A note on the degree for maximal monotone mappings in finite dimensional spaces

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Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space, $T : D(T) \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$ a maximal monotone mapping, and $\Omega \subset \mathbb{R}^n$ an open bounded subset such that $\Omega \cap D(T) \neq \emptyset$ and assume $0 \notin T(\partial \Omega \cap D(T))$. In this note we show an easy way to define the topological degree
$\text{deg}(T, \Omega \cap D(T), 0)$ of $T$ on $\Omega \cap D(T)$ as the limit of the classical Brouwer degree
$\text{deg}(T_\lambda, \Omega, 0)$ as $\lambda \to 0^+$; here $T_\lambda$ is the Yosida approximation of $T$. Furthermore, if $T_i : D \to 2^{\mathbb{R}^n}, i = 1, 2$, are two maximal monotone mappings such that $\Omega \cap D \neq \emptyset$ and $0 \not\in \cup_{i \in [0, 1]} [T_i + (1 - t)T_2](\partial \Omega \cap D)$ and if $T_\lambda + (1 - \lambda)T_2$ is maximal monotone for each $t \in [0, 1]$, we give an easy argument to show $\text{deg}(T_i, D \cap \Omega, 0) = \text{deg}(T_\lambda, D \cap \Omega, 0)$.

1. Introduction and preliminaries

In 1983, F. E. Browder established a degree theory for single-valued mappings of class $(S)_+$ in reflexive Banach spaces and based on this degree Browder also constructed a degree theory for mappings of class $(S)_+$ with a perturbation of a maximal monotone mapping or a pseudo-monotone mapping. Browder’s degree has motivated degree theory for various monotone type mappings; see [1–16,19,20] and the references therein. In [16], Zhang and Chen generalized Browder’s theory to multi-valued mappings of class $(S)_+$ (introduced by Petryshyn [17]) and its perturbations with maximal monotone or pseudo-monotone mappings. In particular they construct a degree theory for maximal monotone mappings. To be precise, let $E$ be a reflexive Banach space, $\Omega \subset E$ a non-empty subset, and $T : D(T) \subseteq E \to 2^E$ a maximal monotone mapping and suppose $\Omega \cap D(T) \neq \emptyset$ and $0 \not\in T(\partial \Omega \cap D(T))$. Then the degree of $T$ on $\Omega \cap D(T)$ is defined by

$$\text{deg}(T, \Omega \cap D(T), 0) = \lim_{\lambda \to 0^+} \text{deg}(T + \lambda J, \Omega \cap D(T), 0),$$

where $J : E \to 2^E$ is the duality mapping which is a demi-continuous mapping of $(S)_+$ (for complete details see [16, 12]). However it is quite difficult to obtain a homotopy property for this degree; see [6]. In this work we show an easy way to define the degree for a maximal monotone mapping in a finite dimensional space by using the classical Brouwer degree. To be precise, let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space, $T : D(T) \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$ a maximal monotone mapping, $\Omega \subset \mathbb{R}^n$ an open bounded subset such that $\Omega \cap D(T) \neq \emptyset$ and assume $0 \not\in T(\partial \Omega \cap D(T))$. Then we define the topological degree
$\text{deg}(T, \Omega \cap D(T), 0)$ of $T$ on $\Omega \cap D(T)$ as the limit of the classical Brouwer degree
$\text{deg}(T_\lambda, \Omega, 0)$ as $\lambda \to 0^+$, where $T_\lambda = (T^{-1} + \lambda I)^{-1}$ is the Yosida approximation of $T$. Moreover a homotopy property for the degree of a maximal monotone mapping is obtained using a very simple argument. A homotopy property for the degree of a sub-differential of a continuous convex function is also presented and finally we obtain a product formula for the topological degree of the composition of a continuous mapping with a maximal monotone mapping.

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2. Main results

In the following, $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, $T : D(T) \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$ a maximal monotone mapping, $T_\lambda = (T^{-1} + \lambda I)^{-1}$ is the Yosida approximation of $T$, and $R_\lambda = I - \lambda T_\lambda$ is the resolvent with respect to $T_\lambda$.

**Lemma 2.1.** Let $T : D(T) \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a maximal monotone mapping and $\Omega \subseteq \mathbb{R}^n$ an open bounded set such that $\Omega \cap D(T) \neq \emptyset$ and assume $0 \not\in T(\partial \Omega \cap D(T))$. Then there exists $\lambda_0 > 0$ such that $0 \not\in T_\lambda(\partial \Omega)$ for all $\lambda \in (0, \lambda_0)$.

