A note on the degree for maximal monotone mappings in finite dimensional spaces

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1. Introduction and preliminaries

In 1983, F. E. Browder established a degree theory for single-valued mappings of class $(S_+)$ in reflexive Banach spaces and based on this degree Browder also constructed a degree theory for mappings of class $(S_+)$ with a perturbation of a maximal monotone mapping or a pseudo-monotone mapping. Browder’s degree has motivated degree theory for various monotone type mappings; see [1–16,19,20] and the references therein. In [16], Zhang and Chen generalized Browder’s theory to multi-valued mappings of class $(S_+)$ (introduced by Petryshyn [17]) and its perturbations with maximal monotone or pseudo-monotone mappings. In particular they construct a degree theory for maximal monotone mappings. To be precise, let $E$ be a reflexive Banach space, $\Omega \subset E$ a non-empty subset, and $T : D(T) \subseteq E \rightarrow 2^E$ a maximal monotone mapping and suppose $\Omega \cap D(T) \neq \emptyset$ and $0 \notin T(\partial \Omega \cap D(T))$. Then the degree of $T$ on $\Omega \cap D(T)$ is defined by

$$\deg(T, \Omega \cap D(T), 0) = \lim_{\lambda \rightarrow 0^+} \deg(T + \lambda J, \Omega \cap D(T), 0),$$

where $J : E \rightarrow 2^E$ is the duality mapping which is a demi-continuous mapping of $(S_+)$ (for complete details see [16, 12]). However it is quite difficult to obtain a homotopy property for this degree; see [6]. In this work we show an easy way to define the degree for a maximal monotone mapping in a finite dimensional space by using the classical Browder degree. To be precise, let $R^d$ be the $n$-dimensional Euclidean space, $T : D(T) \subseteq R^d \rightarrow 2^{R^d}$ a maximal monotone mapping, $\Omega \subset R^d$ an open bounded subset such that $\Omega \cap D(T) \neq \emptyset$ and assume $0 \notin T(\partial \Omega \cap D(T))$. Then we define the topological degree $\deg(T, \Omega \cap D(T), 0)$ of $T$ on $\Omega \cap D(T)$ as the limit of the classical Browder degree $\deg(T_{\lambda}, \Omega, 0)$ as $\lambda \rightarrow 0^+$, where $T_{\lambda} = (T^{-1} + \lambda J)^{-1}$ is the Yosida approximation of $T$. Moreover a homotopy property for the degree of a maximal monotone mapping is obtained using a very simple argument. A homotopy property for the degree of a sub-differential of a continuous convex function is also presented and finally we obtain a product formula for the topological degree of the composition of a continuous mapping with a maximal monotone mapping.
2. Main results

In the following, \( R^n \) is the \( n \)-dimensional Euclidean space, \( T : D(T) \subseteq R^n \rightarrow 2^{R^n} \) a maximal monotone mapping, \( T_\lambda = (T^{-1} + \lambda I)^{-1} \) is the Yosida approximation of \( T \), and \( R_\lambda = I - \lambda T_\lambda \) is the resolvent with respect to \( T_\lambda \).

**Lemma 2.1.** Let \( T : D(T) \subseteq R^n \rightarrow 2^{R^n} \) be a maximal monotone mapping and \( \Omega \subseteq R^n \) an open bounded set such that \( \Omega \cap D(T) \neq \emptyset \) and assume 0 \( \notin \) \( (\partial \Omega \cap D(T)) \). Then there exists \( \lambda_0 > 0 \) such that 0 \( \notin \) \( T_\lambda (\partial \Omega) \) for all \( \lambda \in (0, \lambda_0) \).

**Proof.** Suppose the conclusion is not true. Then there exist \( \lambda_j \rightarrow 0 \), \( x_j \in \partial \Omega \) such that \( x_j \rightarrow x_0 \in \partial \Omega \) and

\[
T_{\lambda_j} x_j = 0.
\]

By the monotonicity of \( T \) we have

\[
(f - T_{\lambda_j} x_j, x - R_{\lambda_j} x_j) \geq 0, \quad \text{for all} \ x \in D(T), \ f \in Tx,
\]

and \( R_{\lambda_j} x_j = x_j - \lambda_j^{-1} T_{\lambda_j} x_j \rightarrow x_0 \), and therefore we have

\[
(f, x - x_0) \geq 0, \quad \text{for all} \ x \in D(T), \ f \in Tx.
\]

The maximal monotonicity of \( T \) implies that \( x_0 \in D(T) \) and 0 \( \notin \) \( Tx_0 \), which is a contradiction. The proof is complete. \( \square \)

