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On the existence of solutions for the Schrödinger–Poisson equations

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ABSTRACT

In this paper, we are concerned with the system of Schrödinger–Poisson equations

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & \text{in } \mathbf{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbf{R}^3. \end{cases} \tag{$*$}$$

Under certain assumptions on V and f, the existence and multiplicity of solutions for $(*)$ are established via variational methods.

1. Introduction

In this paper, we consider the following system of Schrödinger–Poisson equations

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & \text{in } \mathbf{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbf{R}^3. \end{cases} \tag{1.1}$$

System (1.1) was first introduced in [6] as a model describing solitary waves for the nonlinear stationary Schrödinger equations interacting with the electrostatic field. We note that system (1.1) is also called Schrödinger–Maxwell equations, for more details on the physical aspects of this problem, we refer to [6] and references therein.

In recent years, problem (1.1) with $V(x) \equiv 1$ or being radially symmetric, has been widely studied under various conditions on f, see for example [2,8–12,16–18]. The case of positive and bounded nonradial potential V has been considered in [21], when f is asymptotically linear, and in [3,4], when $f(x,u) = |u|^{p-1}u$, with $3 < p < 5$. Moreover, in [3], the existence of ground state solutions for problem (1.1) has been proved in several situations, including the positive constant potential case. In [4], the authors considered problem (1.1) with a class of more general potential which may be unbounded from below, and the existence of ground state solutions was proved. Let $H^1(\mathbf{R}^3)$ and $D^{1,2}(\mathbf{R}^3) = \{u \in L^6(\mathbf{R}^3)\colon |\nabla u| \in L^2(\mathbf{R}^3)\}$ denote the usual Sobolev spaces. We recall here that $(u,\phi) \in H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$ is said to be a ground state solution to problem (1.1), if (u,ϕ) solves (1.1) and minimizes the action functional associated to (1.1) among all possible nontrivial solutions. However, as pointed out in [3,4], because of some technical difficulties, the existence of ground state solutions was established in [3,4] only for $3 < p < 5$ when V was not a constant. In this paper, motivated by techniques used in [14,15], we generalize results in [3,4] to the case $2 < p \leqslant 3$.

In the first part of this paper, we are interested in the existence of ground state solution of (1.1) with pure power type nonlinearity so that (1.1) can be rewritten as

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$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-1}u, & \text{in } \mathbf{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbf{R}^3, \end{cases} \tag{1.2}$$

where $2 < p \leqslant 3$. On the potential V, we make the following assumption

(V_1) $V \in C(\mathbf{R}^3, \mathbf{R})$.
(V_2) $V(\infty) := \liminf_{|x|\to\infty} V(x) \geqslant V(x)$.
(V_3) $\inf\sigma(-\Delta + V(x)) > 0$, where $\sigma(-\Delta + V(x))$ denotes the spectrum of the self-adjoint operator $-\Delta + V(x) : H^2(\mathbf{R}^3) \to L^2(\mathbf{R}^3)$, i.e.,

$$\inf\sigma\big(-\Delta + V(x)\big) = \inf_{u\in H^1(\mathbf{R}^3)\setminus\{0\}} \frac{\int_{\mathbf{R}^3} |\nabla u|^2 + V(x)u^2\,dx}{\int_{\mathbf{R}^3} |u|^2\,dx} > 0.$$

(V_4) V is weakly differentiable, and satisfies $(\nabla V(x), x) \in L^\infty(\mathbf{R}^3) \cup L^{3/2}(\mathbf{R}^3)$, and

$$2V(x) + \big(\nabla V(x), x\big) \geqslant 0 \quad \text{a.e. } x \in \mathbf{R}^3,$$

where $(\cdot,\cdot)$ is the usual inner product in $\mathbf{R}^3$.

Our main result is the following.

Theorem 1.1. *Assume (V_1)–(V_4) and $2 < p \leqslant 3$. Then problem (1.2) has a ground state solution in $H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$.*

We will use variational methods to prove Theorem 1.1. The main difficulty consists in the lack of compactness of the Sobolev embedding, since (1.2) is set on $\mathbf{R}^3$. We recover the compactness by using a version of global compactness lemma, see Lemma 3.6. On the other hand, it seems difficult to get the boundedness of a (*PS*) sequence when $2 < p \leqslant 3$. To overcome the difficulty, motivated by [15,16], we use Jeanjean's result [14] to construct a special bounded (*PS*) sequence then we can complete the proof of Theorem 1.1.

In the second part of this paper, we deal with problem (1.1) with a periodic potential. We assume

(V_5) $V \in C(\mathbf{R}^3, \mathbf{R})$.
(V_6) $V(x)$ is 1-periodic in each x_i, $i = 1, 2, 3$, $\min_{\mathbf{R}^3} V(x) > 0$.
(f_1) $f(x,u)$ is 1-periodic in each x_i, $i = 1, 2, 3$.
(f_2) $f_u(x,u)$ exists everywhere, and $f_{uu}(x,u)$ exists for $u \neq 0$, f_u is a Caratheodory function $f(x,0) = f_u(x,0) = 0$ for all x. There are $C \geqslant 0$ and $p_1, p_2 \in [4,6)$ with $p_1 \leqslant p_2$ such that

$$\big|f_{uu}(x,u)\big| \leqslant C\big(|u|^{p_1-3} + |u|^{p_2-3}\big)$$

holds for every $u \neq 0$ and for every x.
(f_3) There exists $\mu \geqslant 4$ such that

$$0 < \mu F(x,u) \leqslant uf(x,u), \quad \text{for all } (x,u) \in \big(\mathbf{R}^3, \mathbf{R}\big),$$

where $F(x,u) = \int_0^u f(x,t)\,dt$.
(f_4) For every $u \neq 0$ and for every x, it holds that

$$f_u(x,u)u^2 > f(x,u)u.$$

In the following theorem, we consider the multiplicity of solutions of (1.1). We note that if (u_0, ϕ_0) is a solution of (1.1), then so are all elements of the orbit of (u_0, ϕ_0) under the action of $\mathbf{Z}^3$, $\mathcal{O}(u_0, \phi_0) := \{(u_0(\cdot - k), \phi_0(\cdot - k)):\ k \in \mathbf{Z}^3\}$. Two solutions (u_1, ϕ_1) and (u_2, ϕ_2) are said to be geometrically distinct if $\mathcal{O}(u_1, \phi_1)$ and $\mathcal{O}(u_2, \phi_2)$ are disjoint.

Theorem 1.2. *Assume (V_5), (V_6) and (f_1)–(f_4). Then problem (1.1) has infinitely many geometrically distinct solutions in $H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$.*

We remark that assumptions (f_1)–(f_4) were introduced by [5] to study the nonlinear Schrödinger equation and (f_1)–(f_4) hold for $f(x,u) = |u|^{p-1}u$ with $3 \leqslant p < 5$. Since we do not assume here that $f(x,u)$ is odd with respect to u, Theorem 1.2 is a generalization and complement of Theorem 2.1 in [2] in the case $3 \leqslant p < 5$.

The paper is organized as follows. In Section 2, we outline the variational setting and give some preliminary lemmas. The proof of Theorems 1.1 and 1.2 are given respectively in Sections 3 and 4. In Appendix A, we give the proofs of Lemmas 2.2 and 2.3.

2. Preliminary lemmas

In this section we outline the variational framework for problem (1.1) and give some preliminary lemmas.

Throughout the paper, we denote the norm of $H^1(\mathbf{R}^3)$ by

$$\|u\| = \left(\int_{\mathbf{R}^3} \left(|\nabla u|^2 + u^2\right) dx \right)^{1/2},$$

the norm of $D^{1,2}(\mathbf{R}^3)$ by

$$\|u\|_D = \left(\int_{\mathbf{R}^3} |\nabla u|^2 \, dx \right)^{1/2}.$$

We also use the notation

$$\|u\|_V = \left(\int_{\mathbf{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx \right)^{1/2},$$

which is a norm equivalent to $\|\cdot\|$ under (V_1)–(V_3). And by $\|\cdot\|_q$ we denote the usual L^q-norm, C stands for different positive constants.

