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Relative Version of Weyl–Kac Character Formula

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INTRODUCTION

Let G be a Kac–Moody Lie algebra with $3l$ canonical generators $\{e_1, ..., e_l, h_1, ..., h_l, f_1, ..., f_l\}$. Let H be the span of $h_1, ..., h_l$, the Cartan subalgebra of G. As usual, $\rho \in H^*$ is the function satisfying $\rho(h_i) = 1\ \forall i$. Let $\lambda \in H^*$ be an integral function with $\lambda + \rho$ being dominant. If $\pi = \{\alpha_1, ..., \alpha_l\}$ is the set of simple roots of G, denote by $\pi_\lambda = \{\alpha_i \mid \lambda(h_i) \geqslant 0\}$, $\pi^\lambda = \pi \backslash \pi_\lambda$, $G_\lambda =$ the subalgebra of G generated by $\{e_i, h_i, f_i \mid i \in \pi_\lambda\}$, $G^\lambda =$ the subalgebra of G generated by $\{e_i, h_i, f_i \mid i \in \pi^\lambda\}$. Correspondingly, if W is the Weyl group of G, W_λ denotes the subgroup of W generated by those fundamental reflections s_i associated with $\alpha_i \in \pi_\lambda$, $W^\lambda =$ the subgroup of W generated by the fundamental reflections s_i associated with $\alpha_i \in \pi^\lambda$. Let $M(\lambda)$ be the Verma module over G with the highest weight λ and let $L(\lambda)$ be the irreducible quotient of $M(\lambda)$. If $w \in W_\lambda$, $M(w \cdot \lambda)$ $(w \cdot \lambda = w(\lambda + \rho) - \rho)$ is a submodule of $M(\lambda)$. Let $K(\lambda) = \sum_{w \in W_\lambda \setminus \{e\}} M(w \cdot \lambda)$. In this paper, we present a proof for the following generalization of Weyl–Kac character formula:

THEOREM. *For a symmetrizable* G_λ, $\operatorname{ch} M(\lambda)/K(\lambda) = \sum_{w \in W_\lambda} (-1)^{l(w)} \operatorname{ch} M(w \cdot \lambda)$.

When $\pi^\lambda = \phi$, the above theorem reduces to the usual Weyl–Kac character formula. Also, since

$$f(w \cdot W^\lambda) = \begin{cases} (-1)^{l(w)}, & \text{if } w \in W_\lambda \\ 0, & \text{if } w \notin W_\lambda W^\lambda \end{cases}$$

is indeed the relative mobius function of W/W^λ, our theorem states precisely the relative version of Weyl–Kac character formula.

Our proof for the above theorem is based on a counting principle developed in the next section. The symmetrizability assumption on G_λ is to

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guarantee the availability of the Kac–Kazhdan theorem [4]. Other than these, the whole proof is purely module-theoretic and at an elementary level.

1. Key Notions

Recall [5] the category of G-modules O. A module $P \in O$ is called local if P contains a unique (proper) maximal submodule Q. For each $\lambda \in H^*$, denote by $CL_M(\lambda)$ the set of all local submodules P in $M \in O$ such that $P/Q \cong L(\lambda)$. Let $P \in CL_M(\lambda)$, $P'/Q' \simeq L(\lambda)$ be an irreducible subquotient of M. We say P represents P'/Q' if $P \subseteq P'$ and $P \not\subseteq Q'$. Also, if $v \in M$, (v) denotes the submodule of M generated by v. For the sake of self-containedness, we first give a brief summary of concepts and results from [5] of which essential use will be made. Fix a G-module $M \in O$. For $\lambda \in H^*$, $[M: L(\lambda)]$ denotes the multiplicity of $L(\lambda)$ in M.

Proposition 1.1 (Propositions 1.1 and 1.2 [5]). *Let $\lambda \in H^*$ and let $P'/Q' \simeq L(\lambda)$ be an irreducible subquotient of M. Then*

(a) *$P' \in CL_M(\lambda)$ if and only if $P' = (v)$ for all $v \in P' \backslash Q'$.*

(b) *There exists $P \in CL_M(\lambda)$ such that P represents $P' \backslash Q'$.*

Definition. A (finite) subset $\mathfrak{A} = \{P_1, P_2, ..., P_k\} \subseteq CL_M(\lambda)$ is called independent if for every choice of $v_i \in (P_i)_\lambda$, $i = 1, 2, ..., k$, such that $P_i = (v_i)$, the set $\{v_i, v_2, ..., v_k\}$ is linearly independent. $\mathfrak{A}$ is called dependent if it is not independent.