Under the assumption of Lemma 2.1, there exists \( \lambda_0 > 0 \) such that 0 \( \notin \) \( T_\lambda (\partial \Omega) \) for all \( \lambda \in (0, \lambda_0) \). Since \( T_\lambda : R^n \rightarrow R^n \) is continuous (see [6]) the Brouwer degree \( \deg(T_\lambda, \Omega, 0) \) is well defined for \( \lambda \in (0, \lambda_0) \). We define the degree \( \deg(t, \Omega \cap D(T), 0) \) of \( T \) on \( \Omega \cap D(T) \) as

\[
\deg(t, \Omega \cap D(T), 0) = \lim_{\lambda \rightarrow 0^+} \deg(T_\lambda, \Omega, 0).
\]

Note that Lemma 2.6 in [6] guarantees that \( T_{\lambda_1 + (1 - t)\lambda_2} x : [0, 1] \times R^n \rightarrow R^n \) is continuous for \( \lambda_1, \lambda_2 > 0 \). Thus the homotopy property of the Brouwer degree implies that \( \deg(T_{\lambda_1}, \Omega, 0) = \deg(T_{\lambda_2}, \Omega, 0) \) for \( \lambda_1, \lambda_2 \in (0, \lambda_0) \). As a result our degree is well defined.

**Theorem 2.2.** Let \( T : D \subseteq R^n \rightarrow 2^{R^n}, S : D \subseteq R^n \rightarrow 2^{R^n} \) be two maximal monotone mappings and \( \Omega \subseteq R^n \) an open bounded set. Assume that \( iT + (1 - t)S \) is maximal monotone for each \( t \in (0, 1) \) and 0 \( \notin \) \( \cup_{t \in [0, 1]} iT + (1 - t)S(\partial \Omega \cap D) \). Then \( \deg(T, \Omega \cap D, 0) = \deg(S, \Omega \cap D, 0) \).

**Proof.** We first prove that there exists \( \lambda_0 > 0 \) such that

\[
0 \notin \cup_{t \in [0, 1]} iT + (1 - t)S, \quad \text{for all} \ \lambda \in (0, \lambda_0)
\]

where \( iT + (1 - t)S \) is the Yosida approximation of \( iT + (1 - t)S \).

Assume this is not true. Then there exist \( t_0, \lambda_j \rightarrow 0, x_j \in \partial \Omega \cap D \) such that \( x_j \rightarrow x_0 \in \partial \Omega \) and \( [t_0 T + (1 - t_0)S] x_j \neq 0 \). Since

\[
(t_0 f + (1 - t_0) g - [t_0 T + (1 - t_0)S] x_j, x - R_{t_0} x_j) \geq 0,
\]

for all \( x \in D, f \in Tx, g \in Sx, \) where \( R_{t_0} x_j \) is the resolvent with respect to \( [t_0 T + (1 - t_0)S] x_j \), and \( R_{t_0} x_j = x_j - \lambda_j [t_0 T + (1 - t_0)S] x_j \), letting \( j \rightarrow \infty \) we have

\[
(t_0 f + (1 - t_0) g, x - x_0) \geq 0, \quad \text{for all} \ x \in D, \ f \in Tx, \ g \in Sx.
\]

This (see [10, pg. 122]) \( x_0 \in D \) and 0 \( \notin \) \( t_0 Tx_0 + (1 - t_0)Sx_0 \), which is a contradiction. Therefore (2.3) is true. By Lemma 2.6 in [6], \( [iT + (1 - t)S] x : [0, 1] \times R^n \rightarrow R^n \) is continuous, so the homotopy property of the Brouwer degree implies that

\[
\deg(T_{\lambda_1}, \Omega, 0) = \deg(S_{\lambda_2}, \Omega, 0).
\]

Thus \( \deg(T, \Omega \cap D, 0) = \deg(S, \Omega \cap D, 0) \). \( \square \)

**Corollary 2.3.** Let \( T : D \subseteq R^n \rightarrow 2^{R^n}, S : D \subseteq R^n \rightarrow 2^{R^n} \) be two maximal monotone mappings and \( \Omega \subseteq R^n \) an open bounded set. Assume that \( iT + (1 - t)S \) is maximal monotone for each \( t \in (0, 1) \), 0 \( \notin \) \( \partial \Omega \cap D \), 0 \( \notin \) \( S(\partial \Omega \cap D) \), and \( (f, g) \geq 0 \) for all \( f \in Tx, \ g \in Sx, \ x \in \partial \Omega \cap D \). Then \( \deg(T, \Omega \cap D, 0) = \deg(S, \Omega \cap D, 0) \).