For every $u \in H^1(\mathbf{R}^3)$, the Lax–Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbf{R}^3)$ such that

$$-\Delta\phi_u = u^2. \tag{2.1}$$

Moreover, ϕ_u can be expressed by

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbf{R}^3} u^2(y) \frac{1}{|x-y|} \, dy.$$

From (2.1), it is easy to see

$$\|\phi_u\|_D \leqslant C\|u\|_{12/5}^2 \leqslant C\|u\|^2. \tag{2.2}$$

It can be proved that $(u,\phi) \in H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$ is a solution of (1.1) if and only if $u \in H^1(\mathbf{R}^3)$ is a critical point of the functional $I : H^1(\mathbf{R}^3) \to \mathbf{R}$ defined as

$$I(u) = \frac{1}{2} \int_{\mathbf{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx + \frac{1}{4} \int_{\mathbf{R}^3} \phi_u u^2 \, dx - \int_{\mathbf{R}^3} F(x,u) \, dx$$

and $\phi = \phi_u$, where $F(x,u) = \int_0^u f(x,t)\,dt$, see for instance [6].

We collect some properties of the functions ϕ_u, see [9] and [17].

Lemma 2.1. *For any $u \in H^1(\mathbf{R}^3)$, we have*

(i) $\|\phi_u\|_D \leqslant C\|u\|^2$, *where C is independent of u. As a consequence there exists $C' > 0$ such that*

$$\int_{\mathbf{R}^3} \phi_u u^2 \, dx \leqslant C'\|u\|_{12/5}^4;$$

(ii) $\phi_u \geqslant 0$;

(iii) *for any $t > 0$, $\phi_{u_t}(x) = t^2\phi_u(tx)$, where $u_t(x) = t^2u(tx)$.*

Define $N : H^1(\mathbf{R}^3) \to \mathbf{R}$ by

$$N(u) = \int_{\mathbf{R}^3} \phi_u u^2 \, dx.$$

The next lemma shows that the functional N and its derivative N', N'' posses *BL splitting property*, which is similar to the well-known Brezis–Lieb Lemma [7], see [5] for its definition.

Lemma 2.2. *Let $u_n \rightharpoonup u$ in $H^1(\mathbf{R}^3)$ and $u_n \to u$ a.e. in $\mathbf{R}^3$. Then as $n \to \infty$*

(i) $N(u_n - u) = N(u_n) - N(u) + o(1)$;
(ii) $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$, *in* $H^{-1}(\mathbf{R}^3)$;
(iii) $N''(u_n - u) = N''(u_n) - N''(u) + o(1)$ *in* $L(H^1(\mathbf{R}^3), H^{-1}(\mathbf{R}^3))$.

Lemma 2.3.

(i) $N' : H^1(\mathbf{R}^3) \to H^{-1}(\mathbf{R}^3)$ *is weakly sequentially continuous*;
(ii) $N''(u) \in L(H^1(\mathbf{R}^3), H^{-1}(\mathbf{R}^3))$ *is compact for any* $u \in H^1(\mathbf{R}^3)$.

We will give the proofs of Lemmas 2.2 and 2.3 in Appendix A.

3. Proof of Theorem 1.1

In this section we study problem (1.2) and give the proof of Theorem 1.1. The functional I is rewritten as

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx + \frac{1}{4}\int_{\mathbf{R}^3} \phi_u u^2 \, dx - \frac{1}{p+1}\int_{\mathbf{R}^3} |u|^{p+1} \, dx. \tag{3.1}$$

When $3 < p < 5$, in [3,4], the authors proved the existence of ground state solutions by minimizing I restricted to the Nehari manifold

$$N = \left\{u \in H^1\left(\mathbf{R}^3\right) \setminus \{0\}\colon \left\langle I'(u), u\right\rangle = 0\right\}.$$

The key point in their arguments is that N is a natural constraint for the functional I. However, when $2 < p \leqslant 3$, this fact is no longer true and the structure of N is complicated. In this situation, as in [3,17], we introduce another manifold M_λ^∞ which is a natural constraint for the limiting functional of I when $2 < p \leqslant 3$. On the other hand, we find that every (PS) sequence is bounded when $3 < p < 5$ because a variant of global Ambrosetti–Rabinowitz condition is satisfied when $3 < p < 5$, see e.g. [9]. While it seems to be difficult to get the boundedness of a (PS) sequence when $2 < p \leqslant 3$. To overcome this difficulty, motivated by [15,16], we use Jeanjean's result [14], which generalizes Struwe's argument [19,20]. We consider a family of functionals defined by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbf{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) dx + \frac{1}{4}\int_{\mathbf{R}^3} \phi_u u^2 \, dx - \frac{\lambda}{p+1}\int_{\mathbf{R}^3} |u|^{p+1} \, dx, \tag{3.2}$$

for $\lambda \in [1/2, 1]$. By Jeanjean's result [14], we can see that for a.e. $\lambda \in [1/2, 1]$, there exist a bounded $(PS)_{c_\lambda}$ sequence for some $c_\lambda > 0$. Then with the aid of a version of global compactness lemma, we get a nontrivial critical point u_λ of I_λ for a.e. $\lambda \in [1/2, 1]$. By choosing $\lambda_n \to 1$, we get a sequence of $\{u_{\lambda_n}\}$ being the critical points of I_{λ_n}.

Subsequently, a pair of functions $(u_{\lambda_n}, \phi_{u_{\lambda_n}})$ is a solution to the following parameterized system

$$\begin{cases} -\Delta u + V(x)u + \phi u = \lambda_n |u|^{p-1}u, & \text{in } \mathbf{R}^3, \\ -\Delta\phi = u^2, & \text{in } \mathbf{R}^3. \end{cases} \tag{3.3}$$

From the fact that $(u_{\lambda_n}, \phi_{u_{\lambda_n}})$ is a solution to the system (3.3), we know that $\{u_{\lambda_n}\}$ satisfies the Pohozaev identity. We use the Pohozaev identity to show that the sequence $\{u_{\lambda_n}\}$ is bounded in $H^1(\mathbf{R}^3)$ (see also e.g. [15,16]). Then we deduce that $\{u_{\lambda_n}\}$ is a bounded (PS) sequence of I, which yields to Theorem 1.1.

We need the following abstract result which is due to Jeanjean [14].

Proposition 3.1. *Let X be a Banach space equipped with a norm $\|.\|_X$ and let $J \subset \mathbf{R}^+$ be an interval. We consider a family $\{\Phi_\lambda\}_{\lambda\in J}$ of C^1-functional on X of the form*

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \forall\lambda \in J,$$

where $B(u) \geqslant 0, \forall u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\| \to \infty$. We assume that there are two points v_1, v_2 in X such that

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_\lambda\left(\gamma(t)\right) > \max\left\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\right\}, \quad \forall\lambda \in J,$$

where

$$\Gamma = \left\{\gamma \in C\left([0,1], X\right)\colon \gamma(0) = v_1,\ \gamma(1) = v_2\right\}.$$

Then, for almost every $\lambda \in J$, there is a bounded $(PS)_{c_\lambda}$ sequence for Φ_λ, that is, there exists a sequence $\{u_n(\lambda)\} \subset X$ such that

(i) *$\{u_n(\lambda)\}$ is bounded in X,*

(ii) $\Phi_\lambda(u_n(\lambda)) \to c_\lambda$,
(iii) $\Phi'_\lambda(u_n(\lambda)) \to 0$ *in* X^*, *where* X^* *is the dual of* X.

The following lemma ensures that I_λ has the mountain pass geometry [1]. The corresponding mountain pass level is denoted by c_λ.

Lemma 3.2. *Assume* (V_1)–(V_3) *and* $2 < p < 5$. *Then*

(i) *there exists a* $v \in H^1(\mathbf{R}^3) \setminus \{0\}$, *with* $I_\lambda(v) \leqslant 0$, *for all* $\lambda \in [1/2, 1]$,
(ii) $c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda(v)\}$ *for all* $\lambda \in [1/2, 1]$, *where* $\Gamma = \{\gamma \in C([0,1], H^1(\mathbf{R}^3))\colon\ \gamma(0) = 0, \gamma(1) = v\}$.

