, q)$; here, $GB2(\cdot)$ denotes the generalized beta distribution (e.g. see [4,7,12]). The kth moment of R is*

$$E[R^k] = (q)^{k/2\beta}\frac{B\left(q - \frac{k}{2\beta}, \frac{p}{2\beta} + \frac{k}{2\beta}\right)}{B\left(q, \frac{p}{2\beta}\right)} \tag{9}$$

for each positive integer $k \leqslant 2\beta q$.

(iii) *The distribution of the random variable $F = R^{2\beta}$ is the $F(p/\beta, q)$ distribution with p/β and $2q$ degrees of freedom.*

Proof. Part (i) follows from Corollary 1 on p.65 and Theorem 2.5.5 on p.59 of [2]. Proofs of parts (ii) and (iii) can be easily obtained.

Note that $||X|| = (X^T X)^{1/2}$, $||X||^2$ and $X/||X||$ have the same distributions as R, R^2 and $U^{(p)}$, respectively, and $E(U^{(p)}) = 0$, $Var(U^{(p)}) = \frac{1}{p} I_p$, where I_p is the $p \times p$ identity matrix (e.g. see [2,13, p. 57; p. 37]).

Proposition 3. *If the random variable X has a $MGT_p(\mu, \Sigma, \lambda, \beta, q)$ distribution, then,*

(i) *its characteristic function is*

$$\varphi_X(t) = \frac{2\beta(q)^{-p/2\beta}}{B\left(q, \frac{p}{2\beta}\right)} e^{it'\mu} \int_0^\infty \Psi_p\left(r\sqrt{t'\lambda\Sigma t}\right) r^{p-1}\left\{1 + \frac{r^{2\beta}}{q}\right\}^{-q-\frac{p}{2\beta}} dr, \tag{10}$$

where

$$\Psi_p\left(r\sqrt{t'\lambda\Sigma t}\right) = \begin{cases} \frac{1}{B\left(\frac{p-1}{2}, 1/2\right)} \int_0^\pi e^{\left(ir\sqrt{t'\lambda\Sigma t}\cos\theta\right)} \sin^{p-2}\theta\, d\theta, & \textit{for } p>1, \\ \cos\left(r\sqrt{t'\lambda\Sigma t}\right) & \textit{for } p=1. \end{cases}$$

(ii) $E(X) = \mu$,

(iii) $Var(X) = \frac{(q)^{1/\beta}\Gamma\left(\frac{p}{2\beta}+\frac{1}{\beta}\right)\Gamma\left(q-\frac{1}{\beta}\right)}{p\Gamma\left(\frac{p}{2\beta}\right)\Gamma(q)}\lambda\Sigma, \quad$ *for* $q\beta > 1$,

(iv) $\gamma_1(X) = 0$,

(v) $\gamma_2(X) = \frac{p^2 B\left(q, \frac{p}{2\beta}\right) B\left(q-\frac{2}{\beta}, \frac{p}{2\beta}+\frac{2}{\beta}\right)}{\left(B\left(q-\frac{1}{\beta}, \frac{p}{2\beta}+\frac{1}{\beta}\right)\right)^2}, \quad$ *for* $q\beta > 2$,

where $\gamma_1(X)$ *and* $\gamma_2(X)$ *are the multidimensional asymmetry and kurtosis coefficient* (*see* [6, p. 31]).

Proof. (i) Since the characteristic function of $U^{(p)}$ is

$$\Psi_p(t) = \frac{1}{B\left(\frac{p-1}{2}, \frac{1}{2}\right)} \int_0^\pi \exp\left(i\sqrt{t't}\cos\theta\right)\sin^{p-2}\theta\, d\theta$$

[2, p. 54, Theorem 2.5.1], $U^{(p)}$ and R are independent, and the characteristic function of $RU^{(p)}$ is

$$E(e^{(it'RU^{(p)})}) = \int_0^\infty \Psi_p(rt)h_R(r)\, dr,$$

(see [2, p. 56]), then the characteristic function of X given in (10) can be easily obtained.