**Proof.** It is easy to see that 0 \( \notin \) \( \cup_{t \in [0, 1]} iT + (1 - t)S(\partial \Omega \cap D) \). The conclusion follows from Theorem 2.2. \( \square \)

Let \( \phi(x) : D(\phi) \subseteq R^n \rightarrow R \) be a lower semi-continuous convex function. The sub-differential \( \partial \phi(u) \) at \( u \) is defined by

\[
\partial \phi(u) = \{ f \in R^n : \phi(x) - \phi(u) \geq (f, x - u), \ \text{for all} \ x \in D(\phi) \}.
\]

Note that \( \partial \phi \) is a maximal monotone mapping. For \( \lambda > 0 \), let

\[
\phi_\lambda(x) = \inf_{y \in R^n} \left[ \phi(y) + \frac{1}{2\lambda} \| x - y \|^2 \right]
\]

be the so-called Yosida–Moreau regularization of \( \phi \). It is well known that \( \phi_\lambda(x) = \phi(R_\lambda x) + \frac{1}{2\lambda} \| x - R_\lambda x \|^2 \), where \( R_\lambda \) is the resolvent with respect to \( (\partial \phi)_\lambda \), \( \lim_{\lambda \rightarrow 0^+} \phi_\lambda(x) = \phi(x) \) for all \( x \in D(\phi) \), and \( (\partial \phi)_\lambda = \partial \phi_\lambda = \Delta \phi_\lambda \).
Theorem 2.4. Let \( \phi(t, x) : [0, 1] \times \mathbb{R}^n \to \mathbb{R} \) be a continuous function such that \( \phi(t, x) \) is convex in \( x \) for each \( t \in [0, 1] \), and \( \Omega \subset \mathbb{R}^n \) an open bounded set. Suppose \( \phi(t, \cdot) \) does not obtain a minimum on \( \partial \Omega \) for each \( t \in [0, 1] \). Then \( \text{deg}(\partial \phi(t, \cdot), \Omega, 0) \) does not depend on \( t \in [0, 1] \).

Proof. Since \( \phi(t, \cdot) \) does not obtain a minimum on \( \partial \Omega \) for each \( t \in [0, 1] \), \( 0 \not\in \partial \phi(t, \partial \Omega) \) for \( t \in [0, 1] \). We claim that there exists \( \lambda_0 > 0 \) such that
\[
0 \not\in (\partial \phi(t_j, \cdot), (\partial \Omega)), \quad \text{for all } \lambda \in (0, \lambda_0), \ t \in [0, 1].
\]
Assume this is false. Then there exist \( \lambda_j \to 0^+, t_j \to t_0, x_j \in \partial \Omega \) with \( x_j \to x_0 \) such that
\[
(\partial \phi(t_j, \cdot))_j x_j = 0, \quad j = 1, 2, \ldots
\]
Since \( \phi_j(t_j, \cdot)x_j - \phi_j(t_j, \cdot)x_j \geq (\partial \phi_j(t_j, \cdot)x_j, x - x_j) \), for all \( x \in \mathbb{R}^n, j = 1, 2, \ldots \), letting \( j \to \infty \) we get
\[
\phi(t_0, x) - \phi(t_0, x_0) \geq 0, \quad \text{for all } x \in \mathbb{R}^n.
\]
Therefore \( 0 \in \partial \phi(t_0, x_0) \), which is a contradiction. From [18], we know that \( \partial \phi \) is a maximal monotone mapping. By the homotopy property of the Brouwer degree implies that \( \text{deg}(\partial \phi(t_j, \cdot), \partial \Omega, 0) \) does not depend on \( t \in [0, 1] \). Therefore \( \text{deg}(\partial \phi(t, \cdot), \Omega, 0) \) does not depend on \( t \in [0, 1] \). \( \square \)

Let \( \Omega \subset \mathbb{R}^n \) be an open bounded set, \( p \in \mathbb{R}^n \) and \( T : D(T) \subset \mathbb{R}^n \to 2^{\mathbb{R}^n} \) a maximal monotone mapping. By the homotopy property we have that \( \text{deg}(T, \Omega \cap D(T), p) \) has the same value as \( p \) ranges through the same connected component \( U \) of \( \mathbb{R}^n \setminus (\partial \Omega \cap D(T)) \). We denote this value by \( \text{deg}(T, \Omega \cap D(T), U) \).

