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Parameterized complexity of generalized domination problems[☆]

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ABSTRACT

Given two sets σ, ρ of non-negative integers, a set S of vertices of a graph G is (σ, ρ)-*dominating* if $|S \cap N(v)| \in \sigma$ for every vertex $v \in S$, and $|S \cap N(v)| \in \rho$ for every $v \notin S$. This concept, introduced by Telle in 1990's, generalizes and unifies several variants of graph domination studied separately before. We study the parameterized complexity of (σ, ρ)-domination in this general setting. Among other results, we show that the existence of a (σ, ρ)-dominating set of size k (and at most k) are W[1]-complete problems (when parameterized by k) for any pair of finite sets σ and ρ. We further present results on dual parameterization by $n - k$, and results on certain infinite sets (in particular for σ, ρ being the sets of even and odd integers).

1. Introduction

We consider finite undirected graphs without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$ (or simply by V and E if it does not create confusion). Let G be a graph. For a vertex v, we denote by $N_G(v)$ its *(open) neighborhood*, i.e. the set of vertices which are adjacent to v, and we denote by $\deg_G(v)$ the *degree* of this vertex. We may omit the index if the graph under consideration is clear from the context.

Let σ, ρ be a pair of non-empty sets of non-negative integers. A set S of vertices of a graph G is called (σ, ρ)-*dominating* if for every vertex $v \in S$, $|S \cap N(v)| \in \sigma$, and for every $v \notin S$, $|S \cap N(v)| \in \rho$. The concept of (σ, ρ)-domination was introduced by Telle [35,36] (and further elaborated on in [37,23]) as a unifying generalization of many previously studied variants of the notion of dominating sets. See Table 1 for some examples.

It is well known that the optimization problems such as Maximum Independent Set, Minimum Dominating Set, etc. are NP-hard. In many cases of the generalized domination already the existence of a (σ, ρ)-dominating set becomes NP-hard (e.g., when both σ and ρ are finite and non-empty, and $0 \notin \rho$ [35]). Hence attention was paid to special graph classes, e.g. interval graphs ([26] shows polynomial-time solvability for any pair of finite σ, ρ), chordal graphs ([19] shows a P/NP-c dichotomy classification) or degenerate graphs [20].

The generalized domination problems for graphs of bounded width parameters (treewidth, branchwidth, cliquewidth, Boolean-width) were considered in [1,4,5,7,36,37]. The special case when σ and ρ are the sets of non-negative even or odd integers was investigated in [22]. Exact exponential-time algorithms for (σ, ρ)-dominating set were designed in [18,17].

Since the establishment of the parameterized complexity theory by Downey and Fellows [12], domination-type problems have been among the first ones intensively studied in the framework of this theory. (We assume the reader is familiar with the concept of FPT and W[t] classes. For those who are not, we refer to [12,16,34] as excellent textbooks.) It is well known that Independent Set is W[1]-complete [11] and Dominating Set is W[2]-complete [10,12], if parameterized by the size of

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Table 1
Overview of some special cases of (σ, ρ)-domination and their parameterized complexity. (Here $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive and non-negative integers, respectively.)

σ	ρ	Problem name	Parameterized complexity
$\mathbb{N}_0$	$\mathbb{N}$	Dominating set	W[2]-complete [10,12]
$\mathbb{N}$	$\mathbb{N}$	Total dominating set	W[2]-hard [3]
$\mathbb{N}_0$	$\{1\}$	Efficient dominating set	W[1]-hard [3]
$\{0\}$	$\mathbb{N}$	Independent dominating set	W[2]-complete [10]
$\{0\}$	$\mathbb{N}_0$	Independent set	W[1]-complete [11]
$\{0\}$	$\{1\}$	(1-)Perfect code(Indep. Eff. D. S.)	W[1]-complete [11,6]
$\{r\}$	$\mathbb{N}_0$	Induced r-regular subgraph	W[1]-hard [27]
$\{0\}$	$\{0, 1\}$	Strong stable set	W[1]-hard
$\{1\}$	$\{1\}$	Total perfect dominating set	W[1]-complete (Theorem 1)

Table 2
Parameterized complexity of deciding the existence of a (σ, ρ)-dominating set of a given size.

Size	Complexity
$\leq k$	W[1]-c for finite $\sigma, \rho, 0 \notin \rho$ (Theorem 1)
$=k$	W[1]-c for finite $\sigma, \rho, 0 \notin \rho$ (Theorem 1)
$\geq k$	para-NP-c for finite $\sigma, \rho, 0 \notin \rho$ [35]
$\leq n - k$	para-NP-c for finite $\sigma, \rho, 0 \notin \rho$ [35]
$=n - k$	Open
$\geq n - k$	FPT for finite or co-finite σ, ρ (Theorem 13)

the expected set k. A number of domination-type problems is considered in [3], where it is shown (among other results) that Total Dominating Set is W[2]-hard and that Efficient Dominating Set is W[1]-hard. Independent Dominating Set is W[2]-complete [10], while Efficient Independent Dominating Set (also called Perfect Code) is W[1]-complete ([11] shows W[1]-hardness and [6] shows W[1]-membership). The proof of W[1]-hardness of Strong Stable Set is not published (as we know) but easily follows from the W[1]-hardness of Independent Set. Again, all of the results are given with respect to the standard solution-size parameterization k. Treshold Dominating Set is shown to be W[2]-complete in [13]. More results on parameterized complexity of problems from coding theory can be found in [14]. The complexity of finding an r-regular induced subgraph in a graph is studied by Moser and Thilikos in [27]. They proved that the problem is W[1]-hard when parameterized by the solution size but FPT for the dual parameterization.

Parity constraints have been considered in [14]. A subset of a color class of a bipartite graph is called *odd* (*even*) if every vertex from the other class has an odd (even, respectively) number of neighbors in the set. Downey et al. [14] show that deciding the existence of an odd set of size k, an odd set of size at most k, and an even set of size k are W[1]-hard problems; somewhat surprisingly, the complexity of Even Set of Size at Most k remains open.

All these individual results concern special (σ, ρ)-dominating sets, and thus calls for a unifying approach. Our paper attempts to be a starting one by giving general results for large classes of pairs σ, ρ. Our main results are given in Table 2. For completeness, we also included in this table results which immediately follow from the fact proved by Telle [35] that it is NP-hard to test the existence of a (σ, ρ)-dominating set for any finite sets σ and ρ, $0 \notin \rho$. Following the approach of Telle [35] we focus mainly on finite σ and ρ, but we state the results in the most general form we were able to achieve. The second goal of our paper is to study (many of) the above problems from the dual parameterization point of view (looking for a set of size at least $n - k$, where k is the parameter), both for the domination-type and parity-type problems.

We consider the following (σ, ρ)-domination problem

(σ, ρ)-Dominating Set of Size at most k
Input: A graph G.
Parameter: k.
Question: Is there a (σ, ρ)-dominating set in G of size at most k?

and its variants (σ, ρ)-Dominating Set of Size k, (σ, ρ)-Dominating Set of Size at least $n-k$, and (σ, ρ)-Dominating Set of Size $n - k$, whose meaning should be clear. All these problems are parameterized by k, and in the latter two, n denotes the number of vertices of the input graph.

The first of our main results determines the parameterized complexity for finite sets σ and ρ. We prove in Section 2 that both (σ, ρ)-Dominating Set of Size k and (σ, ρ)-Dominating Set of Size at most k are W[1]-complete problems whenever $0 \notin \rho$. The W[1]-membership is proved in a stronger form when σ is only required to be recursive but not necessarily finite. Recall that a set of non-negative integers is called *recursive*, if there is a deterministic algorithm, that given a non-negative integer k decides in finite time, whether k is in the set or not.

We further study the dually parameterized problems and show in an even more general way (also for co-finite sets) that these problems become tractable. In Section 3, we prove that (here and throughout the paper, $\overline{X} = \mathbb{N}_0 \setminus X$ for a set X of integers) for non-empty sets of non-negative integers σ and ρ such that either σ or $\overline{\sigma}$ is finite, and similarly either

ρ or $\overline{\rho}$ is finite, the (σ, ρ)-Dominating Set of Size at least $n - k$ problem is in FPT. We show that a similar result cannot be expected for arbitrary recursive sets σ and ρ. Even for the parity case (when we denote **EVEN** $= \{0, 2, 4, 6, \ldots\}$ and **ODD** $= \{1, 3, 5, \ldots\}$) we prove W[1]-hardness if $\sigma, \rho \in \{$**EVEN**, **ODD**$\}$.

