Nonlinear stability of a shallow unsymmetrical heated orthotropic sandwich shell of double curvature with orthotropic core

A. Chakrabarti a, B. Mukhopadhyay b, R.K. Bera c,*

a 1/14, Ramlal Agarwalla Lane, Kolkata 700050, India
b Bengal Engineering and Science University, Howrah 711103, India
c Heritage Institute of Technology, Chowhaga Road, Anandapur, Kolkata 700107, India

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Abstract

Equations which govern the behavior of an elastic unsymmetrical, orthotropic sandwich shell of double curvature with orthotropic core having different elastic characteristics under uniform heating are derived. The face sheet may be of unequal thickness of different materials. However, a restriction that the radii of curvature of the shell elements be large compared with the overall thickness of the sandwich shell is imposed. The variational procedure has been used to obtain the five equations which govern the behavior of the heated orthotropic sandwich shell for the stability. In case of symmetry the equations resemble with those of Grigolyuk. Finally, the numerical results of a square or a rectangular simply supported curved plate section of a cylindrical shell under thermal loading have been computed and compared with other known results. The graphs have been drawn to show the effects of different sandwich material for immovable and movable edge conditions.

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1. Introduction

The field of sandwich construction, while not new, has become quite important in recent years as a result of improvement in manufacturing techniques. It has long been recognized as an efficient method of obtaining a lightweight compression member, but the prohibited cost of construction has limited its use. However, as new manufacturing methods are now being developed which make the use of sandwiches economically feasible, the collection of more research data is becoming increasingly important.

The first significant contribution to an understanding of the behavior of sandwich shells was presented by Reissner (1950), who showed the effects of shear deformations and core compression that differentiate the
sandwich theory from ordinary shell theory based on the Kirchhoff-Love assumption. Since then numerous papers have been published discussing analytical and experimental results of studies dealing with statically loaded cylindrical shells (Eringen, 1951; Stein and Mayers, 1952; Raville, 1954, 1955a,b; Wang and Desanto, 1955: Haft, 1955; Radkowski, 1957).

More recent investigations have extended the theory of sandwich shells to include doubly curved shells (Korolev, 1955; Grigolyuk, 1957, 1958a,b), fully plastic cores (Grigolyuk, 1958a,b), creep (Hoff et al., 1959), and free vibrations (Yu, 1960; Chu, 1961). Kamiya (1976), using Berger’s (1955) approximations in large deflection analysis, has offered a new set of governing equations to study the nonlinear static behavior of sandwich plates. But his attempt has been restricted to plate geometry due to introduction of a correction factor. Although Berger’s method has been applied to solve ordinary shallow shell problems, this method fails completely for movable edge conditions. Alwan (1964) and Nowanski and Ohnabe (1976) have also derived equations of sandwich plates and shells with orthotropic core for the analysis of large deflection. Recent investigations have also enriched the theory of sandwich shells by including doubly curved shells (Fulton, 1961; Bera, 1996).

The large amplitude vibration of thin elastic plates and stability analysis of heated sandwich plates and shells are also very important in modern design. Unfortunately in this field also not many works have been done so far. The only recent works of Kamiya (1978a,b), Ray et al. (1993), Dutta and Banerjee (1991), Banerjee (1981) and Bera et al. (1996, 1998, and 1999), in the large deflection analysis of heated sandwich plates and shells, can be located. The works of Librescu et al. (1997, 1998 and 2000), on nonlinear modeling of sandwich plates and shells are also noteworthy. But in Kamiya’s works, following Berger’s method, no result for movable edge condition is obtained.

The purpose of the present paper is to develop a simple and yet sufficiently accurate method for stability of heated orthotropic sandwich shells with orthotropic core and faces. In deriving the equations, the idea of Bera (1996), used in the case of isotropic symmetrical sandwich shell has been utilized with purpose and profit. It is interesting to note, however, that the equations obtained by Fulton (1961), for the sandwich shells for symmetrical faces can be easily deduced from the present analysis.

It is assumed that the orthotropic core undergoes only transverse shear deformations and that a line through the undeformed orthotropic core remains straight under deformations but not necessarily perpendicular to the middle surface of the shell. It is further assumed that the total thickness of the shell element is small compared to its radii of curvature. The face sheets, however, are assumed to satisfy the Kirchhoff-Love assumption and their thicknesses, while not equal, are small compared with the overall thickness of the orthotropic sandwich section. It is likewise assumed that the core compression in a direction normal to the middle surface of the orthotropic sandwich shell is negligible while the properties of each layer are different.