Proposition 1.2 (Proposition 2.3 [5]). *Given a subset $\{P_1, P_2, ..., P_k\} \subseteq CL_M(\lambda)$, the following are equivalent:*

(a) *$\{P_1, P_2, ..., P_k\}$ is independent.*

(b) *There exists a permutation σ on $\{1, 2, ..., k\}$ such that $P_{\sigma(i)} \not\subset \sum_{j>i} P_{\sigma(j)}$ for all $i \leq k-1$.*

(c) *For a (nonempty) subset $A \subseteq \{1, 2, ..., k\}$ and any choice of v_i's, $i \in A$, such that $v_i \in (P_i)_\lambda \backslash (Q_i)_\lambda$, $(\sum_{i \in A} v_i)/(\sum_{i \in A} v_i) \cap \sum_{i \in A} Q_i \simeq L(\lambda)$, where Q_i is the unique maximal submodule of P_i for all $i = 1, 2, ..., k$.*

Definition. A subset $\mathfrak{B} \subseteq CL_M(\lambda)$ is called a basis of $CL_M(\lambda)$ if $\mathfrak{B}$ is a maximal independent subset of $CL_M(\lambda)$.

Proposition 1.3 (Proposition 2.6 [5]). *Let $\mathfrak{B} = \{P_1, P_2, ..., P_m\} \subseteq CL_M(\lambda)$ be a basis satisfying $P_i \not\subset \sum_{j>i} P_j$ for all $i = 1, 2, ..., m-1$. Let*

$P \in CL_M(\lambda)$. *If* $P \subset Q_k + \sum_{j>k} P_j$ *for some* $k < m$, $\{P, P_{k+1}, ..., P_m\}$ *is dependent.*

LEMMA 1.4 (Lemma 2.8 [5]). *Let* $\mathfrak{B} = \{P_1, P_2, ..., P_m\} \subseteq CL_M(\lambda)$ *be a basis satisfying* $P_t \not\subset \sum_{s>t} P_s$ *for all* $t = 1, 2, ..., m-1$. *Let* $1 \leqslant i < j \leqslant m$, $v_i \in (P_i)_\lambda - (Q_i)_\lambda$, $v_j \in (P_j)_\lambda - (Q_j)_\lambda$, *where, in general,* Q_k *is the unique maximal submodule of* P_k, $k = 1, 2, ..., m$. *Let* $P_i' \in CL_M(\lambda)$ *represent* $(v_i - v_j)/(v_i - v_j) \cap (Q_i + Q_j)$. *Then* $\mathfrak{B}' = (\mathfrak{B} \backslash \{P_i\}) \cup \{P_i'\}$ *is also a basis of* $CL_M(\lambda)$.

THEOREM 1.5 (Theorem 2.5 [5]). *Any two bases of* $CL_M(\lambda)$ *have the same cardinal number.*

THEOREM 1.6 (Theorem 3.1 [5]). *Let* $\lambda \in H^*$ *and let* $\{P_1, P_2, ..., P_m\}$ *be a basis of* $CL_M(\lambda)$. *Then* $[M: L(\lambda)] = m$.

Based on the foregoing, we now fix a basis $\{P_1, P_2, ..., P_m\}$ of $CL_M(\lambda)$ satisfying $P_i \not\subset \sum_{j>i} P_j$ for all $i \leqslant m-1$ and introduce:

DEFINITION. Let $P \in CL_M(\lambda)$. $\operatorname{ord}(P) = k$ iff $P \subset \sum_{i=k}^m P_i$ and $P \not\subset \sum_{i=k+1}^m P_i$. We call $\operatorname{ord}(P)$ the order of P (relative to the basis $\{P_1, ..., P_m\}$).

DEFINITION. Let N be a submodule of M. A basis $\{R_1, ..., R_n\}$ of $CL_N(\lambda)$ is said to be compatible with $\{P_1, ..., P_m\}$ if $\operatorname{ord}(R_1) < \operatorname{od}(R_2) < \cdots < \operatorname{ord}(R_n)$.

THEOREM 1.7. *Let* N *be a submodule of* M. *Then* $CL_N(\lambda)$ *has a basis compatible with* $\{P_1, ..., P_m\}$.