As a tool for the previous result, we consider the following parity problems on bipartite graphs. Suppose that G is a bipartite graph and R, B is a bipartition of its set of vertices (vertices of R are called *red* and vertices of B are *blue*). A non-empty set $S \subseteq R$ is called *even* if for every vertex $v \in B$, $|N(v) \cap S| \in$ **EVEN**, and it is called *odd* if for every vertex $v \in B$, $|N(v) \cap S| \in$ **ODD**. The following problem

Even Set of Size at least $r - k$
Input: A bipartite graph $G = (R, B, E)$ and $r = |R|$.
Parameter: k.
Question: Is there an even set in R of size at least $r - k$?
and its variants Even Set of Size $r - k$, Odd Set of Size at least $r - k$, and Odd Set of Size $r - k$ are the dually parameterized versions of bipartite parity problems studied in [14]. We prove in Section 4 that all four of them are W[1]-hard. We believe that these results are interesting by themselves. Observe particularly that it is unusual for parameterized problems to have same complexity for dual parameterizations.

We conclude the paper by some observations on FPT results for sparse graphs and open problems.

2. Complexity of the (σ, ρ)-Dominating Set of Size at most k

Here we prove the following theorem.

Theorem 1. *Let σ and ρ be non-empty finite sets of non-negative integers, $0 \notin \rho$. Then both (σ, ρ)-*Dominating Set of Size *k and (σ, ρ)-*Dominating Set of Size at most *k are* W[1]-*complete problems.*

The remaining part of this section contains the proof of this theorem. First, we prove W[1]-hardness. To do it, we introduce and consider an auxiliary problem.

2.1. At most α-satisfiability

To prove the hardness part of Theorem 1, we are going to reduce from a special variant of the Satisfiability problem.
At most α-Satisfiability
Instance: A Boolean formula ϕ in conjunctive normal form, without negated variables.
Parameter: k.
Question: Does ϕ allow a satisfying truth assignment of weight at most k (i.e., at most k variables have value *true*) such that each clause of ϕ contains at most α variables which evaluate to *true*?

So, we start with the proof of W[1]-hardness for this problem.

Lemma 2. *For any $\alpha \geq 1$, the* At most α-Satisfiability *problem is* W[1]-*hard.*

Proof. We provide a reduction from the following problem:
Exact Satisfiability
Instance: A Boolean formula ϕ in conjunctive normal form.
Parameter: k.
Question: Does ϕ have a satisfying truth assignment of weight at most k (i.e., at most k variables have value *true*) such that each clause of ϕ contains exactly one literal which evaluates to *true*?
This problem is W[1]-hard [11] even if all clauses contain only positive variables. It should be noted that in [11] W[1]-hardness was proved for the exact variant of the problem (i.e., for the question: Does ϕ have a satisfying truth assignment of weight exactly k such that each clause of ϕ contains exactly one literal with value *true*?), but it can be easily seen that the proof works for our variant of the question as well (see the proof of W[1]-hardness in [11], also note that Exact Satisfiability is equivalent to the Perfect Code problem, and all perfect codes in a given graph have the same cardinality [25]).

For $\alpha = 1$, At most α-Satisfiability is the same problem as Exact Satisfiability of formulas without negations. Hence the lemma needs to be proved only for $\alpha \geq 2$.

We reduce from Exact Satisfiability. Let $C_1, \ldots, C_m$ be the clauses of ϕ. We introduce α copies of ϕ with different sets of variables, and denote them by $\phi_1, \ldots, \phi_\alpha$. Let $C_{i,1}, \ldots, C_{i,m}$ be the clauses of ϕ_i. Define $\psi = \phi_1 \wedge \cdots \wedge \phi_\alpha \wedge [(C_{1,1} \vee \cdots \vee C_{\alpha,1}) \wedge \cdots \wedge (C_{1,m} \vee \cdots \vee C_{\alpha,m})]$, and let $k' = k\alpha$.

Suppose that ϕ has a satisfying truth assignment of weight at most k such that each clause of ϕ contains exactly one variable with value *true*. Using the same truth assignment for the sets of variables for each ϕ_i we get a satisfying truth assignment of weight at most k' for ψ such that each clause contains at most α variables with value *true*.

For the converse, assume that ψ has a satisfying truth assignment of weight at most k' such that each clause of ψ contains at most α variables with value *true*. Obviously each clause $C_{i,j}$ contains at least one variable with value *true*, but it cannot have two variables with value true, since otherwise the clause $C_{1,j} \vee \cdots \vee C_{\alpha,j}$ of ψ would contain more than α variables of value *true*. So, all formulas ϕ_i have a truth assignment such that each clause contains exactly one variable with value *true*. It remains to note that by a pigeonhole principle at least one formula ϕ_i has a truth assignment of weight at most $(k'/\alpha) = k$, as the formulas have different sets of variables. $\square$

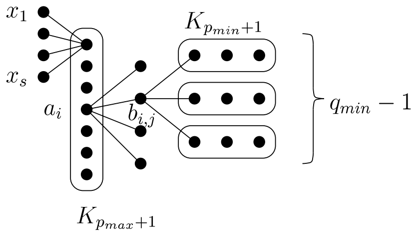


Fig. 1. Construction of $F(s)$.

2.2. *The proof of* W[1]-*hardness*

This section contains the proof of W[1]-hardness of (σ, ρ)-Dominating Set of Size k and (σ, ρ)-Dominating Set of Size at most k by a reduction from At most α-Satisfiability.

Suppose that σ and ρ are non-empty finite sets of non-negative integers, $0 \notin \rho$. Let us denote $p_{\min} = \min \sigma$, $p_{\max} = \max \sigma$, $q_{\min} = \min \rho$ and $q_{\max} = \max \rho$. Further we set $t = \max\{i \in \mathbb{N}_0 : i \notin \rho, i + 1 \in \rho\}$ (since $0 \notin \rho$, t is correctly defined), and $\alpha = q_{\max} - t \geq 1$.

We first construct several auxiliary gadgets. These gadgets "enforce" on a given vertex the property of "not belonging to any (σ, ρ)-dominating set", and at the same time guarantee that this vertex has a given number of neighbors in any (σ, ρ)-dominating set in the gadget. To describe the properties formally, we will consider rooted graphs and introduce the following notion. Let G be a rooted graph with a set of root vertices X. We call a set $S \subset V(G)$ a *(σ, ρ)-dominating set for G* if $|N(v) \cap S| \in \sigma$ for every $v \in S \setminus X$, and $|N(v) \cap S| \in \rho$ for every $v \notin S$, $v \notin X$ (i.e., the conditions from the definition of (σ, ρ)-domination are required for all vertices except the roots).

The $F(s)$ gadget. We take a complete graph $K_{p_{\max}+1}$ with vertices $a_1, \ldots, a_{p_{\max}+1}$. For each vertex a_i, we add $q_{\max} + 1$ vertices $b_{i,1}, \ldots, b_{i,q_{\max}+1}$ and join them to a_i by edges. For each vertex $b_{i,j}$, we add $q_{\min} - 1$ copies of the complete graph $K_{p_{\min}+1}$ and make one vertex of each copy adjacent to $b_{i,j}$. Finally, s vertices $x_1, \ldots, x_s$ are added and joined to a_1. The resulting graph is denoted by $F(s)$ (see Fig. 1) and $x_1, \ldots, x_s$ are its roots.