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a continuous function, \( T : D(T) \subset \mathbb{R}^n \to 2^{\mathbb{R}^n} \) a maximal monotone mapping, \( \Omega \subset \mathbb{R}^n \) an open bounded set such that \( \Omega \cap D(T) \neq \emptyset \) and \( p \in \mathbb{R}^n \). Suppose \( p \not\in f(T(\partial \Omega \cap D(T))) \). Then it is straightforward to prove that there exists \( \lambda_0 > 0 \) such that \( p \not\in f(T_\lambda(\partial \Omega)) \) for \( \lambda \in (0, \lambda_0) \). We can define \( \text{deg}(f(T), \Omega \cap D(T), p) = \lim_{\lambda \to 0^+} \text{deg}(f(T_\lambda), \Omega, p) \). This degree is well defined (use a homotopy argument similar to the one used after (2.2)).

Theorem 2.5. Let \( f, T, \Omega \) be as above, and \( p \in \mathbb{R}^n \) such that \( T(\overline{\Omega} \cap D(T)) \subset B(0, r) \), where \( r > 0 \) is a constant and \( B(0, r) \) is the open ball centered at zero with radius \( r \), and \( p \not\in f(T(\overline{\Omega} \cap D(T))) \). Then
\[
\text{deg}(f(T), \Omega \cap D(T), p) = \bar{\Sigma}_i \text{deg}(f(U_i), p) \text{deg}(T, \Omega \cap D(T), U_i), \quad (2.4)
\]
where the right hand side only has finitely many nonzero terms, and \( U_i \) are bounded connected components of \( \mathbb{R}^n \setminus (\partial \Omega \cap D(T)) \).

Proof. We first prove that (2.4) only has finitely many nonzero terms. Now \( f^{-1}(p) \cap B(0, r) \) is compact, \( f^{-1}(p) \cap \overline{B(0, r)} \subset \mathbb{R}^n \setminus (\partial \Omega \cap D(T)) = \cup_{i=1}^{k+1} U_i \) where \( U_i \) are connected components (note that the maximal monotonicity of \( T \) implies that \( T(\partial \Omega \cap D(T)) \) is closed in \( \mathbb{R}^n \), so \( R^n \setminus (\partial \Omega \cap D(T)) \) is open). Therefore there exist finitely many \( i \), say \( i = 1, 2, \ldots, k, k+1 \), such that
\[
f^{-1}(p) \cap \overline{B(0, r)} \subset \cup_{i=1}^{k+1} U_i \quad \text{and} \quad f^{-1}(p) \cap U_i = \emptyset \text{ for } i \geq k+2
\]
and \( U_{k+1} = U_\infty \cap B(0, r+1) \) where \( U_\infty \) is the unbounded connected component of \( \mathbb{R}^n \setminus (\partial \Omega \cap D(T)) \). Now \( \text{deg}(f, U_i, p) = 0 \) (since \( f^{-1}(p) \cap U_i = \emptyset \) for \( j \geq k+2 \) and \( \text{deg}(T, \Omega \cap D(T), U_{k+1}) = 0 \) (since \( T(\overline{\Omega} \cap D(T)) \subset B(0, r) \)). Thus the right hand side of (2.4) only has finitely many nonzero terms.

Finally from the Brouwer degree (see [12, Theorem 1.2.13]) we have
\[
\Sigma_{i=1}^{k+1} \text{deg}(f(U_i), p) \text{deg}(T, \Omega \cap D(T), U_i) = \sum_{i=1}^{k+1} \text{deg}(f(U_i), p) \lim_{\lambda \to 0^+} \text{deg}(T_\lambda, \Omega, U_i) \\
= \lim_{\lambda \to 0^+} \text{deg}(f(T_\lambda), \Omega, p) = \text{deg}(f(T), \Omega \cap D(T), p). \quad \square
\]

A proof similar to that of Theorem 2.5 yields the following result.

Theorem 2.6. Let \( f, T, \Omega \) be as above, and \( p \in \mathbb{R}^n \) such that \( f^{-1}(p) \subset B(0, r) \), where \( r > 0 \) is a constant and \( B(0, r) \) is the open ball centered at zero with radius \( r \), and \( p \not\in f(T(\overline{\Omega} \cap D(T))) \). Then
\[
\text{deg}(f(T), \Omega \cap D(T), p) = \Sigma \text{deg}(f(U_i), p) \text{deg}(T, \Omega \cap D(T), U_i),
\]
where the right hand side only has finitely many nonzero terms, and \( U_i \) are bounded connected components of \( \mathbb{R}^n \setminus (\partial \Omega \cap D(T)) \subset B(0, r) \).

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References

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