Furthermore, the results for movable edge conditions can be easily derived from the single equation of immovable edge conditions. Numerical results of the critical loads for stabilities of heated sandwich shells have been calculated for movable as well as immovable edge conditions and compared with other available results.

2. Formulation of governing equations

Let us first consider a rectangular Cartesian co-ordinate system \(x, y, z\); \(x, y\) in the middle plane of the core, \(z\) in the thickness direction (positive downward). If the expression for the strain in the \(i\)th face sheet in the \(x\)- and \(y\)-directions are denoted as \(\varepsilon_{1i}, \varepsilon_{2i}\), respectively, the transverse shear strain as \(\gamma_{i}\), curvature in the \(x\) and \(y\) directions as \(\kappa_{1}\) and \(\kappa_{2}\) and the twist as \(\kappa_{12}\), then the relations (1) hold true for each of the separate face sheets (Fulton, 1961):

\[
\begin{align*}
\varepsilon_{1i} &= u_{ix} + \frac{1}{2} \left( w_{x}^{2} - \frac{w}{R_{1}} \right) \\
\varepsilon_{2i} &= v_{iy} + \frac{1}{2} \left( w_{y}^{2} - \frac{w}{R_{2}} \right) \\
\gamma_{i} &= u_{ix} + v_{iy} + w_{x} \cdot w_{y}, \ i = 1, 2.
\end{align*}
\]
where \( u_i, v_i \) and \( w \) are middle surface displacements of the face sheets in the \( x, y \) and \( z \) directions, respectively, and \( R_1 \) and \( R_2 \) are the radii of curvature of the plate elements in the \( x \) and \( y \) directions, respectively. The subscripts \( x \) and \( y \) denote differentiation with respect to \( x \) and \( y \), respectively.

Let the stress–strain relations for each face sheet for the orthotropic material be given by

\[
\begin{align*}
N_x^{(i)} &= B_1 \left( \varepsilon_{11} + v_2 \varepsilon_{22} \right) - \frac{E_i t_i}{1 - v_s} T_i, \\
N_y^{(i)} &= B_2 \left( \varepsilon_{22} + v_1 \varepsilon_{11} \right) - \frac{E_i t_i}{1 - v_s} T_i, \\
N_{xy}^{(i)} &= G_{ii} t_i \gamma_i = \frac{1}{2} (1 - v_s) B_i \gamma_i, B_i = E_i t_i/(1 - v_s^2), v_s = (v_1 v_2)^{\frac{1}{2}},
\end{align*}
\]

where \( E_i, v_i, \bar{\varepsilon}_{ii}, T_i \) and \( t_i \) refer to Young’s modulus, Poisson’s ratio, thermal constant, temperature, the thickness of the face sheet, respectively, \( i = 1 \) and \( 2 \) represent upper and lower faces, respectively, \( G_{ii} \) is the shear modulus.

When dual numerical subscripts are used, the first subscripts refer to the direction of the axis and the second subscript refers to the face sheet under consideration. Thus \( \varepsilon_{21} \) signifies the strain in the \( y \)-direction in the upper face.

Let us now introduce

\[
\begin{align*}
\bar{u} &= \frac{B_1 u_1 + B_2 u_2}{B_1 + B_2}, & \bar{v} &= \frac{B_1 v_1 + B_2 v_2}{B_1 + B_2}, \\
\alpha &= \frac{u_1 - u_2}{h}, & \beta &= \frac{v_1 - v_2}{h},
\end{align*}
\]

where the subscripts \( 1 \) and \( 2 \) refer to the upper and lower face sheets, respectively, and \( h \) is the thickness of the core.

Then the averaged values of both the face strain components can be written as

\[
\begin{align*}
\varepsilon_{11}^m &= \frac{1}{2} (\varepsilon_{11} + \varepsilon_{12}) = \bar{u}_x - \frac{w}{R_1} + \frac{1}{2} w^2 + \frac{B_1 - B_2}{B_1 + B_2} \frac{h}{2} \bar{\alpha}_x, \\
\varepsilon_{22}^m &= \frac{1}{2} (\varepsilon_{21} + \varepsilon_{22}) = \bar{v}_y - \frac{w}{R_2} + \frac{1}{2} w^2 + \frac{B_1 - B_2}{B_1 + B_2} \frac{h}{2} \bar{\beta}_y, \\
\gamma_m &= \frac{1}{2} (\gamma_1 + \gamma_2) = \bar{u}_y + \bar{v}_x + w_x w_y + \frac{B_1 - B_2}{B_1 + B_2} \frac{h}{2} (\bar{\alpha}_y + \bar{\beta}_x).
\end{align*}
\]