Lemma 3. *The graph $F(s)$ has a unique (σ, ρ)-dominating set S for $F(s)$, and this set has the following properties:*

1. $x_1, \ldots, x_s \notin S$,
2. $a_1 \in S$ *(i.e., every root vertex has exactly one neighbor in S),*
3. *S contains $f = (p_{\max} + 1)((q_{\max} + 1)(q_{\min} - 1)(p_{\min} + 1) + 1)$ vertices.*

Proof. Let S consist of all vertices a_i and all vertices of all $(p_{\max} + 1)(q_{\max} + 1)(q_{\min} - 1)$ copies of $K_{p_{\min}+1}$. It is easy to check that S is a (σ, ρ)-dominating set for $F(s)$, and S satisfies properties 1–3. We prove that S is unique. Let S be a (σ, ρ)-dominating set for $F(s)$. Suppose that some vertex a_i is not in S. If a neighbor $b_{i,j}$ is also not in S, then this neighbor can have at most $q_{\min} - 1$ neighbors in S, but this is impossible. So, all vertices $b_{i,j}, j = 1, 2, \ldots, q_{\max} + 1$ are in S, but then a_i has at least $q_{\max} + 1$ neighbors in S, and this is again a contradiction. Hence all vertices $a_i, i = 1, 2, \ldots, p_{\max} + 1$ are in S, and hence all their neighbors (i.e., $x_1, \ldots, x_s$ and $b_{1,1}, \ldots, b_{p_{\max}+1,q_{\max}+1}$) are outside S. It follows that all neighbors of $b_{i,j}$ have to be included in S. This means that each copy of $K_{p_{\min}+1}$ has at least one vertex in S, and consequently all vertices of these copies are included in S. □

The $F'(s)$ gadget. We take $q_{\max}$ copies of $F(1)$ and identify their roots into one vertex x. Then s root vertices $y_1, \ldots, y_s$ of degree one are added and each is made adjacent to x. The resulting graph is denoted by $F'(s)$.

Lemma 4. *The graph $F'(s)$ has a unique (σ, ρ)-dominating set S for $F'(s)$, and this set has the following properties:*

1. $x, y_1, \ldots, y_s \notin S$,
2. *S contains $f' = q_{\max}(p_{\max} + 1)((q_{\max} + 1)(q_{\min} - 1)(p_{\min} + 1) + 1)$ vertices.*

Proof. Clearly, the union of (σ, ρ)-dominating sets for the copies of $F(1)$ is a (σ, ρ)-dominating set for $F'(s)$, and this set has properties 1–2. Now note that any (σ, ρ)-dominating set S for $F'(s)$ must include (σ, ρ)-dominating sets for the copies of $F(1)$, and by Lemma 3, $x \notin S$. Since, by the same lemma, x has a neighbor from S in each copy of $F(1)$ and, thus, it has already $q_{\max}$ neighbors in S, none of $y_1, \ldots, y_s$ is in S. □

Using these gadgets we construct an auxiliary graph $H(l)$ for a positive integer l.

The $H(l)$ gadget. We start with a complete graph K_l with vertices $z_1, \ldots, z_l$ which will be the roots of $H(l)$. We consider three cases:

$q_{\max} = q_{\min} = 1$: A copy of $F'(1)$ with the root y is added, and y is joined to $z_1, \ldots, z_l$ by edges.

$q_{\max} > q_{\min} = 1$: We take $q_{\max} - 1$ copies of $F(1)$ with a common root x, and we add a copy of $F'(1)$ with the root y. Vertices x and y are made adjacent to $z_1, \ldots, z_l$.

$q_{\max} \geq q_{\min} > 1$: We introduce $q_{\max} - 1$ copies of $F(1)$ with a common root x, and we add another $q_{\min} - 1$ copies of $F(1)$ with a common root y. Vertices x and y are made adjacent to $z_1, \ldots, z_l$.

Lemma 5. 1. *For any (σ, ρ)-dominating set S for $H(l)$, each root vertex has no non-root neighbors in S.*
2. *Any (σ, ρ)-dominating set S for $H(l)$ contains exactly one root vertex.*
3. *For each root vertex z_i, there is a (σ, ρ)-dominating set S for $H(l)$ which contains z_i.*
4. *Any (σ, ρ)-dominating set S for $H(l)$ contains h vertices, where*

$$h = h(\sigma, \rho) = \begin{cases} f' + 1, & \text{if } q_{\max} = q_{\min} = 1, \\ (q_{\max} - 1)f + f' + 1, & \text{if } q_{\max} > q_{\min} = 1, \\ (q_{\max} + q_{\min} - 2)f + 1, & \text{if } q_{\max} \geq q_{\min} > 1. \end{cases}$$

Proof. By Lemmas 3 and 4, the vertices x and y do not belong to any (σ, ρ)-dominating set for $H(l)$. Hence the first claim follows. The second claim follows from the observation that every root vertex has an adjacent non-root vertex y with $q_{\min} - 1$ neighbors in S (since $y \notin S$, at least one root vertex has to be in S) and a neighbor (x or y) with $q_{\max} - 1$ neighbors in S (and hence at most one root vertex may be in S). To prove the third and fourth claims, observe that every (σ, ρ)-dominating set for $H(l)$ contains the union of the (σ, ρ)-dominating sets for all copies of $F(1)$ and of $F'(1)$ which were used in the construction, and this union plus one arbitrary root vertex z_i is indeed a (σ, ρ)-dominating set for $H(l)$. □

By the next step for a positive integer l, we construct a selection gadget $R(l)$.

The selection gadget $R(l)$. Take $(p_{\min} + 1)q_{\min}$ copies of $H(l)$ which are denoted $H_{i,j}(l)$ for $i \in \{1, \dots, p_{\min} + 1\}$ and $j \in \{1, \dots, q_{\min}\}$. Let $z_1^{(i,j)}, \dots, z_l^{(i,j)}$ be the roots of $H_{i,j}(l)$. For each $i \in \{1, \dots, p_{\min} + 1\}$, for each pair $j, j' \in \{1, \dots, q_{\min}\}, j \neq j'$, the vertices of the root sets for $H_{i,j}(l)$ and $H_{i,j'}(l)$ are joined by the complements of perfect matchings, i.e., vertices $z_s^{(i,j)}$ and $z_{s'}^{(i,j')}$ are adjacent if and only if $s \neq s'$. Then for each $j \in \{1, \dots, q_{\min}\}$ and any $s \in \{1, \dots, l\}$, the vertices $z_s^{(1,j)}, \dots, z_s^{(p_{\min}+1,j)}$ are made pairwise adjacent to form a clique. The roots of this constructed graph $R(l)$ are the vertices $z_1^{(1,1)}, \dots, z_l^{(1,1)}$.

Lemma 6. 1. *Any (σ, ρ)-dominating set S for $R(l)$ contains exactly one vertex from the set $\{z_1^{(1,1)}, \dots, z_l^{(1,1)}\}$.*
2. *For any vertex $z_s^{(1,1)}$, there is a (σ, ρ)-dominating set in $R(l)$ which contains this vertex.*
3. *Any (σ, ρ)-dominating set in $R(l)$ has $r = r(\sigma, \rho) = (p_{\min} + 1)q_{\min}h$ vertices.*

Proof. Observe that any (σ, ρ)-dominating set S for $R(l)$ induces (σ, ρ)-dominating sets for the graphs $H_{i,j}(l)$ for $i \in \{1, \dots, p_{\min} + 1\}$ and $j \in \{1, \dots, q_{\min}\}$. Hence the first claim of the lemma follows from the second claim of Lemma 5. To prove the second claim, consider the union of (σ, ρ)-dominating sets for the rooted graphs $H_{i,j}$ which contain the vertices $z_s^{(i,j)}$. These sets exist because of the third claim of $z_s^{(1,1)}$. Note explicitly that for this set, $z_s^{(i,j)}$ has exactly $p_{\min}$ neighbors in S and every other root vertex has exactly $q_{\min}$ neighbors in S by the first claim of Lemma 5 and the construction of $R(l)$. Therefore we have a (σ, ρ)-dominating set in $R(l)$ which contain $z_s^{(1,1)}$. The third claim follows immediately from the fourth claim of $z_s^{(1,1)}$. □

Now we are ready to describe the reduction. Let ϕ be a formula as an input of the At most α-Satisfiability problem. Let $x_1, \dots, x_n$ be its variables, and let $C_1, \dots, C_m$ be the clauses.

We take k copies of the graph $R(n+1)$ denoted by $R_1, \dots, R_k$, with the roots of R_i being denoted by $x_{i,j}$. For each clause C_s, a vertex C_s is added and joined by edges to all vertices $x_{i,j}$, $i = 1, \dots, k$ such that the variable x_j occurs in the clause G_s. Observe that vertices $x_{i,n+1}$ are not joined with any C_s. Now we distinguish two cases (recall that $t = \max\{i \in \mathbb{N}_0 : i \notin \rho, i + 1 \in \rho\}$).