Proof. Let $n = [N: L(\lambda)]$. Claim: for each $k = n, ..., 1$, there exists $R_k \in CL_N(\lambda)$ satisfying the following properties:

(i) $\operatorname{ord}(R_k) < \operatorname{ord}(R_{k+1}) < \cdots < \operatorname{ord}(R_n)$.

(ii) For any $R \in CL_N(\lambda)$ such that $\{R, R_{k+1}, ..., R_n\}$ is independent, $\operatorname{ord}(R) \leqslant \operatorname{ord}(R_k)$.

Note that (i) implies that $\{R_k, ..., R_n\}$ is independent using 1.2(b) and the result follows immediately from the claim when $k = 1$.

To prove the claim, we argue by an induction on k. The case when $k = n$ is obvious. In general, let us assume that $R_n, ..., R_{k+1}$ have been found with (i), (ii) being satisfied. Choose R_k in such a way that $\{R_n, ..., R_{k+1}, R_k\}$ is independent and (ii) holds. It remains to show that (i) holds as well.

Since clearly $\{R_n, ..., R_{k+2}, R_k\}$ is independent, $\operatorname{ord}(R_k) \leqslant \operatorname{ord}(R_{k+1})$ by the choice of R_{k+1}. Suppose, arguing by contradiction, $\operatorname{ord}(R_k) =$

$\operatorname{ord}(R_{k+1})=s$ for some $s\leqslant m$. Clearly there exist $v_k\in(R_k)_\lambda\backslash(S_k)_\lambda$, $v_{k+1}\in(R_{k+1})_\lambda\backslash(S_{k+1})_\lambda$ such that $v_k-v_{k+1}\in Q_s+\sum_{i>s}P_i$, where S_i (resp. Q_j) is the maximal submodule of R_i (resp. P_j). By 1.2(c), $(v_k-v_{k+1})/(v_k-v_{k+1})\cap(S_k+S_{k+1})\simeq L(\lambda)$. Let $R\in CL_N(\lambda)$ represent $(v_k-v_{k+1})/(v_k-v_{k+1})\cap(S_k+S_{k+1})$. Then $R\subset Q_s+\sum_{i>s}P_i \Rightarrow \{R, R_{s+1}, ..., R_m\}$ is dependent by 1.3. In particular, $\operatorname{ord}(R)>s$. But the exchange lemma (Lemma 2.4), applied to $\{R_n, ..., R_k\}$, implies that $\{R_n, ..., R_{k+2}, R\}$ is independent. Therefore we obtain a contradiction since $\operatorname{ord}(R)\leqslant\operatorname{ord}(R_{k+1})=s$ by the induction hypothesis. As a result, $\operatorname{ord}(R_k)<\operatorname{ord}(R_{k+1})$, which concludes the theorem.

COROLLARY 1.8. *Let N be a submodule of M. Let $\{R_1, ..., R_n\}$ be a basis of $CL_N(\lambda)$ compatible with $\{P_1, ..., P_m\}$. Then*

(i) *Given any $R\in CL_N(\lambda)$,* $\operatorname{ord}(R)\in\{\operatorname{ord}(R_1), ..., \operatorname{ord}(R_n)\}$.

(ii) *Given another basis $\{R'_1, ..., R'_n\}$ of $CL_N(\lambda)$ compatible with $\{P_1, ..., P_m\}$,* $\operatorname{ord}(R'_i)=\operatorname{ord}(R_i)$ *for all $i=1, ..., n$.*

Proof. Part (ii) is an easy consequence of (i) while (i) follows from the simple fact that $\operatorname{ord}(R)\notin\{\operatorname{ord}(R_1), ..., \operatorname{ord}(R_n)\}\Rightarrow\{R_1, ..., R_n, R\}$ is independent, which is absurd.

DEFINITION. Let N be a submodule of M and let $\{R_1, ..., R_n\}$ be a basis of $CL_N(\lambda)$ compatible with $\{P_1, ..., P_m\}$. Call $\operatorname{ord}(N)=\{\operatorname{ord}(R_1), ..., \operatorname{ord}(R_n)\}$ the $(\lambda-)$ order of N in M relative to $\{P_1, ..., P_m\}$.

COROLLARY 1.9. *Let $K\subset N\subset M$ be submodules of M. Then* $\operatorname{ord}(K)\subseteq\operatorname{ord}(N)$.

Proof. Clear from 1.8.