With the help of Eqs. (3) and (4), we can write

\[
\begin{align*}
\varepsilon_{11} &= \varepsilon_{11}^m + \frac{h}{2} \bar{\alpha}_x, & \varepsilon_{12} &= \varepsilon_{12}^m - \frac{h}{2} \bar{\alpha}_x, \\
\varepsilon_{21} &= \varepsilon_{21}^m + \frac{h}{2} \bar{\beta}_y, & \varepsilon_{22} &= \varepsilon_{22}^m - \frac{h}{2} \bar{\beta}_y, \\
\gamma_1 &= \gamma_m + \frac{h}{2} (\bar{\alpha}_y + \bar{\beta}_x), & \gamma_2 &= \gamma_m - \frac{h}{2} (\bar{\alpha}_y + \bar{\beta}_x).
\end{align*}
\]

By virtue of Hook’s Law, the strain energy of the entire upper and lower faces of orthotropic sandwich shell per unit area, for unsymmetrical elastic material, can be written as
Let us now find the shearing strains \( \gamma_{13} \) and \( \gamma_{23} \) of the core in the \( xz \)- and \( yz \)-planes, respectively. From Eq. (7a), we get (7b) as follows:

\[
\gamma_{13} = u_z + w_x = -(h/c)(x - w_x), \quad \text{and} \quad \gamma_{23} = v_z + w_y = -(h/c)(\beta - w_y),
\]

\( h = c + (t_1 + t_2)/2, \)

where \( \lambda \) and \( c \), the thickness of the core are shown in Fig. 1.

The shearing stresses \( \tau_{13} \) and \( \tau_{23} \) in the orthotropic core are related to the strains by

\[
\tau_{13} = G_{13}\gamma_{13} \quad \text{and} \quad \tau_{23} = G_{23}\gamma_{23},
\]
where $G_{13}$ and $G_{23}$ are the shearing moduli of the orthotropic core. The strain energy per unit area of the orthotropic core due to shear becomes

$$\tilde{V}_0^e = (h^2/2c)[G_{13}(\alpha - w_x)^2 + G_{23}(\beta - w_y)^2].$$  

(9)

In consequence the total strain energy per unit area of the orthotropic sandwich shell with orthotropic core is

$$\tilde{V}_0^h = \tilde{V}_0^e + \tilde{V}_0^c$$  

(10)

Following the assumption made in Banerjee (1981), we write

$$\{1 - (v_1 + v_2)^2/4\} \{\epsilon^m + (h/2)[(B_1 - B_2)/(B_1 + B_2)]\}^2$$

$$+ (1 - v_s)[\gamma_m + (h/2)[(B_1 - B_2)/(B_1 + B_2)](\alpha + \beta_s)]^2$$

$$= \lambda_m[(w_x^2 + w_y^2)/2 - ((1/R_1) + (1/R_2))w]^2,$$  

(11)

where $\lambda_m$ is a quantity which depends upon the material constants of the plate, to be determined later on.

This assumption has been made on the condition that the so-called second invariant is not zero as was taken by Berger on an ad hoc basis, instead its modified form is proportional to the square of the mean square slopes of the normal displacements in $x$ and $y$-directions, respectively. This is reasonable because the contribution of the terms $(1/2)w_x^2$ and $(1/2)w_y^2$ in $\tilde{\varepsilon}_1$, and $\tilde{\varepsilon}_2$, are greater than $u_x$ and $u_y$ in bending and under any type of loading and under any boundary conditions. The proportionality constant $\lambda_m$ is a function of Poisson’s ratio of the material, because $\tilde{\varepsilon}_1$ and $\tilde{\varepsilon}_2$ are perpendicular and $\lambda_m$ can be determined from the condition of minimum potential energy which has now been skillfully modified.

Let us now write the first invariant of the averaged strain in the form

$$I_{1m}^m = \tilde{u}_x + (1/2)(v_1 + v_2)\tilde{v}_x + (1/2)(w_x^2 + ((v_1 + v_2)/2)w_y^2) - (h/2)[(B_1 - B_2)/(B_1 + B_2)]$$

$$\times (\alpha + ((v_1 + v_2)/2)\beta_x),$$  

(12)

so that Eq. (7) can be rewritten as

$$I_{1m}^m = I_1^m + (h/2)\{(B_1 - B_2)/(B_1 + B_2)\} (\alpha + ((v_1 + v_2)/2)\beta_x).$$  

(13)