Proof. (i) First we have

$$I_\lambda(u) \leqslant I^\infty_{1/2}(u) := \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbf{R}^3} V(\infty)u^2\,dx + \frac{1}{4}\int_{\mathbf{R}^3} \phi_u u^2\,dx - \frac{1}{2(p+1)}\int_{\mathbf{R}^3} |u|^{p+1}\,dx.$$

By Lemma 2.1, for $u \in H^1(\mathbf{R}^3)$, $u \neq 0$ fixed, it is easy to see that

$$I^\infty_{1/2}\big(t^2u(tx)\big) = \frac{t^3}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \frac{t}{2}\int_{\mathbf{R}^3} V(\infty)u^2\,dx + \frac{t^3}{4}\int_{\mathbf{R}^3} \phi_u u^2\,dx - \frac{t^{2p-1}}{2(p+1)}\int_{\mathbf{R}^3} |u|^{p+1}\,dx.$$

Since $2p-1 > 3$, we have that $I^\infty_{1/2}(t^2u(tx)) \to -\infty$ as $t \to +\infty$. Taking $v = t^2u(tx)$ for t large, we have $I_\lambda(v) \leqslant I^\infty_{1/2}(v) < 0$. Thus we have (i).

(ii) Since

$$I_\lambda(u) \geqslant \frac{1}{2}\|u\|_V^2 - \frac{\lambda}{p+1}\int_{\mathbf{R}^3} |u|^{p+1}\,dx \geqslant \frac{1}{2}\|u\|_V^2 - C\|u\|_V^{p+1}$$

and $p > 2$, we deduce that I_λ has a strict local minimum in 0 and $c_\lambda > 0$. □

In what follows, we need to consider the associated limit problem

$$\begin{cases} -\Delta u + V(\infty)u + \phi u = \lambda|u|^{p-1}u, & \text{in } \mathbf{R}^3,\\ -\Delta\phi = u^2, & \text{in } \mathbf{R}^3, \end{cases} \tag{3.4}$$

where $2 < p < 5$, $\lambda \in [1/2, 1]$. Define $I^\infty_\lambda : H^1(\mathbf{R}^3) \to \mathbf{R}$ by

$$I^\infty_\lambda(u) = \frac{1}{2}\int_{\mathbf{R}^3} \big(|\nabla u|^2 + V(\infty)u^2\big)\,dx + \frac{1}{4}\int_{\mathbf{R}^3} \phi_u u^2\,dx - \frac{\lambda}{p+1}\int_{\mathbf{R}^3} |u|^{p+1}\,dx. \tag{3.5}$$

As in [3,17], we introduce the following manifold

$$M^\infty_\lambda := \big\{u \in H^1\big(\mathbf{R}^3\big) \setminus \{0\}\colon\ G^\infty_\lambda(u) = 0\big\}$$

where

$$G^\infty_\lambda(u) := \int_{\mathbf{R}^3} \left(\frac{3}{2}|\nabla u|^2 + \frac{1}{2}V(\infty)u^2 + \frac{3}{4}\phi_u u^2 - \frac{(2p-1)\lambda}{p+1}|u|^{p+1}\right)dx.$$

Set

$$m^\infty_\lambda := \inf_{u\in M^\infty_\lambda} I^\infty_\lambda(u). \tag{3.6}$$

The manifold M^∞_λ contains all the nontrivial critical points of I^∞_λ and possesses the following properties, see [17].

Lemma 3.3. *Assume* $2 < p < 5$. *Then for any* $u \in H^1(\mathbf{R}^3)$, $u \neq 0$, *there exists a unique number* $\bar{t} = \bar{t}(u) > 0$, *such that* $u_{\bar{t}} = \bar{t}^2u(\bar{t}x) \in M^\infty_\lambda$. *Moreover*

$$I^\infty_\lambda(u_{\bar{t}}) = \max_{t\geqslant 0} I^\infty_\lambda(u_t).$$

The following lemma is proved in [3].

Lemma 3.4. *Assume $2 < p < 5$. Then m_λ^∞ is achieved at some $u_\lambda^\infty \in M_\lambda^\infty$ for any $\lambda \in [1/2, 1]$. Moreover $I_\lambda'^\infty(u_\lambda^\infty) = 0$, and*

$$I_\lambda^\infty\big(u_\lambda^\infty\big) = m_\lambda^\infty = \inf\big\{I_\lambda^\infty(u):\ u \neq 0,\ I_\lambda'^\infty(u) = 0\big\}.$$

Lemma 3.5. *Assume (V_1)–(V_3) hold and $2 < p < 5$. Then $c_\lambda < m_\lambda^\infty$ for any $\lambda \in [1/2, 1]$.*

Proof. Without loss of generality, we assume $V(x) \not\equiv V(\infty)$. Let u_λ^∞ be the minimizer of m_λ^∞, by Lemma 3.3, we have

$$I_\lambda^\infty\big(u_\lambda^\infty\big) = \max_{t \geqslant 0} I_\lambda^\infty\big(t^2 u_\lambda^\infty(tx)\big).$$

Thus by choosing $v = t^2 u_\lambda^\infty(tx)$ for t large in Lemma 3.2, we have

$$c_\lambda \leqslant \max_{t \geqslant 0} I_\lambda\big(t^2 u_\lambda^\infty(tx)\big) < \max_{t \geqslant 0} I_\lambda^\infty\big(t^2 u_\lambda^\infty(tx)\big) = I_\lambda^\infty\big(u_\lambda^\infty\big) = m_\lambda^\infty. \qquad \Box$$

Lemma 3.6. *Assume (V_1)–(V_3) hold and $2 < p < 5$. Let $\{u_n\}$ be a bounded (PS) sequence for I_λ, for $\lambda \in [1/2, 1]$. Then there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, an integer $l \in \mathbf{N} \cup \{0\}$, sequences $\{y_n^k\} \subset \mathbf{R}^3$, $w^k \in H^1(\mathbf{R}^3)$ for $1 \leqslant k \leqslant l$, such that*

- (i) *$u_n \rightharpoonup u_0$ with $I_\lambda'(u_0) = 0$,*
- (ii) *$|y_n^k| \to \infty$ and $|y_n^k - y_n^{k'}| \to \infty$, for $k \neq k'$,*
- (iii) *$w^k \neq 0$ and $I_\lambda'^\infty(w^k) = 0$, for $1 \leqslant k \leqslant l$,*
- (iv) *$\|u_n - u_0 - \sum_{k=1}^{k=l} w^k(\cdot - y_n^k)\| \to 0$,*
- (v) *$I_\lambda(u_n) \to I_\lambda(u_0) + \sum_{k=1}^{k=l} I_\lambda^\infty(w^k)$,*

where we agree that in the case $l = 0$ the above holds without w^k, $\{y_n^k\}$.

Since we have Lemma 2.2, we can prove Lemma 3.6 in a standard way. We omit it here, see e.g. [15].

We will make use of the following Pohozaev type identity. Since its proof is standard we do not provide it (see e.g. [10]).

Lemma 3.7. *Let u be a critical point of I_λ in $H^1(\mathbf{R}^3)$, then*

$$\frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\,dx + \frac{3}{2}\int_{\mathbf{R}^3} V(x)u^2\,dx + \frac{1}{2}\int_{\mathbf{R}^3} \big(\nabla V(x), x\big)u^2\,dx + \frac{5}{4}\int_{\mathbf{R}^3} \phi_u u^2\,dx - \frac{3\lambda}{p+1}\int_{\mathbf{R}^3} |u|^{p+1}\,dx = 0. \tag{3.7}$$

Lemma 3.8. *Assume (V_1)–(V_4) hold and $2 < p < 5$. Let $\{u_n\}$ be a bounded $(PS)_{c_\lambda}$ sequence for I_λ, i.e. $\lim_{n\to\infty} I_\lambda(u_n) = c_\lambda$, $I_\lambda'(u_n) \to 0$. Then, up to a subsequence, $\{u_n\}$ converges to a nontrivial critical point u_λ of I_λ with $I_\lambda(u_\lambda) = c_\lambda$.*

Proof. By Lemma 3.6, there exist $l \in \mathbf{N} \cup \{0\}$ and $u_\lambda \in H^1(\mathbf{R}^3)$ such that $I_\lambda'(u_\lambda) = 0$ and

$$u_n \rightharpoonup u_\lambda \quad \text{in } H^1\big(\mathbf{R}^3\big),$$
$$I_\lambda(u_n) \to I_\lambda(u_\lambda) + \sum_{k=1}^{k=l} I_\lambda^\infty\big(w^k\big),$$

where $\{w^k\}_{k=1}^l$ are the critical points of I_λ^∞.