$t = 0$: In this case a copy of $F'(m)$ is introduced, and the m roots of this gadget are identified with vertices $C_1, \dots, C_m$. In this case we set $k' = kr + f'$.

$t > 0$: We construct t copies of $F(m)$, and the roots of each copy are identified with $C_1, \dots, C_m$. In this case we set $k' = kr + tf$.

The resulting graph is called G. The proof of W[1]-hardness is then concluded by the following lemma.

Lemma 7. *The formula ϕ allows a satisfying truth assignment of weight at most k such that each clause of ϕ contains at most α variables with value true if and only if G has a (σ, ρ)-dominating set of size at most k'. Moreover, in such a case the size of any (σ, ρ)-dominating set is exactly k'.*

Proof. Suppose that the variables $x_1, \dots, x_n$ have a satisfying truth assignment of weight at most k such that each clause of ϕ contains at least one variable, and at most α variables with value *true*. Without loss of generality we assume that $x_j = true$ for $j \in \{1, \dots, l\}$, $x_j = false$ for $j \in \{l + 1, \dots, n\}$ and $l \leq k$. We construct a (σ, ρ)-dominating set S for G as follows. For each $j \in \{1, \dots, l\}$, all vertices of the (σ, ρ)-dominating set of R_j which contains $x_{j,j}$ are included in S (see Lemma 6). Notice that the satisfying truth assignment can have weight strictly lower that k, i.e. $l < k$. In this case for each $j \in \{l + 1, \dots, k\}$, all vertices of the (σ, ρ)-dominating set of R_j which contains $x_{j,n+1}$ are included in S. For each copy of $F(m)$ (or $F'(m)$), all vertices of corresponding (σ, ρ)-dominating sets (see Lemmas 3 and 4) for these rooted graphs are added to S. By Lemmas 3, 4 and 6, $|S| = k'$. By the same lemmas, for any vertex $v \neq C_1, \dots, C_m$, (σ, ρ)-conditions are satisfied, and we have to check them only for vertices C_s. Since $C_s \notin S$ (see Lemmas 3 and 4), it is necessary to prove that $|S \cap N(C_s)| \in \rho$. One more time using Lemmas 3 and 4 we note that C_s has t neighbors in S from gadgets $F(m)$ or $F'(m)$. Each clause C_j contains at least one variable and at most $\alpha = q_{\max} - t$ variables with value *true*. By the construction of S and Lemma 6, each vertex C_s has at least one and at most $\alpha = q_{\max} - t$ neighbors in S from gadgets $R_1, \ldots, R_k$. Therefore $t + 1 \leq |S \cap N(C_s)| \leq t + \alpha = q_{\max}$, and $\{t + 1, \ldots, q_{\max}\} \subseteq \rho$.

Now assume that S is a (σ, ρ)-dominating set of size at most k' in G. By Lemmas 3, 4 and 6, S is the union of the (σ, ρ)-dominating sets of the graphs $R_1, \ldots, R_k$ and the (σ, ρ)-dominating sets for the gadgets $F(m)$ (or $F'(m)$) (note that it means that $|S| = k'$). It follows from Lemma 5 that for each $i \in \{1, \ldots, k\}$, S contains exactly one vertex from the set $\{x_{i,1}, \ldots, x_{i,n+1}\}$. For each $j \in \{1, \ldots, n\}$, we set the variable $x_j = \mathit{true}$ if $x_{i,j} \in S$ for some $i \in \{1, \ldots, k\}$ and $x_j = \mathit{false}$ otherwise. Clearly we have a truth assignment of weight at most k. By Lemmas 3 and 4, vertices C_s are not in S, and each vertex C_s has t neighbors in S from gadgets $F(m)$ or $F'(m)$. Since S is a (σ, ρ)-dominating set, C_s has at least one and at most $q_{\max} - t = \alpha$ neighbors in S from the graphs $R_1, \ldots, R_k$. Recall that C_s is not adjacent with vertices $x_{i,n+1}$. Hence the neighbors of C_s in S are vertices $x_{i,j}$ for $j \in \{1, \ldots, n\}$ which correspond to the variables that were set *true*. It follows immediately that each clause C_s contains at least one and at most α variables with value *true*. □

2.3. W[1]-*membership*

To complete the proof of Theorem 1, it remains to prove that our problems are included in W[1]. Here we prove a slightly stronger claim.

Theorem 8. *Let σ be recursive, and suppose that ρ is finite or $\rho = \mathbb{N}_0$. Then the (σ, ρ)-*Dominating Set of Size at most *k and (σ, ρ)-*Dominating Set of Size *k problems are in* W[1].

To show the membership of the problems in W[1], we use the characterization of W[1] by non-deterministic random access machines as proposed in [16].

A non-deterministic random access machine (NRAM) model is based on the standard deterministic random access machine (RAM) model. A single non-deterministic instruction "GUESS" is added, whose semantics is: *Guess a natural number less than or equal to the number stored in the accumulator and store it in the accumulator*. Acceptance of an input by an NRAM is defined as usually for non-deterministic machines, that is the program accepts the particular input if there is a computation on it that ends by an execution of the ACCEPT instruction. The steps of computation of an NRAM that execute a GUESS instruction are called *non-deterministic steps*.

Definition 9. An NRAM program $\mathbb{P}$ is *tail-non-deterministic k-r*estricted if there are computable functions f and g and a polynomial p such that on every run with input $(x, k) \in \Sigma^* \times \mathbb{N}$ the program $\mathbb{P}$

- performs at most $f(k) \cdot p(n)$ steps;
- uses at most the first $f(k) \cdot p(n)$ registers;
- contains numbers $\leq f(k) \cdot p(n)$ in any register at any time;

and all non-deterministic steps are among the last $g(k)$ steps of the computation. Here $n = |x|$.

The following characterization is crucial for our proof.

Theorem 10 (*[16]*). *A parameterized problem P is in* W[1] *if and only if there is a tail-non-deterministic k-restricted NRAM program deciding P.*

Now we introduce our program `SigmaRho` for the case ρ is finite, that takes a graph G and a positive integer k as an input and there is an accepting computation of `SigmaRho` on G and k if and only if there is a (σ, ρ)-dominating set of size exactly k in G. We present it in a higher level language that can be easily translated to the NRAM instructions. Then we will show that this program is tail-non-deterministic k-restricted. Recall that $q_{\max} = \max \rho$. $\binom{V}{r}$ denotes the set $\{R \subseteq V \mid |R| = r\}$.

Lemma 11. *Let G be a graph and $k \in \mathbb{N}$. There is an accepting computation of* `SigmaRho` *on G and k if and only if there is a (σ, ρ)-dominating set of size (exactly) k in G.*

Proof. We will show that the program `SigmaRho` accepts the input if and only if the set S guessed in step 2 is a (σ, ρ)-dominating set of size k for the input graph G. Clearly, if the program accepts, then S contains k distinct vertices, otherwise it would have been rejected in step 3. It is easy to see that the members of the set S must satisfy the σ-condition due to step 4. Now observe that the number $D(r)$ computed in step 5 denotes the number of pairs (R, v) such that R is a subset of S of size r and v is a vertex not in S that has all vertices from R as neighbors (the first term counts all such vertices v in V and the second term subtracts such vertices v that are in S). Hence this $D(r)$ represents the number of vertices outside S which have at least r neighbors in S with multiplicities, in particular a vertex with t neighbors in S is counted $\binom{t}{r}$ times. Since in the first run of the cycle 5 with $r = q_{\max} + 1$ we check that there is no vertex outside S with more than $q_{\max}$ neighbors in S, $C(r)$ represents the number of vertices outside S which have exactly r neighbors in S. Now it is clear that if $r \notin \rho$ and there is a vertex outside S with r neighbors in S (i.e., $C(r) > 0$), then S cannot form a (σ, ρ)-dominating set. In the last step 6 we sum up the number of vertices outside S that satisfy the ρ-condition and thus S (which satisfies all the conditions checked by the previous steps) is (σ, ρ)-dominating if and only if this sum is equal to the total number of vertices outside S, i.e., $n - k$, or $0 \in \rho$. □