Now, introducing Eqs. (6), (9) and (11) into Eq. (10), we obtain the total potential energy as

$$\tilde{V}_0^h = \frac{1}{2}(B_1 + B_2)\{(I_{1m}^m)^2 + \lambda_m\left[\frac{w_x^2 + w_y^2}{2} - \left(\frac{1}{R_1} + \frac{1}{R_2}\right)w\right]^2$$

$$+ \frac{h^2B_1B_2}{(B_1 + B_2)^2} \left[\left(\frac{\alpha + \frac{v_1}{2} + \frac{v_2}{1 + v_1}}{1 + v_1}\right)\beta_x^2 + \left(1 - \frac{v_1}{2} \left(1 + \frac{v_1}{1 + v_1}\right)\right)\beta_y^2 + \left(\frac{1}{2} - (1 - v_s)(\alpha + \beta_s)^2\right)\right]^2$$

$$- 2\alpha_11(1 + v_s)\left(\frac{(B_1 + \tilde{\varepsilon}_{21}B_2)}{(B_1 + B_2)}\right)^{1/m}T_m + h(\alpha + \beta_s)\frac{T_w}{4} - 2\alpha_11(1 - v_s)\frac{(B_1 + \tilde{\varepsilon}_{21}B_2)}{(B_1 + B_2)}$$

$$\times \sqrt{\lambda_m}\left[\frac{w_x^2 + w_y^2}{2} - \left(\frac{1}{R_1} + \frac{1}{R_2}\right)w\right]T_m + \frac{h^2}{2c}[G_{13}(\alpha - w_x)^2 + G_{23}(\beta - w_y)^2].$$

(14)

Applying Euler’s variational principle so as to minimize the total potential energy per unit area of the present elastic orthotropic system of the heated sandwich shell with orthotropic faces and core, we arrive at the following five differential Eqs. (15)–(19):
From Eqs. (15) and (16), we get

\[
\frac{\partial}{\partial x} \left[ I_{1m}^{LM} - \frac{(B_1 + z_{21}B_2)}{(B_1 + B_2)} z_{11}(1 + v_e) T_m \right] = 0, \tag{15}
\]

\[
\frac{\partial}{\partial x} \left[ I_{1m}^{LM} - \frac{(B_1 + z_{21}B_2)}{(B_1 + B_2)} \frac{(v_1 + v_2)}{2} \beta_{xx} + \frac{1 - v_e}{2} (\alpha_{xy} + \beta_{yy}) \right] - (1 + v_e) z_{11} \frac{(B_1 + z_{21}B_2)}{(B_1 + B_2)} \frac{\partial T_d}{\partial x} - \frac{G_{13}}{c} (\alpha - w_x) = 0, \tag{17}
\]

\[
\frac{\partial}{\partial y} \left[ I_{1m}^{LM} - \frac{(B_1 + z_{21}B_2)}{(B_1 + B_2)} \frac{(v_1 + v_2)}{2} \beta_{yy} + \frac{1 - v_e}{2} (\alpha_{xy} + \beta_{yy}) \right] - (1 + v_e) z_{11} \frac{(B_1 + z_{21}B_2)}{(B_1 + B_2)} \frac{\partial T_d}{\partial y} - \frac{G_{23}}{c} (\beta - w_y) = 0, \tag{18}
\]

\[
(B_1 + B_2) \left[ \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \left\{ I_{1m}^{LM} - \frac{(B_1 + z_{21}B_2)}{(B_1 + B_2)} z_{11}(1 + v_e) T_m \right\} \right]
+ \frac{1}{R_1} + \frac{1}{R_2} \left\{ \lambda_m \left( w_x^2 + w_y^2 \right) \right. \left. - \frac{(1}{R_1} + \frac{1}{R_2}) w \right) - z_{11} \frac{(B_1 + z_{21}B_2)}{(B_1 + B_2)} \sqrt{\lambda_m(1 - v_1v_2) T_m} \right\}
+ \frac{1}{R_1} + \frac{1}{R_2} \left( w_x^2 + w_y^2 \right) \right.
+ \frac{1}{R_1} + \frac{1}{R_2} \left( \frac{w_x^2 + w_y^2}{2} \right) \frac{\partial^2 w}{\partial x^2} + w_x w_y \frac{\partial w}{\partial x} + w_y \frac{\partial w}{\partial y} \right) + w_x w_y \frac{\partial w}{\partial y}
+ \frac{1}{R_1} + \frac{1}{R_2} \left( \frac{w_x^2 + w_y^2}{2} \right) \frac{\partial^2 w}{\partial y^2} + w_x \frac{\partial w}{\partial y} + w_y \frac{\partial w}{\partial x}
+ \frac{1}{R_1} + \frac{1}{R_2} \left( \frac{w_x^2 + w_y^2}{2} \right) \frac{\partial^2 w}{\partial x \partial y} + w_x \frac{\partial w}{\partial x} + w_y \frac{\partial w}{\partial y}
+ \frac{1}{R_1} + \frac{1}{R_2} \left( \frac{w_x^2 + w_y^2}{2} \right) \frac{\partial^2 w}{\partial y \partial x} + w_x \frac{\partial w}{\partial y} + w_y \frac{\partial w}{\partial x}
+ \frac{1}{R_1} + \frac{1}{R_2} \left( \frac{w_x^2 + w_y^2}{2} \right) \frac{\partial^2 w}{\partial x \partial y} + w_x \frac{\partial w}{\partial x} + w_y \frac{\partial w}{\partial y} \right) \frac{\partial^2 T_d}{\partial y^2} = 0. \tag{21}
\]