2. MAIN RESULT

We shall need a couple of more facts towards the end:

PROPOSITION 2.1. *Let M be a G-module, $\mu\in H^*$. Given an irreducible subquotient $P/Q\simeq L(\mu)$, and subquotients $P_1/Q_1, ..., P_n/Q_n$ of M satisfying the conditions*

$$P\subset\sum_{i=1}^{n}P_i,\qquad P\not\subset Q_j+\sum_{\substack{i=1\\ i\neq j}}^{n}P_i\qquad \forall j=1,2,...,n,$$

we have $[P_j/Q_j; L(\mu)]\neq 0$, $\forall j$.

Proof. Clearly, the assumption implies that $(P+Q_j+\sum_{i\neq j} P_i)/(Q+Q_j+\sum_{i\neq j} P_i)\simeq L(\mu)$. Since $\sum_{i=1}^{n} P_i/Q_j+\sum_{i\neq j} P_i$ is a homomorphic image of P_j/Q_j, the result is obvious.

PROPOSITION 2.2. *Let A be a subset of W_λ. If there exists $\alpha_i\in\pi_\lambda$ such that $w<s_iw$ $\forall w\in A$, then $\sum_{w'\in\bar{A}}\det w'=0$, where $\bar{A}=\{w'\in W_\lambda\mid w\leqslant w'$ for some $w\in A\}$.*

Proof. The result is an easy consequence of the fact that $s_i\bar{A}=\bar{A}$, using Deodhar's characterization of the Bruhat ordering in W_λ [1].

We now prove the main theorem stated as in the Introduction.

Proof of Main Theorem. To prove the theorem, it suffices to show that

$$-\operatorname{ch} K(\lambda)=\sum_{\substack{w\in W_\lambda\\ w\neq e}}(-1)^{l(w)}\operatorname{ch} M(w\cdot\lambda).$$

Equivalently, we shall show that

$$-[K(\lambda):L(\mu)]=\sum_{\substack{w\in W_\lambda\\ w\neq e}}(-1)^{l(w)}[M(w\cdot\lambda):L(\mu)],\qquad \forall\mu\in H^*.$$

Fix $\mu\in H^*$. Write $\mu=\lambda-\mu_1-\mu_2$, where $\mu_1\in Z_+\cdot\pi_\lambda=\{\sum_{\alpha_i\in\pi_\lambda}k_i\alpha_i\mid k_i\in Z_+\}$, $\mu_2\in Z_+\cdot\pi^\lambda$. Identifying $M_0=U(G_\lambda)\cap U(G_-)$ as a subspace of $M(\lambda)$ and viewing it as a G_λ-module, let $M_0\supset M_1\supset M_2\supset\cdots$ be a local composition series of M_0 at $\lambda-\mu_1$. Without loss of generality, we may assume that this local composition series refines a decreasing sequence of submodules of M_0 of the form

$$M_0\supset\sum_{\substack{w\in W_\lambda\\ w\neq w_0=e}}M(w\cdot\lambda)\cap U(G_\lambda)$$

$$\supset\sum_{\substack{w\in W_\lambda\\ w\neq w_0,w_1}}M(w\cdot\lambda)\cap U(G_\lambda)$$

$$\supset\sum_{\substack{w\in W_\lambda\\ w\neq w_0,w_1,w_2}}M(w\cdot\lambda)\cap U(G_\lambda)\supset\cdots,$$

where w_0, w_1, w_2, ... is an enumeration of elements in W_λ with increasing length. Let $\{P_1,...,P_n\}$ be a basis of $CL_{K(\lambda)}(\mu)$. For each $i=1,...,n$ let d_i be the largest integer s with $P_i\subset U(G^\lambda)M_s$. Note that we may assume

$i > j \Rightarrow d_i \geqslant d_j$. Fix $i \leqslant n$. Suppose $i \in \operatorname{ord}(M(w \cdot \lambda))$ for some $w \in W_\lambda$ and $i \notin \operatorname{ord}(K(w \cdot \lambda))$, where

$$K(w \cdot \lambda) = \sum_{\substack{w' \in W_\lambda \\ w' < w}} M(w' \cdot \lambda).$$

This means that $\exists$ a local submodule $P \in CL_{M(w \cdot \lambda)}(\mu)$ with $P \subset K(w \cdot \lambda)$ and $\operatorname{ord} P = i$. As before, let $N_0 = M(w \cdot \lambda) \cap U(G_\lambda) \supset N_1 \supset N_2 \supset \cdots \supset N_{\tau-1} \supset N_t = K(w \cdot \lambda) \cap U(G_\lambda)$ be a local composition series of $M(w \cdot \lambda) \cap U(G_\lambda)/K(w \cdot \lambda) \cap U(G_\lambda)$ at μ_1. Suppose $P \subset N_q$, $P \not\subset N_{q+1}$ for some $q < t$.