Denote

$$\begin{cases} a := \displaystyle\int_{\mathbf{R}^3} |\nabla u_\lambda|^2\,dx, \qquad b := \int_{\mathbf{R}^3} V(x)u_\lambda^2\,dx, \\ \bar{b} := \displaystyle\int_{\mathbf{R}^3} \big(\nabla V(x), x\big)u_\lambda^2\,dx, \qquad c := \int_{\mathbf{R}^3} \phi_{u_\lambda} u_\lambda^2\,dx, \qquad d := \int_{\mathbf{R}^3} |u_\lambda|^{p+1}\,dx. \end{cases} \tag{3.8}$$

It follows from (V_4) that $2b + \bar{b} \geqslant 0$. Then by Lemma 3.7, we have

$$\begin{cases} \dfrac{1}{2}a + \dfrac{3}{2}b + \dfrac{1}{2}\bar{b} + \dfrac{5}{4}c - \dfrac{3\lambda}{p+1}d = 0, \\ \dfrac{1}{2}a + \dfrac{1}{2}b + \dfrac{1}{4}c - \dfrac{\lambda}{p+1}d = I_\lambda(u_\lambda), \\ a + b + c - d = 0. \end{cases} \tag{3.9}$$

The first equation comes from the Pohozaev equality (3.7). The second one is from the definition of I_λ, and the last one is $\langle I'_\lambda(u_\lambda), u_\lambda\rangle = 0$. From these relations, we have

$$3I_\lambda(u_\lambda) = b + \frac{1}{2}\bar{b} + \frac{2(p-2)}{p+1}d,$$

that is

$$3I_\lambda(u_\lambda) = \int_{\mathbf{R}^3}\left(V(x) + \frac{1}{2}\big(\nabla V(x), x\big)\right)u_\lambda^2\,dx + \frac{2(p-2)}{p+1}\int_{\mathbf{R}^3}|u_\lambda|^{p+1}\,dx \geqslant \frac{2(p-2)}{p+1}\int_{\mathbf{R}^3}|u_\lambda|^{p+1}\,dx \geqslant 0.$$

If $l \neq 0$, then

$$c_\lambda = \lim_{n\to\infty} I_\lambda(u_n) = I_\lambda(u_\lambda) + \sum_{k=1}^{k=l} I_\lambda^\infty\big(w^k\big) \geqslant m_\lambda^\infty,$$

which is a contradiction by Lemma 3.5. Thus $l = 0$ then Lemma 3.6 implies that $u_n \to u_\lambda$ in $H^1(\mathbf{R}^3)$ and $I_\lambda(u_\lambda) = c_\lambda$. □

Now we are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. The first step is to show that I has a nontrivial critical point. By Lemma 3.8, it is sufficient to construct a bounded $(PS)_{c_1}$ sequence for I, where c_1 is the mountain pass value of $I_1 = I$.

We use Proposition 3.1 with $X = H^1(\mathbf{R}^3)$, $J = [1/2, 1]$, $\Phi_\lambda = I_\lambda$ and $A(u) = \frac{1}{2}\int_{\mathbf{R}^3}(|\nabla u|^2 + V(x)u^2)\,dx + \frac{1}{4}\int_{\mathbf{R}^3}\phi_u u^2\,dx$, $B(u) = \frac{1}{p+1}\int_{\mathbf{R}^3}|u|^{p+1}\,dx$. By Lemma 3.2, I_λ satisfies the assumptions of Proposition 3.1. Thus for almost every $\lambda \in [1/2, 1]$, there exists a bounded $(PS)_{c_\lambda}$ sequence for I_λ. Then Lemma 3.8 implies that there exists $u_\lambda \in H^1(\mathbf{R}^3)$, $u_\lambda \neq 0$, such that

$$I'_\lambda(u_\lambda) = 0, \qquad I_\lambda(u_\lambda) = c_\lambda \quad \text{for a.e. } \lambda \in [1/2, 1].$$

Choosing $\lambda_n \to 1$, then we have a sequence of u_{λ_n}, denoted by $\{u_n\}$, the critical points of I_{λ_n}. Next, we show that $\{u_n\}$ is bounded.

Define $a_n, b_n, \bar{b}_n, c_n$ and d_n as in (3.8) by replacing u with u_n respectively. As (3.9), we have

$$\begin{cases} \frac{1}{2}a_n + \frac{3}{2}b_n + \frac{1}{2}\bar{b}_n + \frac{5}{4}c_n - \frac{3\lambda_n}{p+1}d_n = 0, \\ \frac{1}{2}a_n + \frac{1}{2}b_n + \frac{1}{4}c_n - \frac{\lambda_n}{p+1}d_n = c_{\lambda_n}, \\ a_n + b_n + c_n - d_n = 0. \end{cases}$$

From these relations, we have

$$\frac{1}{4}a_n + \frac{1}{4}b_n + \left(\frac{1}{4} - \frac{1}{p+1}\right)\lambda_n d_n = c_{\lambda_n} \tag{3.10}$$

and

$$b_n + \frac{1}{2}\bar{b}_n + \frac{2(p-2)}{p+1}\lambda_n d_n = 3c_{\lambda_n} \leqslant 3c_{1/2}. \tag{3.11}$$

Note that (V_4) implies $b_n + \frac{1}{2}\bar{b}_n \geqslant 0$. Thus (3.11) implies d_n is bounded since $2 < p < 5$. It turn out that $a_n + b_n$ is bounded by (3.10), that is, $\{u_n\}$ is bounded in $H^1(\mathbf{R}^3)$. Therefore

$$\lim_{n\to\infty} I_\lambda(u_n) = \lim_{n\to\infty}\left(I_{\lambda_n}(u_n) + (\lambda_n - 1)\int_{\mathbf{R}^3}|u_n|^{p+1}\,dx\right) = \lim_{n\to\infty} c_{\lambda_n} = c_1.$$

Here we use the fact that the map $\lambda :\to c_\lambda$ is left-continuous, see [14]. Similarly, $I'(u_n) \to 0$ in $H^{-1}(\mathbf{R}^3)$. Then Lemma 3.8 yields that there exists $u_0 \in H^1(\mathbf{R}^3)$, $u_0 \neq 0$, being a critical point of I.

Next we prove the existence of ground state solution of problem (1.2). Denote

$$m := \inf\big\{I(u):\ u \neq 0,\ I'(u) = 0\big\}.$$

As in the proof of Lemma 3.8, we can see that every critical point of I has nonnegative energy. Thus $0 \leqslant m \leqslant I(u_0) < +\infty$. To complete the proof, it suffices to prove m can be achieved in $H^1(\mathbf{R}^3)$. Let $\{u_n\}$ be a sequence of nontrivial critical points of I satisfying $I(u_n) \to m$. Since $I(u_n)$ is bounded, using the argument in Lemma 3.8, we can deduce that $\{u_n\}$ is bounded. Thus, in particular, $\{u_n\}$ is a bounded (PS) sequence of I. Applying Lemma 3.6, we have

$$I(u_n) \to I(\bar{u}_0) + \sum_{k=1}^{k=l} I^\infty\bigl(w^k\bigr) \tag{3.12}$$

with $l \geqslant 0$, $\bar{u}_0$ a critical point of I and w^k critical points of I^∞, where $I^\infty = I_1^\infty$ defined in (3.5). Let

$$m^\infty := \inf\bigl\{I^\infty(u):\ u \neq 0,\ I'^\infty(u) = 0\bigr\},$$

which can be achieved in $H^1(\mathbf{R}^3)$ by Lemma 3.4. As in the proof of Lemma 3.5, we assume $V(x) \not\equiv V(\infty)$, thus, we have $m \leqslant c_1 < m^\infty$. Since $I^\infty(w^k) \geqslant m^\infty > 0$, for each k, we deduce $\bar{u}_0 \neq 0$ and $l = 0$ from (3.12). Thus $I(\bar{u}_0) = m$, and $(\bar{u}_0, \phi_{\bar{u}_0})$ is the ground state solution of problem (1.2). □

Remark 3.9. In [4], the authors prove the existence of a ground state solution to problem (1.2) with $3 < p < 5$ under the assumptions

(V_1') $V : \mathbf{R}^3 \to \mathbf{R}$ is measurable function;
(V_2') $V(\infty) := \liminf_{|x|\to\infty} V(x) > V(x)$, for almost every $x \in \mathbf{R}^3$, and the inequality is strict in a nonzero measure domain;
(V_3') there exists $C > 0$ such that, for any $u \in H^1(\mathbf{R}^3)$,

$$\int_{\mathbf{R}^3} |\nabla u|^2 + V(x)u^2\,dx \geqslant C\|u\|^2.$$

As pointed out in [4], under conditions (V_1'), (V_2') and (V_3'), V is allowed to be unbounded from below. Since we can easily obtain the boundedness of (PS) sequence when $3 < p < 5$, it is not difficult to give an alternative proof of Theorem 1.2 in [4] using Lemma 3.6.

4. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. Following [5], we show Theorem 1.2 by a superposition principle established in [5]. We recall it below.

Let E be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_E$. Let $\mathcal{G}$ be a Abelian group acting isometrically on E. Assume $\mathcal{G}$ satisfies

(G) Every infinite subset $\mathcal{A}$ of $\mathcal{G}$ contains a cofinal sequence.

For the definition of cofinal sequence, we refer to [5]. Remark that $\mathcal{G} = \mathbf{Z}^3$ satisfies (G).

Define the functional $\Phi : E \to \mathbf{R}$ by

$$\Phi(u) := \frac{1}{2}\langle Lu, u\rangle_E - \Psi(u).$$

We consider $L \in \mathcal{L}(E)$ with the following properties

(L_1) L is a selfadjoint isomorphism. Its spectrum is a finite set.
(L_2) L is equivariant under the action of $\mathcal{G}$.

And we consider a differentiable map $\Psi : E \to \mathbf{R}$ and denote its gradient by Λ. We assume the following properties for Ψ

(F_1) There is α in $(0,1]$ such that $\Psi \in C^{2+\alpha}(E,\mathbf{R})$, uniformly on bounded subsets.
(F_2) Ψ, Ψ' and Ψ'' have the *BL-splitting property*.
(F_3) Λ is weakly sequentially continuous.
(F_4) For every u in E, the operator $\Lambda'(u)$ is compact.
(F_5) Ψ' is invariant under the action of $\mathcal{G}$.

We need to recall some notions, see [5]. If X, Y are normed spaces and $f : X \to Y$ is a map, we call f is bounded if it maps bounded subsets of X into bounded subsets of Y. We say that $f \in C^n(X,Y)$ uniformly on bounded subsets if all derivatives up to order n are bounded in this sense. For $\alpha \in (0,1)$ we say that $f \in C^{n+\alpha}(X,Y)$ uniformly on bounded subsets if $f \in C^n(X,Y)$ uniformly on bounded sunsets and if the nth derivative of f is uniformly Hölder continuous with exponent α on bounded subsets of X. And we say that f has the *BL-splitting property*, if for every weakly convergent sequence $\{x_n\}$ in X with $x_n \rightharpoonup x$ it holds that

$$f(x_n) - f(x_n - x) \to f(x) \quad \text{in } Y \text{ as } n \to \infty.$$

Let us recall the superposition principle in [5].

Proposition 4.1. *Suppose that* (G), (L_1), (L_2) *and* (F_1)–(F_5) *hold. Let* $\bar{u} \neq 0$ *is an isolated critical point of* Φ, *such that* Φ *has nonzero reduced local degree at* $\bar{u}$. *Then* $\mathcal{K}_{kc-\varepsilon}^{kc+\varepsilon}/\mathcal{G}$ *is infinite for* $c := \Phi(\bar{u})$ *and for every* $\varepsilon > 0$ *and every* k *in* $\mathbf{N} \setminus \{1\}$, *where* $\mathcal{K}_a^b = \{u \in E\colon \Phi'(u) = 0,\ a \leqslant \Phi(u) \leqslant b\}$.

In $H^1(\mathbf{R}^3)$, we introduce the inner product

$$\langle u, v\rangle_V = \int_{\mathbf{R}^3} \big(\nabla u \nabla v + V(x)uv\big)\,dx.$$

Define $\widehat{N} : H^1(\mathbf{R}^3) \to \mathbf{R}$ by

$$\widehat{N}(u) = \int_{\mathbf{R}^3} F(x,u)\,dx.$$

Thus the functional I defined in (2.3) is rewritten as

$$I(u) = \frac{1}{2}\|u\|_V^2 + \frac{1}{4}N(u) - \widehat{N}(u).$$

Similar to N, under the assumptions (f_1)–(f_4), the functional $\widehat{N}$ and its derivative $\widehat{N}'$, $\widehat{N}''$ posses *BL splitting property*. For the proofs of Lemmas 4.2 and 4.3 we refer to [5].

Lemma 4.2. *Assume* (f_1)–(f_4) *hold. Let* $u_n \rightharpoonup u$ *in* $H^1(\mathbf{R}^3)$ *and* $u_n \to u$ *a.e. in* $\mathbf{R}^3$. *Then as* $n \to \infty$

(i) $\widehat{N}(u_n - u) = \widehat{N}(u_n) - \widehat{N}(u) + o(1)$;
(ii) $\widehat{N}'(u_n - u) = \widehat{N}'(u_n) - \widehat{N}'(u) + o(1)$, *in* $H^{-1}(\mathbf{R}^3)$;
(iii) $\widehat{N}''(u_n - u) = \widehat{N}''(u_n) - \widehat{N}''(u) + o(1)$ *in* $L(H^1(\mathbf{R}^3), H^{-1}(\mathbf{R}^3))$.

Lemma 4.3. *Assume* (f_1)–(f_4) *hold. Then*

(i) $\widehat{N}' : H^1(\mathbf{R}^3) \to H^{-1}(\mathbf{R}^3)$ *is weakly sequentially continuous*;
(ii) $\widehat{N}''(u) \in L(H^1(\mathbf{R}^3), H^{-1}(\mathbf{R}^3))$ *is compact for any* $u \in H^1(\mathbf{R}^3)$.

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. We use Proposition 4.1 with $E = H^1(\mathbf{R}^3)$, $\mathcal{G} = \mathbf{Z}^3$, $L = Id$, $\Psi(u) = -\frac{1}{4}N(u) + \widehat{N}(u)$ and $\Phi(u) = I(u)$. Then

$$I(u) = \frac{1}{2}\|u\|_V^2 - \Psi(u).$$

As in the proof of Lemma 3.2, we can see that I possesses the mountain pass geometry. By Ekeland's variational principle [13], there exists $\{u_n\} \subset H^1(\mathbf{R}^3)$ and $c > 0$ such that

$$I(u_n) \to c, \qquad I'(u_n) \to 0, \quad \text{in } H^{-1}\big(\mathbf{R}^3\big). \tag{4.1}$$

By (f_3),

$$c + 1 + \|u_n\|_V \geqslant I(u_n) - \frac{1}{4}\big\langle I'(u_n), u_n\big\rangle \geqslant \frac{1}{4}\|u_n\|_V^2,$$

which implies that $\{u_n\} \subset H^1(\mathbf{R}^3)$ is bounded. Since I is invariant under the action of $\mathbf{Z}^3$, we claim that after an appropriate translation, $u_n \rightharpoonup \bar{u}$, in $H^1(\mathbf{R}^3)$, for some $\bar{u} \in H^1(\mathbf{R}^3)$, $\bar{u} \neq 0$. Otherwise, by a standard argument, we can deduce $c = 0$, a contradiction. From (4.1), it is easy to see $I'(\bar{u}) = 0$. And as in the proof of Proposition 4.1 in [5], I has nonzero reduced local degree at $\bar{u} \neq 0$. Thus to apply Proposition 4.1, we need to verify that Ψ satisfies (F_1)–(F_5). In fact, by simple computation, we can see N satisfies (F_1), and by (f_2), $\widehat{N}$ satisfies (F_1), see [5]. Thus (F_1) holds for Ψ. (F_2) follows from Lemmas 2.2 and 4.2. (F_3)–(F_4) hold by Lemmas 2.3 and 4.3. (F_5) is obvious since V and $f(\cdot,u)$ is 1-periodic. By Proposition 4.1, we complete the proof of Theorem 1.2. □

Appendix A

In this section, we give the proof of Lemmas 2.2 and 2.3. We need the well-known Hardy–Littlewood–Sobolev inequality.