Eliminating \( \left( \alpha_x + \frac{G_{13}}{G_{13}} \beta_y \right) \) from Eqs. (19) and (21) and using Eq. (20), we obtain

\[
\left( \frac{B_1 B_2}{B_1 + B_2} \frac{\nabla^2 - \frac{G_{13}}{c}}{n} \right) \left( \alpha_x + \frac{G_{23}}{G_{13}} \beta_y \right) + \left( w_{xx} + \frac{G_{23}}{G_{13}} w_{yy} \right) - \frac{z_{11}}{4h} \frac{(B_1 + z_{21}B_2)}{(B_1 + B_2)^2} \nabla^2 T_d = 0. \tag{22}
\]

It is to be noted that the movable edge condition...
\[ I_1^m - \bar{\alpha}_{11} \left( \frac{B_1 + \bar{\alpha}_{21} B_2}{B_1 + B_2} \right) (1 - v_2) T_m = 0 \]  

(23)

must be satisfied.

If \( v_1 = v_2 = v \) and \( \bar{\alpha}_{11} = \bar{\alpha}_{22} = \bar{\alpha}_f \), then Eq. (22) is exactly the same as that obtained earlier by Bera (1998). Along with it, if \( B_1 = B_2 \) and \( T_m = T_d = 0 \), then the equation is also same as that obtained by Fulton (1961) and Grigolyuk (1958a,b) following different approaches.

Finally, minimizing the potential energy, the value of \( \lambda_m \) can be obtained as

\[ \lambda_m = 2v_1v_2, \text{ for clamped edge} \]

\[ \lambda_m = v_1v_2, \text{ for simply supported edge}. \]  

(24)

**Stability of shallow shell under uniform heating:**

Let us consider a square simply supported curve plate (a section of the shell) under uniform heating subjected to a normal force \( N_x \) parallel to its directrix along the edge \( x = 0, x = a \). It is required to determine the critical load for the problem and to investigate the post buckling behavior of the shallow shell.

For this problem \( R_1 = \infty, R_2 = R, a = b, \bar{u} = 0, \bar{v} = 0, \bar{\alpha} = 0 = \beta, M_x = 0 = M_y \) where \( M_x \) and \( M_y \) are the bending moments. The governing equation (19) can be rewritten and adding the contribution for \( N_x \) we obtain

\[ - \left[ A \left( \frac{1}{2R} v_1 \left( 1 + \frac{v_2}{v_1} \right) + w_{xx} + \frac{1}{2} v_1 \left( 1 + \frac{v_2}{v_1} \right) w_{yy} \right) + \lambda_m \left( \frac{1}{R} w_{xx} + w_{yy} \right) \right] \times \left( \frac{w_x^2 + \frac{v_1}{2} w_{xx}^2 + w_{yy} + 2w_xw_yw_{xy} - \frac{1}{R} (w_x^2 + w_y^2)}{2} \right) \]

\[ + \frac{G_{13} h^2}{c(B_1 + B_2)} \left\{ \phi - \left( w_{xx} + \frac{G_{23}}{G_{13}} w_{yy} \right) \right\} + \frac{N_x}{B_1 + B_2} w_{xx} = 0. \]  

(25)

In this example the temperature distributions at each face sheets are assumed as follows:

\[ T_m = \frac{(T_1 + T_2)}{2}, T_d = T_1 - T_2, \text{ and } T_d = T_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{a}. \]  

(26)

where \( T_0 \) is a constant. \( T_1 \) and \( T_2 \) are temperatures at the upper and lower faces, respectively, as mentioned earlier.