Now suppose $d_i = d_{i+1} = \cdots = d_{i+m} < d_{i+m+1}$ for some m. By the choice of μ_1, $M_{d_i}/M_{d_{i+1}}$ is an irreducible G_λ-module. So let $y + M_{d_i+1}$ be a highest weight vector of M_{d_i}/M_{d_i+1}. Since $\operatorname{ord}(P) = i$, $P \subset \sum_{j \geqslant i} P_j$. Let $x, x_1, ..., x_n$ be highest weight vectors of $P, P_i, ..., P_n$, respectively. Then $\exists$ scalars $b_i, .., b_n$ such that $x = \sum_{j \geqslant i} b_j x_j$. Since for each $j = i, i+1, ..., i+m$, $\exists z_j \in U(G_-)$ such that $x_j = z_j y \pmod{U(G_-) M_{d_i+1}}$, we have $\sum_{i \geqslant j \geqslant i+m} b_j z_j y = x - \sum_{j > i+m} b_j x_j \pmod{U(G_-) M_{d_i+1}}$. This easily implies that $y \in M_{d_i+1} + N_q$.

Pick a pair of integers (r, k) satisfying the following conditions:

(i) $r \geqslant d_i + 1$, $q \leqslant k \leqslant t$.

(ii) $y \in M_r + N_k$.

(iii) $y \notin M_r + N_{k+1}$, $y \notin M_{r+1} + N_k$.

Case 1. $k < t$. Apply Proposition 2.1, and we get $[N_k/N_{k+1} : L_{G_\lambda}(\theta)] \neq 0$, where θ is the highest weight of M_{d_i}/M_{d_i-1} and $L_{G_\lambda}(\theta)$ denotes the irreducible G_λ-module with highest weight θ. In particular, $[M(w \cdot \lambda) \cap U(G_\lambda)/K(w \cdot \lambda) \cap U(G_\lambda) \colon L_{G_\lambda}(\theta)] \neq 0$.

Case 2. $k = t$. That is, $y \in M_r + K(w \cdot \lambda)$. By our assumption together with the fact $r > d_i$, $i \notin \operatorname{ord}(U(G) M_r) \cup \operatorname{ord}(K(w \cdot \lambda)) \Rightarrow i \notin \operatorname{ord}(U(G) M_r + K(w \cdot \lambda))$, a contradiction since $x_i = z_i y \pmod{U(G^\lambda) M_{d_i+1}}$ and $\operatorname{ord}(P_i) = i$. Therefore, this cannot happen.

To conclude, since G_λ is symmetrizable, $\theta = w_0 \cdot \lambda$ for some $w_0 \in W_\lambda - \{e\}$. In particular, $\exists j$ such that $w_0 \cdot \lambda(h_j) < 0$, or equivalently, $s_j w_0 > w_0$. Now, for each $w \in A^i_\mu = \{w \in W_\lambda \mid i \in \operatorname{ord} M(w \cdot \lambda),\ i \notin \operatorname{ord}(K(w \cdot \lambda))\}$, $[M(w \cdot \lambda)/K(w \cdot \lambda) \colon L_{G_\lambda}(\theta)] \neq 0$ by the foregoing argument. Thus we have $s_j w > w$, $\forall w \in A^i_\mu$. Thus

$$\sum_{\substack{w \in \bar{A}^i_\mu \\ w \neq e}} (-1)^{l(w)} = -1 \qquad \forall i$$

using Proposition 2.2. As a result,

$$\begin{aligned} -[K(\lambda):L(\mu)] = -n &= \sum_{i=1}^{n} \sum_{\substack{w \in \bar{A}_\mu^i \\ w \neq e}} (-1)^{l(w)} \\ &= \sum_{i=1}^{n} \sum_{\substack{i \in \operatorname{ord}(M(w \cdot \lambda)) \\ w \in W_\lambda \\ w \neq e}} (-1)^{l(w)} \\ &= \sum_{\substack{w \in W_\lambda \\ w \neq e}} (-1)^{l(w)} [M(w \cdot \lambda): L(\mu)]. \end{aligned}$$

This completes the proof.

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