Program SigmaRho($G = (V, E), k$)

1 **for** $r := 1$ **to** $q_{max} + 1$ **do forall the** $R \in \binom{V}{r}$ **do**

$$B(R) := |\bigcap_{u \in R} N_G(u)| = |\{v | v \in V, \forall u \in R : uv \in E\}|;$$

2 **Guess** k **vertices** $v_1, \ldots, v_k$, **denote** $S = \{v_1, \ldots, v_k\}$**;**

3 **forall the** i, j, $1 \le i < j \le k$ **do if** $v_i = v_j$ **then REJECT;**

4 **for** $i := 1$ **to** k **do if** $|\{v_j | v_i v_j \in E\}| \notin \sigma$ **then REJECT;**

5 **for** $r := q_{max} + 1$ ***downto*** 1 **do**

$$\begin{aligned} D(r) &:= \sum_{R \in \binom{S}{r}} (B(R) - |\bigcap_{u \in R} N_G(u) \cap S|) = \\ &= \sum_{R \in \binom{S}{r}} |\{v | v \in V \setminus S, \forall u \in R : uv \in E\}|; \\ C(r) &:= D(r) - \sum_{t=r+1}^{q_{max}} \left(\binom{t}{r} \cdot C(t) \right); \end{aligned}$$

if $r \notin \rho$ *and* $C(r) \neq 0$ **then REJECT;**

6 **if** $0 \notin \rho$ *and* $\sum_{r \in \rho} C(r) \neq n - k$ **then REJECT; else ACCEPT;**

Lemma 12. *Program SigmaRho is tail-non-deterministic k-restricted.*

Proof. To prove the lemma, we prove the following claims.

Claim 1: *There is a function $g(k)$ such that steps* 2–6 *can be performed in at most $g(k)$ steps*. Step 2 is a simple k times execution of the "GUESS" instruction. Hence it is carried out in $O(k)$ time. Step 3 takes $O(k^2)$ time. If $a(k)$ denotes the maximum time needed for $l \le k$ to decide whether $l \in \sigma$ (such a function exists since σ is recursive), then step 4 can be carried out in $O(k^2 \cdot a(k))$. We can also suppose that $a(k)$ bounds any numbers involved in this computation and the number of registers used. The cycle in 5 is executed constantly $(q_{\max} + 1)$ many times. The value $D(r)$ is computed according to the first expression, where there is at most $O(k^{q_{\max}+1})$ different indices for the sum, the first term means just a table lookup and the second term can be determined in $O((q_{\max} + 1) \cdot k)$ steps for each fixed index. The expression for $C(r)$ contains constantly many (at most $q_{\max}$) terms. With the last step taking also constant time, this means that steps 2–6 altogether can take at most $g(k) = O(k^{q_{\max}+2} + k^2 \cdot a(k))$ time. Since the first non-deterministic instruction is in step 2, non-deterministic steps are among the last $g(k)$ steps.

Claim 2: *Step* 1 *can be carried out in $O(n^{q_{\max}+2})$ time*. There are at most $O(n^{q_{\max}+1})$ subsets of size at most $q_{\max}+1$ in V, $|V| = n$ and for each subset R the computation of $B(R)$ can be performed in $O((q_{\max} + 1) \cdot n)$ time. Together with Claim 1 this shows the first condition of the definition.

Claim 3: *During the computation the numbers involved and the number of registers used are bounded by $O(n^{q_{\max}+2})$, except for step* 4, *where the bound is $a(k)$*. There is $n^{q_{\max}+1}$ many $B(R)$'s, a constant number of $D(r)$'s and $C(r)$'s, k vertices of S, the input and an additional constant number of variables for the indices stored during the computation. The $B(R)$'s are bounded by n since they contain the number of vertices satisfying certain conditions. Each $D(r)$ is a sum of at most $n^{q_{\max}+1} B(R)$'s and hence bounded by $n^{q_{\max}+2}$. The $C(r)$ and the sum in the last step are sums of constantly many $D(r)$'s and $C(r)$'s, respectively, hence bounded by $O(n^{q_{\max}+2})$. The bound for step 4 is proved in Claim 1. This shows the second and third conditions of the definition, completing the proof. □

Proof of Theorem 8. The W[1]-membership of (σ, ρ)-Dominating Set of Size k for the case ρ is finite is a direct consequence of Theorem 10 together with Lemmas 11 and 12. To show the membership of (σ, ρ)-Dominating Set of Size at most k it suffices to add one more non-deterministic step "Guess $l \le k$; $k := l$" before Step 2 of SigmaRho. To modify the program SigmaRho for the case $\rho = \mathbb{N}_0$ it is enough to omit Steps 1, 5 and 6. □

This completes the proof of Theorem 1.

3. Complexity of the (σ, ρ)-Dominating Set of Size at least $n - k$ problems

Now we consider our problems for the dual parameterization. Note that the studied class also contains (except for others) Vertex Cover (as a dual of Independent Set), probably the most studied problem in parameterized complexity that is well known to be FPT [12]. The dual parameterization of r-Regular Induced Subgraph was shown to be FPT in [27]. We provide a common generalization for these results.

Theorem 13. *Let σ and ρ be sets of non-negative integers such that either σ or $\overline{\sigma}$ is finite, and similarly either ρ or $\overline{\rho}$ is finite. Then the (σ, ρ)-Dominating Set of Size at least $n - k$ problem is in* FPT.

Proof. We present an algorithm that is based on the bounded search tree technique. At the beginning, the algorithm includes all vertices into the set S and then tries recursively excluding some of the vertices to make S (σ, ρ)-dominating. Once a vertex is excluded, it is never included in the set again (in the same branch of the algorithm). Obviously at most k vertices can be excluded from S to fulfill the size constraint.

We call a vertex v *satisfied* (with respect to the current set S) if it has the right number of neighbors in S (i.e., $v \in S$ and $|N(v) \cap S| \in \sigma$ or $v \notin S$ and $|N(v) \cap S| \in \rho$), otherwise we call it *unsatisfied*. Let $\widetilde{p}_{\max}$ denote $\max \sigma$ if σ is finite and $\max \overline{\sigma}$ if $\overline{\sigma}$ is finite. Similarly, let $\widetilde{q}_{\max}$ denote $\max \rho$ or $\max \overline{\rho}$. (Here it is assumed that $\max \emptyset = -\infty$.) Finally, let b denote $\max\{\widetilde{p}_{\max}, \widetilde{q}_{\max}\}$. We call a vertex v *big* if $\deg(v) > b + k$ and *small* otherwise.

The main idea of the algorithm is that there is at most one way to make an unsatisfied big vertex satisfied (to exclude it from S) and if this does not work, there is no (σ, ρ)-dominating set at all. On the other hand to satisfy a small vertex, we must either exclude it or one of its first b neighbors that were in S.

```
Procedure Exclude(S)
  if there is no unsatisfied vertex then Return(S);Exit;
  if |S| = n − k then Halt;
  let v be an unsatisfied vertex;
  if v is big then
  |  if v ∈ S and ρ is infinite then Exclude(S \ v);
  |_ else Halt;
  else
  |  if v ∈ S then Exclude(S \ v);
  |  let {u_1, ..., u_r} = S ∩ N(v) be the set of included neighbors of v;
  |  if r = 0 then Halt;
  |_ for i := 1 to min{b + 1, r} do Exclude(S \ {u_i}).
```

The algorithm consists of a single call Exclude(V) and returns the set S returned by the procedure or NO if no set was returned.

Claim 1: *The algorithm* Exclude(V) *runs in* $O((b+2)^k \cdot n + m)$ *time*. First observe that since each recursive call reduces the size of S by one, there can be at most k nested calls. With at most $b+2$ recursive calls made by one call this means altogether $O((b+2)^k)$ calls of Exclude. Note that we can decide which vertices are satisfied at the beginning in $O(m)$ time and then update this before each recursive call in $O(n)$ time. Since all the other operations in one call can be also carried out in $O(n)$ time, we get the claimed running time.