Let us assume \( w = w_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \).

(27)

Introducing Eq. (27) into Eq. (20) and integrating over the entire area and remembering that \( \bar{u}, \bar{v} \) vanishes at the boundaries, we get the separation constant

\[ A = - \frac{4}{\pi} v_1 \left( 1 + \frac{v_2}{v_1} \right) \frac{1}{R} w_0 + \frac{\pi^2}{8a^2} \left( 1 + \frac{v_1}{2} \left( 1 + \frac{v_2}{v_1} \right) \right) w_0^2 - \bar{\alpha}_{11} \frac{1}{1 + \frac{v_2}{v_1}} (1 + v_s) T_m = 0. \]  

(28)

Also, introducing Eqs. (26) and (27) into Eq. (21), the resulting equation can be easily integrated and \( \phi = \bar{\alpha}_x + \frac{G_{23}}{G_{13}} \beta_x \) can be expressed as

\[ \phi = \phi_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{a}. \]  

(29)

where

\[ \phi_0 = [\bar{\alpha}_{11}((1 + v_s)/4h)(B_1 c/G_{13})(1 + \bar{\alpha}_{21} B_2/B_1)(2\pi^2/a^2)T_0 - (\pi^2/a^2)(1 + (G_{23}/G_{13})w_0)[1 + 2K \pi^2/a^2]^{-1}, \]  

(30)

and

\[ K = \frac{cB_1(B_2/B_1)}{G_{13}(1 + B_2/B_1)}. \]  

(31)

Introducing Eqs. (26)–(31) in Eq. (25), it can be solved approximately by applying Galerkin’s method, and after simplification, we obtain
\[
\frac{\pi^2}{8a^2} \left[ \left( 1 + \frac{v_1}{2} \left( 1 + \frac{v_2}{v_1} \right) \right)^2 + 5\pi^2 \lambda_m \right] w_0^2 - \frac{4}{\pi^2} \left[ \frac{3}{2} \left( 1 + \frac{v_2}{v_1} \right) \left( 1 + \frac{v_1}{2} \left( 1 + \frac{v_2}{v_1} \right) \right)^2 \right] \left( 1 + \frac{8}{3} \lambda_m \right) w_0 \]
\]
\[
+ \frac{64a^2}{\pi^6R^2} \left( 1 + \frac{v_2}{v_1} \right)^2 + \frac{a^2\lambda_m}{\pi^2R^2} + \frac{16}{\pi^2} cB_1 \left( 1 + B_2/B_1 \right) \left( 1 + \frac{G_{23}}{G_{13}} \right) a^4 \left[ 1 + 2K\pi^2/a^2 \right]^{-1} \]
\[- \left( 1 + \frac{1}{2} \left( 1 + \frac{v_2}{v_1} \right) \right) \bar{z}_{11} \left[ 1 + \frac{\bar{z}_{21}B_2/B_1}{1 + B_2/B_1} \right] \left( 1 + v_1 \right) T_m \right].
\]

where the parameter \( \lambda_m \) can be obtained from (24). The critical load is obtained by dropping the nonlinear terms as

\[
N_x^u = \left( B_1 + B_2 \right) \frac{64a^2}{\pi^6R^2} \left( \frac{v_1}{2} \left( 1 + \frac{v_2}{v_1} \right) \right)^2 + \frac{a^2\lambda_m}{\pi^2R^2} + \frac{16}{\pi^2} cB_1 \left( 1 + B_2/B_1 \right) \left( 1 + \frac{G_{23}}{G_{13}} \right) a^4 \left[ 1 + 2K\pi^2/a^2 \right]^{-1} \]
\[- \left( 1 + \frac{v_2}{v_1} \right) \frac{1}{1 + B_2/B_1} \left( 1 + v_1 \right) T_m \right].
\]

Eq. (33) gives the upper critical load at the instant the shell snaps through. The lower critical load, which corresponds to the snap through condition, may be obtained by considering the non-linear terms.