Lemma A.1 *(Hardy–Littlewood–Sobolev inequality). Let*

$$I_r f(x) = \int_{\mathbf{R}^n} |x-y|^{-n/r} f(y)\,dy.$$

If $r>1$ *and* $\frac{1}{r} = 1 - (\frac{1}{p} - \frac{1}{q})$ *for some* $1<p<q<\infty$, *then there exists a* $C = C(p,q,r,n)$ *such that*

$$\|I_r f\|_q \leqslant C\|f\|_p.$$

Proof of Lemma 2.2. Let $u_n \rightharpoonup u$ in $H^1(\mathbf{R}^3)$ and $u_n \to u$ a.e. in $\mathbf{R}^3$. We use these notations

$$G(x,y) := \frac{1}{4\pi}\frac{1}{|x-y|}, \qquad A := \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u^2(y)u^2(x)G(x,y)\,dx\,dy,$$

$$I_n^{(1)} := \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u_n^2(y)u^2(x)G(x,y)\,dx\,dy, \qquad I_n^{(2)} := \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u_n(y)u(y)u_n(x)u(x)G(x,y)\,dx\,dy,$$

$$I_n^{(3)} := \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u_n^2(y)u_n(x)u(x)G(x,y)\,dx\,dy, \qquad I_n^{(4)} := \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u_n(y)u(y)u^2(x)G(x,y)\,dx\,dy,$$

(i) Using above notations, we have

$$N(u_n - u) - \big(N(u_n) - N(u)\big) = 2I_n^{(1)} + 4I_n^{(2)} - 4I_n^{(3)} - 4I_n^{(4)} + 2A,$$

thus it suffices to show that

$$\lim_{n\to\infty} I_n^{(i)} = A, \quad i = 1,2,3,4. \tag{A.1}$$

Set

$$v_n(x) := \frac{1}{4\pi}\int_{\mathbf{R}^3} u_n^2(y)\frac{1}{|x-y|}\,dy, \qquad v(x) := \frac{1}{4\pi}\int_{\mathbf{R}^3} u^2(y)\frac{1}{|x-y|}\,dy. \tag{A.2}$$

First, we have

$$v_n \to v \quad \text{a.e. on } \mathbf{R}^3. \tag{A.3}$$

In fact,

$$\begin{aligned}\big|v_n(x) - v(x)\big| &\leqslant \int_{\mathbf{R}^3} \big|u_n^2(y) - u^2(y)\big| G(x,y)\,dy \\ &\leqslant C\big\|u_n^2 - u^2\big\|_{L^2(B_R(x))}\bigg(\int_{|y-x|\leqslant R} \frac{1}{|x-y|^2}\,dy\bigg)^{1/2} + C\big\|u_n^2 - u^2\big\|_{L^{4/3}(B_R^c(x))}\bigg(\int_{|y-x|\geqslant R} \frac{1}{|x-y|^4}\,dy\bigg)^{1/4}.\end{aligned}$$

Letting $n\to\infty$ and then $R\to\infty$, we get $v_n(x) \to v(x)$, as $n\to\infty$.

We note that $v_n = \phi_{u_n} \in D^{1,2}(R^3)$, and

$$\|v_n\|_6 \leqslant C\|v_n\|_D \leqslant C\|u_n\|_{12/5}^2 \leqslant C.$$

Therefore we can assume, $v_n \rightharpoonup v$ in $L^6(\mathbf{R}^3)$. Since $u \in H^1(\mathbf{R}^3)$ thus $u^2 \in L^{6/5}(\mathbf{R}^3)$, the dual space of $L^6(\mathbf{R}^3)$, we have

$$\int_{\mathbf{R}^3} v_n u^2\,dx \to \int_{\mathbf{R}^3} v u^2\,dx,$$

that is $\lim_{n\to\infty} I_n^{(1)} = A$.

To prove $\lim_{n\to\infty} I_n^{(2)} = A$, we set

$$\widetilde{v}_n(x) := \frac{1}{4\pi}\int_{\mathbf{R}^3} u_n(y)u(y)\frac{1}{|x-y|}\,dy. \tag{A.4}$$

As before it is easy to verify that $\widetilde{v}_n \to v$ a.e. in $\mathbf{R}^3$.

Now by Hardy–Littlewood–Sobolev inequality, $\widetilde{v}_n \in L^6(\mathbf{R}^3)$ and

$$\|\widetilde{v}_n\|_6 \leqslant C\|u_n u\|_{6/5} \leqslant C\|u_n\|_{12/5}\|u\|_{12/5},$$

which implies that

$$\|\widetilde{v}_n u_n\|_2 \leqslant \|\widetilde{v}_n\|_6 \|u_n\|_3 \leqslant C\|u_n\|^2\|u\| \leqslant C.$$

Therefore up to a subsequence $\widetilde{v}_n u_n \rightharpoonup vu$ in $L^2(\mathbf{R}^3)$. Since $u \in L^2(\mathbf{R}^3)$,

$$\int_{\mathbf{R}^3} \widetilde{v}_n u_n u\,dx \to \int_{\mathbf{R}^3} vu^2\,dx,$$

that is $\lim_{n\to\infty} I_n^{(2)} = A$. In a similar way, we can verify (5.1) with $i = 3, 4$. We omit the details. Thus we have (i).

(ii) For any $\varphi \in H^1(\mathbf{R}^3)$ with $\|\varphi\| \leqslant 1$, we need to show

$$\big\langle N'(u_n - u) - N'(u_n) + N'(u), \varphi\big\rangle \to 0, \quad \text{uniformly with respect to } \varphi.$$

In fact,

$$\begin{aligned}\big\langle N'(u_n - u) - N'(u_n) + N'(u), \varphi\big\rangle &= \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \big(-u_n^2(y)G(x,y)u(x)\varphi(x) - 2u_n(y)u(y)G(x,y)u_n(x)\varphi(x) \\ &\quad + 2u_n(y)u(y)G(x,y)u(x)\varphi(x) + u^2(y)G(x,y)u_n(x)\varphi(x)\big)\,dx\,dy \\ &= -\widetilde{I}_n^{(1)} - 2\widetilde{I}_n^{(2)} + 2\widetilde{I}_n^{(3)} + \widetilde{I}_n^{(4)}\end{aligned}$$

where

$$\widetilde{I}_n^{(1)} = \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u_n^2(y)G(x,y)u(x)\varphi(x)\,dx\,dy, \qquad \widetilde{I}_n^{(2)} = \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u_n(y)u(y)G(x,y)u_n(x)\varphi(x)\,dx\,dy,$$

$$\widetilde{I}_n^{(3)} = \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u_n(y)u(y)G(x,y)u(x)\varphi(x)\,dx\,dy, \qquad \widetilde{I}_n^{(4)} = \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u^2(y)G(x,y)u_n(x)\varphi(x)\,dx\,dy.$$

Thus it suffices to show

$$\lim_{n\to\infty} \widetilde{I}_n^{(i)} = \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u^2(y)G(x,y)u(x)\varphi(x)\,dx\,dy, \quad i = 1,2,3,4, \tag{A.5}$$

uniformly with respect to φ.

First,

$$\begin{aligned}\left|\widetilde{I}_n^{(1)} - \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u^2(y)G(x,y)u(x)\varphi(x)\,dx\,dy\right| &\leqslant \int_{\mathbf{R}^3} \big|\big(v_n(x) - v(x)\big)u(x)\varphi(x)\big|\,dx \\ &\leqslant \|\varphi\|_2 \bigg(\int_{\mathbf{R}^3} (v_n - v)^2 u^2\,dx\bigg)^{1/2} \\ &\leqslant C\bigg(\int_{\mathbf{R}^3} (v_n - v)^2 u^2\,dx\bigg)^{1/2}\end{aligned}$$

where v_n and v are defined by (5.2). Since $\{(v_n - v)^2\} \subset L^3(\mathbf{R}^3)$ is bounded and $v_n \to v$ a.e. in $\mathbf{R}^3$, up to a subsequence,

$$(v_n - v)^2 \rightharpoonup 0 \quad \text{in } L^3\big(\mathbf{R}^3\big).$$

Since $u^2 \in L^{3/2}(\mathbf{R}^3)$, we have

$$\int_{\mathbf{R}^3} (v_n - v)^2 u^2\,dx \to 0,$$

that is, (5.5) with $i = 1$.

The verification of (5.5) with $i = 2, 3, 4$ can be done in a similar way. For the sake of simplicity, we prove only the case $i = 2$, since it is the most complicated one.