Claim 2: *If there is a* (σ, ρ)*-dominating set* O *of size at least* $n - k$*, then the algorithm returns some* (σ, ρ)*-dominating set of size at least* $n - k$. We show that there is always a branch of the algorithm that keeps $O \subset S$, thus we cannot miss O. A big vertex has always at least b neighbors in any set of vertices of size at least $n - k$. Hence if the unsatisfied vertex v is big, this means that σ is finite, and since it is satisfied by O, this means $v \notin O$, which is the only branch tested by the algorithm (ρ must be infinite, because otherwise there would be no (σ, ρ)-dominating set). On the other hand, if v is small and in S, then either $v \notin O$ or $\{u_1, \ldots, u_{\min\{b+1,r\}}\} \not\subset O$ because otherwise v would have either the same neighborhood or at least $b + 1$ neighbors in O and it would remain unsatisfied, since σ is finite and thus $b + 1 \notin \sigma$. Similarly for the other case.

Clearly, if the algorithm returns a set S, then S is a (σ, ρ)-dominating set of size at least $n - k$, since all vertices are satisfied. This together with the two claims completes the proof. □

4. Complexity for the case $\sigma, \rho \in$ {EVEN, ODD}

We proved that our problems are in FPT for the dual parameterization if σ, ρ are finite or co-finite. Now we show that it cannot be expected that similar results could be established in more general cases. Particularly, these problems are W[1]-hard for $\sigma, \rho \in$ {**EVEN**, **ODD**}. Note that the sets **EVEN** and **ODD** constitute the simplest examples of sets that are neither finite nor co-finite. This was also one of the reasons why similar problems were studied in [22].

4.1. Complexity for the parity problems

Recall that it was shown by Downey et al. [14] that for a bipartite graph $G = (R, B, E)$, deciding the existence of an odd set of red vertices (i.e. of a subset of R such that each blue vertex from B has an odd number of neighbors in this set) of size k, an odd set of size at most k, and an even set of size k are W[1]-hard problems. As a counterpart to these results, we first show that all four parity problems for red/blue bipartite graphs are hard under the dual parameterization.

Theorem 14. *The* Even Set of Size $r - k$, Even Set of Size at least $r - k$, Odd Set of Size $r - k$, *and* Odd Set of Size at least $r - k$ *problems are all* W[1]*-hard.*

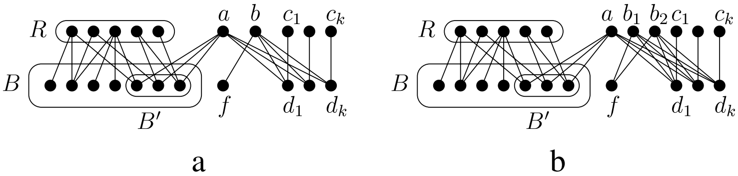


Fig. 2. Construction of H in the proof of Lemma 15.

Proof. We reduce from Odd Set of Size at most k:

Input: A bipartite graph $G = (R, B, E)$.

Parameter: k.

Question: Is there an odd set in R of size at most k?

It should be noted that W[1]-hardness was stated in [14] for the exact variant of the problem (i.e. for the question: Is there an odd set in R of size k?), but for our variant of the question, the proof of [14] works the same. We show that the problem remains W[1]-hard if all blue vertices have odd degrees and also if all of them have even degrees. Then we deduce the claims by considering the set $R \setminus S$ for a would-be odd set $S \subset R$.

Lemma 15. *The* Odd Set of Size at most k *problem remains* W[1]*-hard even if*

1. *all blue vertices have odd degrees;*
2. *all blue vertices have even degrees.*

Proof. To prove the first claim, we reduce from Odd Set of Size at most k. Let $G = (R \cup B, E)$ be an instance of the problem and let $B' \subseteq B$ be the set of vertices of even degree. If $B' = \emptyset$, then we let $H = G$. Otherwise the graph H is constructed as follows. Red vertices $a, b, c_1, \dots, c_k$ and blue vertices $f, d_1, \dots, d_k$ are added to G. Then all vertices of B' are joined by edges with a, vertices a and b are joined with $d_1, \dots, d_k$, vertex b is connected with f, and finally each vertex c_i is joined with d_i. The construction of H is shown in Fig. 2(a). Clearly, all blue vertices of H have odd degrees. Let $k' = k$ if $B' = \emptyset$ and $k' = k+1$ otherwise. We prove that G has an odd set of size k if and only if H has an odd set of size k'.

If $B' = \emptyset$, then the claim is trivial. Suppose that B' contains at least one vertex. If $S \subseteq R$ is an odd set in G of size at most k, then $S \cup \{b\}$ is an odd set in H which contains at most $k + 1$ vertices. Now assume that H has an odd set S of size at most k'. It is easy to see that $b \in S$. Suppose that $a \in S$. But in this case $c_1, \dots, c_k \in S$, and S contains at least $k + 2$ vertices. This contradiction proves that $a \notin S$, and therefore $c_1, \dots, c_k \notin S$. It remains to note that $S \setminus \{b\}$ is an odd set in G of size at most k.

For the proof of the second claim, we again reduce from Odd Set of Size at most k, and the reduction uses same ideas. Here let $B' \subseteq B$ be the set of vertices of odd degree. We construct the graph H (see Fig. 2(b)) in a similar way as it was done above. The only difference is that for the case $B' \neq \emptyset$, vertex b is replaced by two vertices b_1 and b_2 with same neighborhoods. Parameter k' is defined as before. We prove that G has an odd set of size k if and only if H has an odd set of size k'.

If $B' = \emptyset$, then the claim is trivial. Suppose that B' contains at least one vertex. If $S \subseteq R$ is an odd set in G of size at most k, then $S \cup \{b_1\}$ is an odd set in H which contains at most $k + 1$ vertices. Now assume that H has an odd set S of size at most k'. It is easy to see that either $b_1 \in S, b_2 \notin S$ or $b_1 \notin S, b_2 \in S$. Suppose that $a \in S$. But in this case $c_1, \dots, c_k \in S$, and S contains at least $k + 2$ vertices. This contradiction proves that $a \notin S$, and therefore $c_1, \dots, c_k \notin S$. It remains to note that $S \setminus \{b_1, b_2\}$ is an odd set in G of size at most k. □

To complete the proof W[1]-hardness of Even Set of Size at least $r - k$ it is enough to observe that if all blue vertices have odd degrees then S is an odd set of size at most k if and only if $R \setminus S$ is an even set of size at least $r - k$. The proof of W[1]-hardness for the Even Set of Size $r - k$ problem is similar, we reduce from the exact variant of the Odd Set of Size k problem.

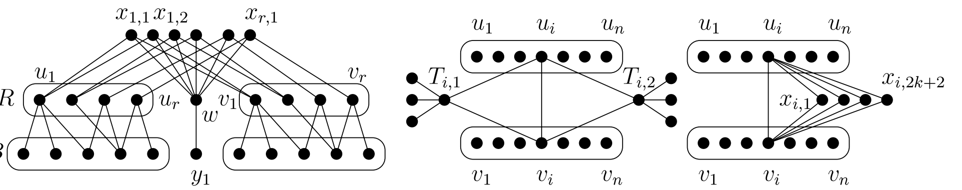
For the proof of W[1]-hardness of Odd Set of Size at least $r - k$, it is sufficient to notice that if all blue vertices have even degrees, then S is an odd set of size at most k if and only if $R \setminus S$ is an odd set of size at least $r - k$. The proof of W[1]-hardness for the Odd Set of Size $r - k$ problem is the same, we only reduce from the exact variant of the Odd Set of Size at most k problem. □

*4.2. Complexity of the (**EVEN–ODD**)-domination problems*

The main result of this section is the hardness of the (**EVEN–ODD**)-domination problems under the dual parameterization. Note that, in contrast to the previous subsection, these results are for general graphs.

Theorem 16. *Let* $\sigma, \rho \in \{$**EVEN**, **ODD**$\}$. *Then the* (σ, ρ)-Dominating Set of Size $n - k$ *and* (σ, ρ)-Dominating Set of Size at least $n - k$ *problems are* W[1]*-hard.*

Proof. We prove this theorem for the (σ, ρ)-Dominating Set of Size at least $n - k$ problem. The proof for the (σ, ρ)-Dominating Set of Size $n - k$ is done by similar arguments. We consider several cases.