Differentiation of the Eq. (32) with respect to \( w_0 \) followed by solution for \( w_0 \) yields

\[
w_0 = \frac{16a^2}{\pi^4} \left[ \left( 3/2 \right) \left( \left( v_1/2 \right) \left( 1 + v_2/v_1 \right) \right) \left( 1 + \left( v_1/2 \right) \left( 1 + v_2/v_1 \right) \right) + \frac{8/3}{\lambda_m} \right].
\]

Substitution of Eq. (34) into Eq. (32) gives the lower value of the critical load \( N_x^l \) which results after loss of stability as

\[
N_x^l = N_x^u - \left( B_1 + B_2 \right) \frac{32a^2}{\pi^6R^2} \left[ \left( 3/2 \right) \left( \left( v_1/2 \right) \left( 1 + v_2/v_1 \right) \right) \left( 1 + \left( v_1/2 \right) \left( 1 + v_2/v_1 \right) \right) + \frac{8/3}{\lambda_m} \right]^2.
\]

A measure of the energy loss resulting from shell buckling may be obtained by investigating the ratio of upper and lower critical loads for the various parameter of the shell

\[
\frac{N_x^u}{N_x^l} = \frac{N_x^u}{N_x^u - \left( B_1 + B_2 \right) \frac{32a^2}{\pi^6R^2} \left( \frac{8/3}{\lambda_m} \right)}\left[ \left( v_1/2 \right) \left( 1 + v_2/v_1 \right) \right] \left( 1 + \left( v_1/2 \right) \left( 1 + v_2/v_1 \right) \right) + \frac{8/3}{\lambda_m} \right]^2.
\]

or

\[
\frac{N_x^u}{N_x^l} = \frac{1}{1 - \varepsilon},
\]

where

\[
\varepsilon = \frac{1}{N_x^l} \left( B_1 + B_2 \right) \frac{32a^2}{\pi^6R^2} \left[ \frac{\left( v_1/2 \right) \left( 1 + v_2/v_1 \right) \left( 1 + \left( v_1/2 \right) \left( 1 + v_2/v_1 \right) \right) + \frac{8/3}{\lambda_m} \right]^2}{\left( v_1/2 \right) \left( 1 + v_2/v_1 \right) \left( 1 + \left( v_1/2 \right) \left( 1 + v_2/v_1 \right) \right) + \frac{8/3}{\lambda_m} \right]}.
\]

A maximum value of \( \varepsilon \) is determined by considering only the term

\[\left( B_1 + B_2 \right) \frac{a^2}{\pi^6R^2} \left[ \frac{v_1}{2} \left( 1 + \frac{v_2}{v_1} \right) \right]^2 + \pi^4\lambda_m \right] - \left( 1 + \frac{v_1}{2} \left( 1 + \frac{v_2}{v_1} \right) \right) \bar{z}_{11} \left[ 1 + \frac{\bar{z}_{21}B_2/B_1}{1 + B_2/B_1} \right] \left( 1 + v_1 \right) T_m \]

of \( N_x^u \) in Eq. (37). For \( v_1 = v_2 = v = 0.25 \) and \( \lambda_m = 0.0625 \) and \( T_m = 10^5C \), we get \( \varepsilon = 0.34 \) and

\[
\frac{N_x^u}{N_x^l} = 1.51.
\]

It is to be noted that value obtained in non-thermal case by Chakrabarti and Bera (2002) was 1.31.
Solution for movable edge ($\Lambda=0$):

From Eq. (32), we obtain

$$\frac{N_x}{B_1 + B_2} = \frac{\pi^2}{8d^2} [5\pi^2 \lambda m] w_0^2 - \frac{4}{\pi^2 R} \left[ \frac{8}{3} \lambda m \right] w_0 + \frac{a^2 \lambda_m}{\pi^2 R^2} \left( 1 + \frac{v_1}{2} \left( 1 + \frac{v_2}{v_1} \right) \right) z_{11}$$

$$1 + \frac{a_{21} B_2 / B_1}{1 + B_2 / B_1} \left( 1 + v_x \right) T_m + \frac{16}{\pi^2} \frac{G_{13} h^2}{cB_1 (1 + B_2 / B_1)}.$$  \hspace{1cm} (39)

Following the same procedure as in the case of immovable edge above, the upper critical load may be obtained here as

$$N_x^u = (B_1 + B_2) \left[ \frac{a^2 \lambda_m}{\pi^2 R^2} \left( 1 + \frac{v_1}{2} \left( 1 + \frac{v_2}{v_1} \right) \right) z_{11} \right] \frac{1 + a_{21} B_2 / B_1}{1 + B_2 / B_1} \left( 1 + v_x \right) T_m + \frac{16}{\pi^2} \frac{G_{13} h^2}{cB_1 (1 + B_2 / B_1)}.$$  \hspace{1cm} (40)

Differentiation of Eq. (39) with respect to $w_0$ followed by a solution for $w_0$ yields

$$w_0 = \frac{128 a^2}{15 \pi^6 R^4}.$$  \hspace{1cm} (41)