By Minkowski inequality, we have

$$\left|\widetilde{I}_n^{(2)} - \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u^2(y)G(x,y)u(x)\varphi(x)\,dx\,dy\right| \leqslant \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \left|\big(u_n(y)u_n(x) - u(y)u(x)\big)G(x,y)u(y)\varphi(x)\right|dx\,dy$$

$$\leqslant \|\varphi\|_3\left(\int_{\mathbf{R}^3}\left(\int_{\mathbf{R}^3}\left|\big(u_n(y)u_n(x) - u(y)u(x)\big)G(x,y)u(y)\right|dy\right)^{3/2}dx\right)^{2/3}$$

$$\leqslant C\int_{\mathbf{R}^3}\left(\int_{\mathbf{R}^3}\left|\big(u_n(y)u_n(x) - u(y)u(x)\big)G(x,y)u(y)\right|^{3/2}dx\right)^{2/3}dy$$

$$= C\int_{\mathbf{R}^3}\left(\int_{\mathbf{R}^3}\left|\big(u_n(y)u_n(x) - u(y)u(x)\big)\right|^{3/2}G^{3/2}(x,y)\,dx\right)^{2/3}\left|u(y)\right|dy$$

$$= C\int_{\mathbf{R}^3} w_n^{2/3}(y)\left|u(y)\right|dy,$$

where w_n is defined by

$$w_n(y) := \int_{\mathbf{R}^3}\left|\big(u_n(y)u_n(x) - u(y)u(x)\big)\right|^{3/2}G^{3/2}(x,y)\,dx. \tag{A.6}$$

First we claim $w_n \to 0$ a.e. in $\mathbf{R}^3$.
Set

$$\overline{w}_n(y) := \int_{\mathbf{R}^3}\left|u_n(x) - u(x)\right|^{3/2}G^{3/2}(x,y)\,dx,$$

$$\widetilde{w}_n(y) := \int_{\mathbf{R}^3}\left|u_n(x)\right|^{3/2}G^{3/2}(x,y)\,dx, \qquad \widetilde{w}(y) := \int_{\mathbf{R}^3}\left|u(x)\right|^{3/2}G^{3/2}(x,y)\,dx.$$

In fact,

$$w_n(y) = \int_{\mathbf{R}^3}\left|\big(u_n(x) - u(x)\big)u_n(y) + \big(u_n(y) - u(y)\big)u(x)\big)\right|^{3/2}G^{3/2}(x,y)\,dx$$

$$\leqslant \left|u_n(y)\right|^{3/2}\overline{w}_n(y) + \left|u_n(y) - u(y)\right|^{3/2}\widetilde{w}(y). \tag{A.7}$$

For $y \in \mathbf{R}^3$ fixed,

$$\overline{w}_n(y) \leqslant \left(\int_{|x-y|\leqslant R}\big(u_n(x) - u(x)\big)^{9/2}dx\right)^{1/3}\left(\int_{|x-y|\leqslant R}G^{9/4}(x,y)\,dx\right)^{2/3} + \left(\int_{\mathbf{R}^3}|u_n - u|^2\,dx\right)^{3/4}\left(\int_{|x-y|\geqslant R}G^6(x,y)\,dx\right)^{1/4}. \tag{A.8}$$

Letting $n \to \infty$ and then $R \to \infty$, we get $\overline{w}_n(y) \to 0$, as $n \to \infty$. By (5.7), $w_n(y) \to 0$.

Next we consider the integrability of w_n. By Hardy–Littlewood–Sobolev inequality, $\widetilde{w}_n \in L^4(\mathbf{R}^3)$, for all $q \geqslant 4$, and

$$\|\widetilde{w}_n\|_4 \leqslant \left\|u_n^{3/2}\right\|_{4/3} \leqslant \|u_n\|_2^{3/2} \leqslant C.$$

Similarly, $\widetilde{w} \in L^4(\mathbf{R}^3)$. Thus

$$\int_{\mathbf{R}^3}\left|w_n^{2/3}(y)\right|^2 dy \leqslant \int_{\mathbf{R}^3}\big(|u_n|^2|\widetilde{w}_n|^{4/3} + |u|^2|\widetilde{w}|^{4/3}\big)\,dy$$

$$\leqslant \left(\int_{\mathbf{R}^3}|u_n|^3\,dy\right)^{2/3}\left(\int_{\mathbf{R}^3}|\widetilde{w}_n|^4\,dy\right)^{1/3} + \left(\int_{\mathbf{R}^3}|u|^3\,dy\right)^{2/3}\left(\int_{\mathbf{R}^3}|\widetilde{w}|^4\,dy\right)^{1/3}$$

$$\leqslant C.$$

Thus $w_n^{2/3} \in L^2(\mathbf{R}^3)$ is bounded and by $w_n \to 0$ a.e. in $\mathbf{R}^3$, we have

$$w_n^{2/3} \rightharpoonup 0 \quad \text{in } L^2\big(\mathbf{R}^3\big).$$

Since $u \in L^2(\mathbf{R}^3)$, $\int_{\mathbf{R}^3} w_n^{2/3} u\,dx \to 0$, that is (5.5) with $i = 2$.

(iii) For any $\varphi, h \in H^1(\mathbf{R}^3)$ with $\|\varphi\| \leqslant 1$ and $\|h\| \leqslant 1$, we need to show

$$\langle N''(u_n - u)h, \varphi\rangle - \langle N''(u_n)h, \varphi\rangle + \langle N''(u)h, \varphi\rangle \to 0,$$

uniformly with respect to φ and h. In fact,

$$\langle N''(u)h, \varphi\rangle = \int_{\mathbf{R}^3} \big(8u(y)h(y)G(x,y)u(x)\varphi(x) + 4u^2(y)G(x,y)h(x)\varphi(x)\big)\,dx, \tag{A.9}$$

thus

$$\begin{aligned}
&\langle N''(u_n - u)h, \varphi\rangle - \langle N''(u_n)h, \varphi\rangle + \langle N''(u)h, \varphi\rangle \\
&\quad = \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \Big(\big(-8u_n(y)u(x) - 8u(y)u_n(x) + 16u(y)u(x)\big)h(y)G(x,y)\varphi(x) + \big(-8u_n(y)u(y) + 8u^2(y)\big)G(x,y)h(x)\varphi(x)\Big)\,dx\,dy \\
&\quad = -8\widehat{I}_n^{(1)} - 8\widehat{I}_n^{(2)} - 8\widehat{I}_n^{(3)} + 16B + 8\widehat{B},
\end{aligned}$$

where

$$\begin{aligned}
\widehat{I}_n^{(1)} &:= \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u_n(y)u(x)h(y)G(x,y)\varphi(x)\,dx\,dy, \\
\widehat{I}_n^{(2)} &:= \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u(y)u_n(x)h(y)G(x,y)\varphi(x)\,dx\,dy, \\
\widehat{I}_n^{(3)} &:= \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u_n(y)u(y)G(x,y)h(x)\varphi(x)\,dx\,dy, \\
B &:= \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u(y)u(x)h(y)G(x,y)\varphi(x)\,dx\,dy, \\
\widehat{B} &:= \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} u(y)^2 G(x,y)h(x)\varphi(x)\,dx\,dy.
\end{aligned}$$

Thus it suffices to show

$$\lim_{n\to\infty} \widehat{I}_n^{(i)} = B, \quad i = 1, 2, \qquad \lim_{n\to\infty} \widehat{I}_n^{(3)} = \widehat{B},$$

uniformly with respect to φ and h.