(ii) $E(X) = E(\mu + RA^T U^{(p)}) = \mu + E(R)A^T E(U^{(p)})$, and since $E(U^{(p)}) = 0$, we get $E(X) = \mu$.

(iii) Similarly, $Var(X) = E(R^2)A^T Var(U^{(p)})A = \frac{E(R^2)}{p}\lambda\Sigma$, and since $E(R^2) = (q)^{1/\beta}\frac{B\left(q-\frac{1}{\beta}, \frac{p}{2\beta}+\frac{1}{\beta}\right)}{B\left(q, \frac{p}{2\beta}\right)}$, we get $Var(X) = \frac{(q)^{1/\beta}\Gamma\left(\frac{p}{2\beta}+\frac{1}{\beta}\right)\Gamma\left(q-\frac{1}{\beta}\right)}{p\Gamma\left(\frac{p}{2\beta}\right)\Gamma(q)}\lambda\Sigma$.

(iv) Let $X = \mu + q^{1/2\beta}\Sigma^{1/2}V_1^{1/2}Z_1$ and $Y = \mu + q^{1/2\beta}\Sigma^{1/2}V_2^{1/2}Z_2$ be two independent random variables distributed as $MGT_p(\mu, \Sigma, \lambda, \beta, q)$, where V_1, Z_1, V_2 and Z_2 are all independent to each other. Let $a = \frac{(q)^{1/\beta}\Gamma\left(\frac{p}{2\beta}+\frac{1}{\beta}\right)\Gamma\left(q-\frac{1}{\beta}\right)}{p\Gamma\left(\frac{p}{2\beta}\right)\Gamma(q)}$. Then, we get $E(\{(X-\mu)^T Var(X)^{-1}(Y-\mu)\}^3) = (a/\lambda)^{-3} q^{3/\beta} E((V_1V_2)^3) E((Z_1^T Z_2)^3)$, and since $E((Z_1^T Z_2)^3) = 0$ (see [3, Proposition 3.2]), we obtain $\gamma_1(X) = E(\{(X-\mu)^T Var(X)^{-1}(Y-\mu)\}^3) = 0$.

(v) $E(\{(X-\mu)^T Var(X)^{-1}(X-\mu)\}^2) = a^{-2} E(\{(X-\mu)^T (\lambda\Sigma)^{-1}(X-\mu)\}^2) = a^{-2}E((R^2)^2)$, from (9) we have $E((R^2)^2) = (q)^{2/\beta} \frac{B\left(q-\frac{2}{\beta}, \frac{p}{2\beta}+\frac{2}{\beta}\right)}{B\left(q, \frac{p}{2\beta}\right)}$, and hence we get

$$\gamma_2(X) = E(\{(X-\mu)^T Var(X)^{-1}(X-\mu)\}^2) = \frac{p^2 B\left(q, \frac{p}{2\beta}\right) B\left(q-\frac{2}{\beta}, \frac{p}{2\beta}+\frac{2}{\beta}\right)}{\left(B\left(q-\frac{1}{\beta}, \frac{p}{2\beta}+\frac{1}{\beta}\right)\right)^2}, \text{ for } q\beta > 2. \quad \square$$

Finally, the marginal and the conditional distributions of a random variable $X \sim MGT_p(\mu, \Sigma, \lambda, \beta, q)$ are given in the following proposition. Since these results can easily be obtained using the properties of the elliptically contoured distributions (e.g. [2]), the proof is omitted.

Proposition 4. *Let* $X \sim MGT_p(\mu, \Sigma, \lambda, \beta, q)$ *and* $Y = CX + b$.