From Eqs. (39) and (41), as in the present case, the lower value of the critical load $N_x^l$, which results after loss of stability, can now be obtained as

$$N_x^l = N_x^u - (B_1 + B_2) \frac{2048 a^2}{45 \pi^6 R^2} \lambda_m.$$  \hspace{1cm} (42)

Thus we obtain

$$\frac{N_x^u}{N_x^l} = \frac{N_x^u}{N_x^u - (B_1 + B_2) \frac{2048 a^2}{45 \pi^6 R^2} \lambda_m} = \frac{1}{1 - \varepsilon},$$  \hspace{1cm} (43)

where

$$\varepsilon = \frac{(B_1 + B_2) \left( \frac{2048 a^2}{45 \pi^6 R^2} \right) \lambda_m}{N_x^u}.$$  \hspace{1cm} (44)

A maximum value of $\varepsilon$ may be determined by considering only the term

$$(B_1 + B_2) \frac{a^2 \lambda_m}{\pi^2 R^2} \left( 1 + \frac{v_1}{2} \left( 1 + \frac{v_2}{v_1} \right) \right) z_{11} \frac{1 + a_{21} B_2 / B_1}{1 + B_2 / B_1} \left( 1 + v_x \right) T_m$$

of Eq. (40) as

$$\varepsilon = 0.081 \text{ at } T_m = 10^\circ C$$  \hspace{1cm} (46)

and substituting the value of $\varepsilon$ in Eq. (43), we obtain

$$\frac{N_x^u}{N_x^l} = 1.09.$$  \hspace{1cm} (47)

It is not out of place to mention that the value obtained by Chakrabarti and Bera (2002) in non-thermal condition for movable edge was 1.05 which was also obtained by Fulton (1961) for an immovable edge condition following a completely different method.

3. Numerical discussion

Figs. 2–6 present a comparative study of the ratio of the upper and lower critical loads vs. temperature for a square orthotropic sandwich shell with orthotropic core for both immovable and movable edges in case of materials like steel, high strength concrete, brass and glass epoxy as obtained in the present study with the parameters involved having the following numerical values:
Fig. 2. Ratio of upper and lower critical load vs. temperature.

Fig. 3. Ratio of upper and lower critical load vs. temperature.

Fig. 4. Ratio of upper and lower critical load vs. temperature.
\( a = 0.254 \text{ m}, \quad h = 1.7135 \times 10^{-2} \text{ m}, \quad E_1 = 7347.201 \times 10^6 \text{ kg/m}^2, \)
\( G_{13} = 4218.4884 \times 10^3 \text{ kg/m}^2, \quad \frac{G_{23}}{G_{13}} = 0.995, \)
\( v_1 = 0.25, \quad v_2 = 0.30, \quad \lambda_m = 0.075, \quad R = 1 \text{ m}. \)
\( \frac{c}{G_{13}} = 0.001, \quad \frac{B_2}{B_1} = 0.5, \quad B_1 = 5.172 \times 10^6 \text{ kg/m}. \)
\[ \bar{\alpha}_{11} = 11 \times 10^{-6}/\text{oC (steel),} \quad 4.5 \times 10^{-6}/\text{oC (concrete),} \quad 18 \times 10^{-6}/\text{oC (brass) and} \quad 6.6 \times 10^{-6}/\text{oC (glass epoxy) with} \quad \bar{\alpha}_{21} = 0.9. \]

The graphs for the materials like steel, concrete, brass and glass epoxy have been drawn to show the behavior of the materials at different temperatures for their critical loads. From the present analysis which exhibits enough light towards the behavior of stability of an orthotropic sandwich shell with orthotropic core for different materials under thermal loading. It is observed that the results of the present analysis are sufficiently accurate for both movable as well as for immovable edge conditions to the experimental results for an orthotropic sandwich shell with orthotropic core under thermal loading. Furthermore, the present analysis provides the results for movable edge conditions which require separate analysis when other methods are used. It has also been noted that Berger’s method fails for obtaining results in case of movable edge conditions.

4. Conclusion

It is observed from the figures that the value of ratio of upper and lower critical loads of steel is higher than that of concrete and that of brass is higher than glass epoxy. As a result, the increase in the value of \( \alpha \) is due to
an increase in temperature up to a certain stage, but changes its nature after that stage and a negative value of the ratio will be obtained in case of steel. Similar nature will be found in case of brass. Lastly, we would like to mention that in our present study, we have used the same differential equation for the analysis of both immovable as well as for movable edge conditions. This is an added advantage of the present analysis over other methods. It may also be noted here that the graphs corresponding to Berger’s approximation cannot be obtained in case of movable edge conditions.

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References

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