$$\begin{aligned}
\big|\widehat{I}_n^{(1)} - B\big| &\leqslant \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \big|\big(u_n(y) - u(y)\big)u(x)G(x,y)h(y)\varphi(x)\big|\,dx\,dy \\
&\leqslant \|\varphi\|_2 \bigg(\int_{\mathbf{R}^3}\bigg(\int_{\mathbf{R}^3} \big|\big(u_n(y) - u(y)\big)u(x)G(x,y)h(y)\big|\,dy\bigg)^2 dx\bigg)^{1/2} \\
&\leqslant \|\varphi\|_2 \int_{\mathbf{R}^3}\bigg(\int_{\mathbf{R}^3} \big|\big(u_n(y) - u(y)\big)u(x)G(x,y)h(y)\big|^2 dx\bigg)^{1/2} dy \\
&= \|\varphi\|_2 \int_{\mathbf{R}^3}\bigg(\int_{\mathbf{R}^3} \big|u(x)G(x,y)\big|^2 dx\bigg)^{1/2} \big|u_n(y) - u(y)\big|\big|h(y)\big|\,dy \\
&\leqslant \|\varphi\|_2\|h\|_2 \bigg(\int_{\mathbf{R}^3} \widehat{v}(y)\big(u_n(y) - u(y)\big)^2 dy\bigg)^{1/2},
\end{aligned}$$

where $\widehat{v}(y)$ is defined by

$$\widehat{v}(y) := \int_{\mathbf{R}^3} \big|u(x)G(x,y)\big|^2 dx.$$

Again, by Hardy–Littlewood–Sobolev inequality, we have $\widehat{v} \in L^2(\mathbf{R}^3)$. Since $\{(u_n-u)^2\}$ is bounded in $L^2(\mathbf{R}^3)$, and $u_n \to u$ a.e. in $\mathbf{R}^3$, we have $(u_n-u)^2 \rightharpoonup 0$ in $L^2(\mathbf{R}^3)$. Thus

$$\int_{\mathbf{R}^3} \widehat{v}(u_n-u)^2\,dy \to 0,$$

that is $\lim_{n\to\infty} \widehat{I}_n^{(1)} = B$. Similarly, we can verify that $\lim_{n\to\infty} \widehat{I}_n^{(2)} = B$. Next we show that $\lim_{n\to\infty} \widehat{I}_n^{(3)} = \widehat{B}$. First, by Hölder inequality yields that

$$\int_{\mathbf{R}^3} G(x,y)h(x)\varphi(x)\,dx \leqslant \left(\int_{\mathbf{R}^3} h^2(x)G(x,y)\,dx\right)^{1/2}\left(\int_{\mathbf{R}^3} \varphi^2(x)G(x,y)\,dx\right)^{1/2} = \phi_h^{1/2}(y)\phi_\varphi^{1/2}(y).$$

Thus

$$\begin{aligned}
\left|\widehat{I}_n^{(3)} - \widehat{B}\right| &\leqslant \int_{\mathbf{R}^3} \left|\big(u_n(y)-u(y)\big)u(y)\right|\left(\int_{\mathbf{R}^3} G(x,y)h(x)\varphi(x)\,dx\right)dy \\
&\leqslant \int_{\mathbf{R}^3} \left|\big(u_n(y)-u(y)\big)u(y)\right|\phi_h^{1/2}(y)\phi_\varphi^{1/2}(y)\,dy \\
&\leqslant \left(\int_{\mathbf{R}^3} \left|\big(u_n(y)-u(y)\big)\right|^{6/5}\left|u(y)\right|^{6/5}dy\right)^{5/6}\left(\int_{\mathbf{R}^3} \phi_h^6(y)\,dy\right)^{1/12}\left(\int_{\mathbf{R}^3} \phi_\varphi^6(y)\,dy\right)^{1/12} \\
&\leqslant C\|h\|_{12/5}\|\varphi\|_{12/5}\left(\int_{\mathbf{R}^3} \left|\big(u_n(y)-u(y)\big)\right|^{6/5}\left|u(y)\right|^{6/5}dy\right)^{5/6}.
\end{aligned}$$

Since $\{(u_n-u)^{6/5}\}$ is bounded in $L^2(\mathbf{R}^3)$, and $u_n \to u$ a.e. in $\mathbf{R}^3$, we have $(u_n-u)^{6/5} \rightharpoonup 0$ in $L^2(\mathbf{R}^3)$. Thus

$$\int_{\mathbf{R}^3} (u_n-u)^{6/5}u^{6/5}\,dy \to 0,$$

and then $\lim_{n\to\infty} \widehat{I}_n^{(3)} = \widehat{B}$. □

Proof of Lemma 2.3. (i) Let $\{u_n\} \subset H^1(\mathbf{R}^3)$ and $u_n \rightharpoonup u$ in $H^1(\mathbf{R}^3)$, we need to show for any $\varphi \in H^1(\mathbf{R}^3)$, $\langle N'(u_n) - N'(u), \varphi\rangle \to 0$.

In fact

$$\left\langle N'(u_n) - N'(u), \varphi\right\rangle = 4\int_{\mathbf{R}^3} (\phi_{u_n}u_n\varphi - \phi_u u\varphi)\,dx = 4\int_{\mathbf{R}^3} \big(\phi_{u_n}(u_n-u)\varphi + (\phi_{u_n}-\phi_u)u\varphi\big)\,dx.$$

Since

$$\int_{\mathbf{R}^3} \left|\phi_{u_n}(u_n-u)\right|^2 dx \leqslant \left\|\phi_{u_n}\right\|_6^2\|u_n-u\|_3^2 \leqslant C\|u_n\|^4\|u_n-u\|^2 \leqslant C$$

and $u_n \to u$ a.e. in $\mathbf{R}^3$, we have $\phi_{u_n}(u_n-u) \rightharpoonup 0$ in $L^2(\mathbf{R}^3)$. Thus

$$\int_{\mathbf{R}^3} \phi_{u_n}(u_n-u)\varphi\,dx \to 0.$$

Similarly,

$$\int_{\mathbf{R}^3} (\phi_{u_n}-\phi_u)u\varphi\,dx \to 0.$$

Thus we have (i).

(ii) Let $\{v_n\} \subset H^1(\mathbf{R}^3)$ be a bounded sequence. Without loss of generality, we may assume, up to a subsequence, $v_n \rightharpoonup 0$. We need to prove for $\varphi \in H^1(\mathbf{R}^3)$, $\|\varphi\| \leqslant 1$

$$\left\langle N''(u)v_n, \varphi\right\rangle \to 0, \quad \text{uniformly with respect to } \varphi.$$

In fact, by (5.9),

$$\begin{aligned}\langle N''(u)v_n,\varphi\rangle &= \int_{\mathbf{R}^3}\big(8u(y)v_n(y)G(x,y)u(x)\varphi(x)+4u^2(y)G(x,y)v_n(x)\varphi(x)\big)\,dx\\ &\leqslant 8\|\varphi\|_2\bigg(\int_{\mathbf{R}^3}\bigg(\int_{\mathbf{R}^3}\big|u(y)v_n(y)G(x,y)u(x)\big|\,dy\bigg)^2dx\bigg)^{1/2}+4\|\varphi\|_3\bigg(\int_{\mathbf{R}^3}\bigg(\int_{\mathbf{R}^3}\big|u^2(y)G(x,y)v_n(x)\big|\,dy\bigg)^{3/2}dx\bigg)^{2/3}\\ &\leqslant 8\|\varphi\|_2\int_{\mathbf{R}^3}\widehat{v}^{1/2}(y)u(y)v_n(y)\,dy+4\|\varphi\|_3\int_{\mathbf{R}^3}\widehat{w}_n^{2/3}u^2(y)\,dy\end{aligned}$$

where $\widehat{v}(y)$ and $\widehat{w}(y)$ are defined respectively by

$$\widehat{v}(y):=\int_{\mathbf{R}^3}\big|u(x)G(x,y)\big|^2\,dx,\qquad \widehat{w}_n(y):=\int_{\mathbf{R}^3}\big|v_n(x)G(x,y)\big|^{3/2}\,dx.$$

It is easy to see $\widehat{v}\in L^2(\mathbf{R}^3)$ by Hardy–Littlewood–Sobolev inequality. Thus

$$\int_{\mathbf{R}^3}\big(\widehat{v}^{1/2}(y)u(y)\big)^2\,dy\leqslant\|\widehat{v}\|_2\|u\|_4^2<+\infty,$$

which implies

$$\int_{\mathbf{R}^3}\widehat{v}^{1/2}(y)u(y)v_n(y)\,dy\to 0.$$

And similarly, $\widehat{w}_n\in L^4(\mathbf{R}^3)$ and is bounded in $L^4(\mathbf{R}^3)$. As in the proof of (5.8), we have $\widehat{w}_n\to 0$ a.e. in $\mathbf{R}^3$, thus

$$\widehat{w}_n^{2/3}\rightharpoonup 0\quad\text{in } L^6\big(\mathbf{R}^3\big),$$

and from $u^2\in L^{6/5}(\mathbf{R}^3)$, we have

$$\int_{\mathbf{R}^3}\widehat{w}_n^{2/3}u^2(y)\,dy\to 0.$$

Thus

$$\big\langle N''(u)v_n,\varphi\big\rangle\to 0,\quad\text{uniformly with respect to }\varphi.\qquad\square$$

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