(a) $\sigma = \rho = \mathbf{ODD}$ (b) $\sigma = \mathbf{ODD}, \rho = \mathbf{EVEN}$ (c) $\sigma = \mathbf{EVEN}, \rho = \mathbf{ODD}$

Fig. 3. Construction of H.

1. $\sigma = \rho =$ **EVEN**. We use the following lemma.

Lemma 17. *The* Even Set of Size at least $r - k$ *problem remains* W[1]-*hard if all red vertices have even degrees.*

Proof. We reduce from the Even Set of Size at least $r - k$ problem by replacing each blue vertex by two vertices with the same neighborhoods. Trivially $S \subseteq R$ is an even set in the obtained graph if and only if it is an even set in the original graph. □

If all red vertices have even degrees then $S \subseteq R$ is an even set if and only if $S \cup B$ is an (**EVEN**, **EVEN**)-dominating set. It follows immediately that G has an even set of size at least $r - k$ if and only if G has a (σ, ρ)-dominating set of size at least $n - k$ for $\sigma = \rho =$ **EVEN**.

For the exact variant of the problem it is necessary to force all vertices of B (more precisely all vertices of $V \setminus R$) to be in any (**EVEN**, **EVEN**)-dominating set of size $n - k$. This can be done by adding to each vertex $b \in B$ an adjacent clique with an even number of at least $k + 1$ vertices. Any vertex of the clique not included in the dominating set would have one neighbor less than any included vertex of the clique, which is impossible. Since at least one vertex of the clique is included, whole of the clique must be included in any (**EVEN**, **EVEN**)-dominating set of size $n - k$ and hence b must be included as well.

2. $\sigma = \rho =$ **ODD**.

Lemma 18. *The* Odd Set of Size at least $r - k$ *problem remains* W[1]-*hard if all red vertices have odd degrees.*

Proof. We reduce from the Odd Set of Size at least $r - k$ problem. Consider two copies of the instance of this problem. Denote by $u_1, \ldots, u_r$ the red vertices of the first graph, and by $v_1, \ldots, v_r$ the red vertices of the second graph. Assume that vertices $u_1, \ldots, u_s$ (vertices $v_1, \ldots, v_s$ correspondingly) have odd degrees, and the other red vertices have even degrees. We introduce one additional red vertex w. For each $i \in \{1, \ldots, s\}$, two blue vertices $x_{i,1}$ and $x_{i,2}$ are added and joined by edges with $u >_i$, v_i and w. For each $i \in \{s + 1, \ldots, r\}$, we add one blue vertex $x_{i,1}$ and join it with u_i, v_i and w. If $r - s$ is even, then one blue vertex y_1 is introduced and joined with w, and otherwise two blue vertices y_1, y_2 are added and joined with w. Denote the obtained graph by H (see Fig. 3(a)). Clearly all red vertices of H have odd degrees. Let $k' = 2k$. Now we prove that the original graph G has an odd set of size at least $r - k$ if and only if H has an odd set of size at least $2r + 1 - k'$.

Suppose that G has an odd set S of size at least $r - k$. Denote by S_1 the copy of this set for the first copy of G, and by S_2 the odd set for the second copy. It can be easily checked that $S_1 \cup S_2 \cup \{w\}$ is an odd set in H of size at least $2r - 2k + 1$. Now assume that S is an odd set in H of size at least $2r + 1 - k'$. Note that $w \in S$ because of the presence of the vertex y_1. Let $S_1 = S \cap \{u_1, \ldots, u_r\}$ and $S_2 = S \cap \{v_1, \ldots, v_r\}$. We claim that $u_i \in S_1$ if and only if $v_i \in S_2$. Suppose that $u_i \in S_1$. Since $x_{i,1}$ has three red neighbors u_i, v_i, w and $w, u_i \in S$, $v_i \in S$ also. It remains to note that S_1 (or S_2) is an odd set of size at least $r - k$ in G. □

It is easy to see that if all red vertices in the red/blue bipartite graph G have odd degrees then $S \subset R$ is an odd set if and only if $S \cup B$ is an (**ODD**, **ODD**)-dominating set in G. Correspondingly G has an odd set of size at least $r - k$ if and only if G has a (σ, ρ)-dominating set of size at least $n - k$ for $\sigma = \rho =$ **ODD**.

For the exact variant it is again necessary to force all vertices of B to be in any (**ODD**, **ODD**)-dominating set of size $n - k$. This is done in similar way as in the case 1 (the case $\sigma = \rho =$ **EVEN**) by adding to each vertex $b \in B$ two cliques adjacent with b each having an odd number of at least $k + 1$ vertices.

3. $\sigma =$ **ODD** and $\rho =$ **EVEN**. We reduce from the (σ, ρ)-Dominating Set of Size at least $n - k$ problem for $\sigma = \rho =$ **EVEN**. Consider two copies of the instance of this problem. Denote by $u_1, \ldots, u_n$ the vertices of the first copy of the graph G, and by $v_1, \ldots, v_n$ the vertices of the second copy. For each $i \in \{1, \ldots, n\}$, vertices u_i and v_i are connected by an edge, two copies of stars $K_{1,2k+1}$ denoted by $T_{i,1}$ and $T_{i,2}$ are introduced, and central vertices of these stars are joined with u_i and v_i. Denote the obtained graph by H (see Fig. 3(b)). It is easy to see that H has $n' = 2n(2k + 3)$ vertices. Let $k' = 2k$. We claim that G has an (**EVEN**, **EVEN**)-dominating set of size at least $n - k$ if and only if H has an (**ODD**, **EVEN**)-dominating set of size at least $n' - k'$.

Suppose that S is an (**EVEN**, **EVEN**)-dominating set of size at least $n - k$ in G. Let S_1 be the set of vertices of S in the first copy of G, and correspondingly let S_2 be this set in the second copy. It can be straightforwardly checked that $S' = S_1 \cup S_2 \cup \bigcup_{i=1}^{n} (V(T_{i,1}) \cup V(T_{i,2}))$ is an (**ODD**, **EVEN**)-dominating set of size at least $n' - k'$ in H.

Now assume that S' is an (**ODD**, **EVEN**)-dominating set of size at least $n' - k'$ in H. Consider some star $T_{i,j}$. Since this star has $2k + 1$ leaves and $n' - |S'| \le 2k$, at least one leaf of $T_{i,j}$ is included in S'. Thus the central vertex of the star is in S' as $\sigma = \mathbf{ODD}$, and hence every leaf is in S' as $\rho = \mathbf{EVEN}$. So all vertices of the star are included in S'. It means that $\bigcup_{i=1}^{n}(V(T_{i,1}) \cup V(T_{i,2})) \subseteq S'$. Since the central vertex of each star has odd degree, it has an even number of neighbors in $\{u_i, v_i\}$, i.e. either $u_i, v_i \in S'$ or $u_i, v_i \notin S'$. Therefore each vertex $u_i \in S'$ if and only if $v_i \in S'$. It remains to note that $S = \{u_1, \dots, u_n\} \cap S'$ is an (**EVEN**, **EVEN**)-dominating set in G of size at least $n - k$.

4. $\sigma = \mathbf{EVEN}$ and $\rho = \mathbf{ODD}$. We reduce from the (σ, ρ)-Dominating Set of Size at least $n - k$ problem for $\sigma = \rho = \mathbf{ODD}$. Similarly as in the case 3, consider two copies of the instance of this problem. Denote by $u_1, \dots, u_n$ the vertices of the first copy of the graph G, and by $v_1, \dots, v_n$ the vertices of the second copy. For each $i \in \{1, \dots, n\}$, vertices u_i and v_i are joined by an edge, and then $2k + 2$ vertices $x_{i,1}, \dots, x_{i,2k+2}$ are introduced and joined with u_i and v_i. Denote the obtained graph by H (see Fig. 3(c)). This graph has $n' = 2n(k + 2)$ vertices. Let $k' = 2k$. We prove that G has an (**ODD**, **ODD**)-dominating set of size at least $n - k$ if and only if H has an (**EVEN**, **ODD**)-dominating set of size at least $n' - k'$.