(i) *If C is a $p \times p$ nonsingular matrix, and $b \in \mathbf{R}^p$, then the distribution of Y is $MGT_p(C\mu + b, C\Sigma C^T, \lambda, \beta, q)$.*

(ii) *If C is a $k \times p$ matrix with $k < p$ and $rank(C) = k$, and $b \in \mathbf{R}^k$, then Y has an $EC_k(\mu_Y, \Sigma_Y, g_1)$ with $\mu_Y = C\mu + b$, $\Sigma_Y = C\Sigma C^T$, and*

$$g_1(t) = t^{\frac{p-k}{2}} \int_0^1 w^{\frac{k-p}{2}-1} (1-w)^{\frac{p-k}{2}-1} \left\{1 + \frac{t^\beta}{q w^\beta}\right\}^{-q-\frac{p}{2\beta}} dw, \quad t > 0. \tag{11}$$

(iii) *Partition X, μ and Σ as*

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},$$

where X_1 and μ_1 are $k \times 1$ vectors and Σ_{11} is a $k \times k$ matrix. Then, X_1 has an $EC_k(\mu_1, \Sigma_{11}, g_1)$. Further, $E(X_1) = \mu_1$ and $Var(X_1) = \frac{(q)^{1/\beta}\Gamma\left(\frac{p}{2\beta}+\frac{1}{\beta}\right)\Gamma\left(q-\frac{1}{\beta}\right)}{p\Gamma\left(\frac{p}{2\beta}\right)\Gamma(q)} \lambda\Sigma_{11}$.

(iv) *The conditional distribution of X_2 given $X_1 = x_1$ is an $EC_k(\mu_{2.1}, \Sigma_{22.1}, g_{2.1})$ with*

$$\mu_{2.1} = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1),$$

$$\Sigma_{22.1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12},$$

$$g_{2.1}(t) = \{q + [t + (x_1-\mu_1)^T \Sigma_{11}^{-1}(x_1-\mu_1)]^\beta\}^{-\left(q+\frac{p}{2\beta}\right)}. \tag{12}$$

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References

[1] R.B. Arellano-Valle, H. Bolfarine, On some characterizations of the *t* distribution, Statist. Probab. Lett. 25 (1995) 79–85.
[2] K. Fang, Y. Zhang, Generalized Multivariate Analysis, Springer, Beijing, 1990.
[3] E. Gomez, M.A. Gomez-Villegas, J.M. Marin, A multivariate generalization of the power exponential family of distributions, Comm. Statist. Theory Methods 27 (3) (1998) 589–600.
[4] N.L. Johnson, S. Kotz, N. Balakrishnan, Continuous Univariate Distributions, Vol. 2, Wiley, New York, 1995.
[5] J.N. Lye, V.L. Martin, Robust estimation, nonnormalities and generalized exponential distributions, J. Amer. Statist. Assoc. 88 (421) (1993) 261–267.
[6] K.V. Mardia, J.T. Kent, J.M. Bibby, Multivariate Analysis, Academic Press, London, 1979.
[7] J.B. McDonald, R.J. Butler, Some generalized mixture distributions with an application to unemployment duration, Rev. Econom. Statist. 69 (1987) 232–240.
[8] R.J. Butler, J.B. McDonald, R.D. Nelson, S.B. White, Robust and partially adaptive estimation of regression models, Rev. Econom. Statist. 72 (1990) 321–327.
[9] J.B. McDonald, R.D. Nelson, Alternative beta estimation for the market model using partially adaptive techniques, Comm. Statist. Theory Methods 18 (11) (1989) 4039–4058.
[10] J.B. McDonald, R.D. Nelson, Beta estimation in the market model: skewness and leptokurtosis, Comm. Statist. Theory Methods 22 (10) (1993) 2843–2862.
[11] J.B. McDonald, W.K. Newey, Partially adaptive estimation of regression models via the generalized *t* distribution, Econometric Theory 4 (1988) 428–457.
[12] J.B. McDonald, D.O. Richards, Model selection: some generalized distributions, Comm. Statist. Theory Methods 16 (4) (1987) 1049–1074.
[13] R.J. Muirhead, Aspects of Multivariate Statistical Theory, Wiley, New York, 1982.