Suppose that S is an (**ODD**, **ODD**)-dominating set of size at least $n - k$ in G. Let S_1 be the set of vertices of S in the first copy of G, and correspondingly let S_2 be this set in the second copy. It is easy to see that $S' = S_1 \cup S_2 \cup \bigcup_{i=1}^{n}\{x_{i,1}, \dots, x_{i,2k+2}\}$ is an (**EVEN**, **ODD**)-dominating set of size at least $n' - k'$ in H.

Now assume that S' is an (**EVEN**, **ODD**)-dominating set of size at least $n' - k'$ in H. For each $i \in \{1, \dots, n\}$, at least one vertex $x_{i,j}$ is in S', since otherwise there are at least $k' + 2$ vertices that are not included in S'. Therefore, either $u_i, v_i \in S'$ or $u_i, v_i \notin S'$ since $\sigma = \mathbf{EVEN}$. Hence all vertices $x_{i,1}, \dots, x_{i,2k+2}$ are in S' since otherwise $x_{i,j} \notin S'$ would have an even number of neighbors in S' contrary to $\rho = \mathbf{ODD}$. Also for each vertex u_i, $u_i \in S'$ if and only if $v_i \in S'$. Now note that $S = \{u_1, \dots, u_n\} \cap S'$ is an (**ODD**, **ODD**)-dominating set in G of size at least $n - k$. □

5. Complexity of the (σ, ρ)-Dominating Set of Size (at most) k problem for sparse graphs

It is well known that many problems which are difficult for general graphs can be solved efficiently for sparse graphs.

It can be easily seen (by an argument somewhat similar to the one below) that due to the results of Courcelle et al. [8] (see also [1,4,5]), that for both σ and ρ either finite or co-finite on graphs of bounded cliquewidth one can in linear time decide the existence of a (σ, ρ)-dominating set or even find the minimum and maximum cardinality of such a set.

We prove that the existence of (σ, ρ)-dominating set of a particular size can be efficiently decided on much more general class of sparse graphs, for which we use very general results established in [9]. To state the results, the following concept of classes of graphs of bounded expansion, introduced by Nešetřil and Ossona de Mendez in [28] and in the series of journal papers [30–32], and also the concept of nowhere dense graphs introduced in [29,33] are needed.

An *r-shallow minor* of a graph G is a graph that can be obtained from G by removing some of the vertices and edges of G and then contracting vertex-disjoint subgraphs of radius at most r to single vertices (removing arising parallel edges to keep the graph simple). The *density* of a graph is the number of its edges divided by the number of its vertices. The *grad (greatest reduced average density) of rank r* of a graph G is equal to the largest average density of an r-shallow minor of G. The grad of rank r of G is denoted by $\triangledown_r(G)$. A class $\mathcal{C}$ of graphs has *bounded expansion* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $G \in \mathcal{C}$ and every $r \ge 0$ integer $\triangledown_r(G) \le f(r)$. A class of graphs $\mathcal{C}$ is said to be *nowhere dense* if for every r there is a graph H such that H is not an r-shallow minor of any $G \in \mathcal{C}$.

Examples of classes of graphs with bounded expansion include proper minor-closed classes of graphs, classes of graphs with bounded maximum degree, classes of graphs excluding a subdivision of a fixed graph, classes of graphs that can be embedded on a fixed surface with bounded number of crossings per each edge and many others. It can be shown, that every class of graphs with locally bounded treewidth or locally excluding a minor is nowhere dense.

The following results were proved in [9], similar results also appeared independently in [15].

Theorem 19 ([9]). *Let $\mathcal{C}$ be a nowhere dense class of graphs. For every $\epsilon > 0$ there is a computable function $f : \mathbb{N} \to \mathbb{N}$ and an algorithm which, given $G \in \mathcal{C}$ and $\phi \in FO$, decides whether G satisfies ϕ in time $f(|\phi|) \cdot |G|^{1+\epsilon}$.*

Theorem 20 ([9]). *Let $\mathcal{C}$ be a class of graphs of bounded expansion. There is a computable function $f : \mathbb{N} \to \mathbb{N}$ and a linear-time parameterized algorithm which, given $G \in \mathcal{C}$ and $\phi \in FO$, decides whether G satisfies ϕ.*

As the corollary of the above theorems we get the following.

Theorem 21. *Let σ and ρ be recursive sets of non-negative integers. Then the (σ, ρ)-*Dominating Set of Size (at most) *k problem is* FPT *(with parameter k) on classes of graphs of bounded expansion and nowhere dense graph classes.*

Proof. It suffices to provide for each k the FOL formula for "there is a (σ, ρ)-dominating set of size (at most) k". First note that a set of size (at most) k is (σ, ρ)-dominating if and only if it is (σ_k, ρ_k)-dominating, where $\sigma_k = \sigma \cap \{0, \dots, k\}$ and $\rho_k = \rho \cap \{0, \dots, k\}$ as no vertex can have more than k neighbors in a set of size (at most) k. As both σ and ρ are recursive, sets σ_k and ρ_k can be computed in time $a(k)$ for some $a : \mathbb{N} \to \mathbb{N}$ solely depending on k.

Now we will gradually build a vocabulary, which will at the end allow us to formulate the desired formula, deciding, whether vertices $x_1, \dots, x_k$ form a (σ_k, ρ_k)-dominating set of size (exactly) k. Then the "at most" k formula can be easily obtained as a conjunction of formulas for sizes 0 to k. In what follows, $1 \leq r \leq k$, and formulas (except for those stated) also contain free variables $x_1, \ldots, x_k$.

$$\text{selected}(x) = \bigvee_{i=1}^{k} (x = x_i)$$

$$\text{at\_least\_}r\text{\_sel\_neighs}(v) = \exists y_1 \exists y_2 \ldots \exists y_r \left(\bigwedge_{1 \leq i < j \leq r} \neg(y_i = y_j) \right) \wedge \left(\bigwedge_{i=1}^{r} \text{selected}(y_i) \wedge \text{adj}(v, y_i) \right)$$

$$\text{exact\_}r\text{\_sel\_neighs}(v) = (\text{at\_least\_}r\text{\_sel\_neighs}(v)) \wedge \neg(\text{at\_least\_}(r+1)\text{\_sel\_neighs}(v))$$

$$\text{exact\_0\_sel\_neighs}(v) = \neg(\text{at\_least\_1\_sel\_neighs}(v))$$

$$\text{is\_satisfied}(v) = \left(\text{selected(v)} \wedge \bigvee_{r \in \sigma_k} (\text{exact\_}r\text{\_sel\_neighs}(v)) \right) \vee \left(\neg(\text{selected}(v)) \wedge \bigvee_{r \in \rho_k} (\text{exact\_}r\text{\_sel\_neighs}(v)) \right).$$

Now the desired formula can be expressed as

$$\text{there\_is\_a\_}(\sigma, \rho)\text{-dominating\_set\_of\_size\_}k = \exists x_1 \exists x_2 \ldots \exists x_k \left(\bigwedge_{1 \leq i < j \leq k} \neg(x_i = x_j) \right) \wedge \forall v(\text{is\_satisfied}(v)).$$

It is easy to see, that the formula can be constructed in a time solely dependent on k and, hence, the result follows as a corollary of Theorems 19 and 20. □

6. Conclusion

In this work we studied the parameterized complexity of the (σ, ρ)-domination problems. Our results give more or less general picture for finite sets σ and ρ. Still, it would be interesting to extend these results for (σ, ρ)-Dominating Set of Size k in the case $0 \in \rho$. We suppose that it would be a challenging task to investigate the case of (possibly) infinite sets. We presented some partial results when σ or ρ can be co-finite, but we leave open the question about the parameterized complexity of (σ, ρ)-Dominating Set of Size (at most) k. Table 1 indicates that we can expect that these problems are W[1] or W[2]-hard for majority of sets.

Another direction of the research is to consider the parameterized complexity for different graphs classes. Particularly, we proved that the (σ, ρ)-domination problems are FPT for graphs of bounded expansion and nowhere dense graph classes when parameterized by the solution size. It is known that some domination problems are FPT for the more general class of *degenerate* graphs (see e.g. [2,24]). These results can be easily generalized for (σ, ρ)-domination problems for some special sets σ and ρ. It is an interesting open problem whether the results of Theorem 21 can be extended to degenerate graphs.

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