**Proof.** Suppose the conclusion is not true. Then there exist $\lambda_j \to 0$, $x_j \in \partial \Omega$ such that $x_j \to x_0 \in \partial \Omega$ and

$$T_{\lambda_j}x_j = 0.$$  \hspace{1cm} (2.1)

By the monotonicity of $T$ we have

$$(f - T_{\lambda_j}x_j, x - R_{\lambda_j}x_j) \geq 0, \quad \text{for all } f \in D(T), \; x \in Tx,$$

and $R_{\lambda_j}x_j = x_j - \lambda_j^{-1}T_{\lambda_j}x_j \to x_0$, and therefore we have

$$(f, x - x_0) \geq 0, \quad \text{for all } f \in D(T), \; x \in Tx.$$  

The maximal monotonicity of $T$ implies that $x_0 \in D(T)$ and $0 \in Tx_0$, which is a contradiction. The proof is complete. \hspace{1cm} $\square$

Under the assumption of Lemma 2.1, there exists $\lambda_0 > 0$ such that $0 \not\in T_\lambda(\partial \Omega)$ for all $\lambda \in (0, \lambda_0)$. Since $T_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is continuous (see [6]) the Brouwer degree $\deg(T_\lambda, \Omega, 0)$ is well defined for $\lambda \in (0, \lambda_0)$.

We define the degree $\deg(T, \lambda_0 \cap D(T), 0)$ on $\lambda \cap D(T)$ as

$$\deg(T, \lambda_0 \cap D(T), 0) = \lim_{\lambda \to 0^+} \deg(T_\lambda, \Omega, 0).$$

Note that Lemma 2.6 in [6] guarantees that $T_{\lambda_1 + (1 - \lambda_2)x} : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous for $\lambda_1, \lambda_2 > 0$. Thus the homotopy property of the Brouwer degree implies that $\deg(T_{\lambda_1}, \lambda_0, 0) = \deg(T_{\lambda_2}, \lambda_0, 0)$ for $\lambda_1, \lambda_2 \in (0, \lambda_0)$. As a result our degree is well defined.

**Theorem 2.2.** Let $T : D \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$, $S : D \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be two maximal monotone mappings and $\Omega \subseteq \mathbb{R}^n$ an open bounded set. Assume that $IT + (1 - t)S$ is maximal monotone for each $t \in (0, 1)$ and $0 \not\in \cup_{t \in [0, 1]}(IT + (1 - t)S)(\partial \Omega \cap D(T))$. Then

$$\deg(T, \lambda_0 \cap D(T), 0) = \deg(S, \lambda_0 \cap D(T), 0).$$

**Proof.** We first prove that there exists $\lambda_0 > 0$ such that

$$0 \not\in \cup_{t \in (0, 1]}(IT + (1 - t)S)_t(\partial \Omega \cap D)$$

for all $\lambda \in (0, \lambda_0)$ \hspace{1cm} (2.3)

where $[IT + (1 - t)S]_t$ is the Yosida approximation of $IT + (1 - t)S$.

Assume this is not true. Then there exist $t_j \to t_0, \lambda_j \to 0, x_j \in \partial \Omega \cap D$ such that $x_j \to x_0 \in \partial \Omega$ and $[t_jT + (1 - t_j)S]_{\lambda_j}x_j = 0$.

Since $(tf + (1 - t_j)g - [t_jT + (1 - t_j)S]_{\lambda_j}x_j, x - R_{\lambda_j}^{g}x_j) \geq 0$, for all $x \in D, f \in Tx, g \in Sx$, where $R_{\lambda_j}^{g}x_j$ is the resolvent with respect to $[t_jT + (1 - t_j)S]_{\lambda_j}$, and $R_{\lambda_j}^{g}x_j - x_j = -\lambda_j[tf + (1 - t_j)S]_{\lambda_j}x_j = 0$, letting $j \to \infty$ we have

$$(t_0f + (1 - t_0)g, x - x_0) \geq 0, \quad \text{for all } x \in D, \; f \in Tx, \; g \in Sx.$$  

Thus (see [10, pg. 122]) $x_0 \in D$ and $0 \in t_0T_{x_0} + (1 - t_0)S_{x_0}$, which is a contradiction. Therefore (2.3) is true. By Lemma 2.6 in [6], $[IT + (1 - t)S]_t : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, so the homotopy property of the Brouwer degree implies that

$$\deg(T_{\lambda_j}, \Omega, 0) = \deg(S_{\lambda_j}, \Omega, 0).$$

Thus $\deg(T, \Omega \cap D(T), 0) = \deg(S, \Omega \cap D(T), 0)$. \hspace{1cm} $\square$

**Corollary 2.3.** Let $T : D \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$, $S : D \subseteq \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be two maximal monotone mappings and $\Omega \subseteq \mathbb{R}^n$ an open bounded set. Assume that $IT + (1 - t)S$ is maximal monotone for each $t \in (0, 1)$, $0 \not\in T(\partial \Omega \cap D)$, $0 \not\in S(\partial \Omega \cap D)$, and $(f, g) \geq 0$ for all $f \in Tx, g \in Sx, x \in \partial \Omega \cap D$. Then $\deg(T, \Omega \cap D(T), 0) = \deg(S, \Omega \cap D(T), 0)$.

**Proof.** It is easy to see that $0 \not\in \cup_{t \in (0, 1]}(IT + (1 - t)S)(\partial \Omega \cap D)$. The conclusion follows from Theorem 2.2. \hspace{1cm} $\square$

Let $\phi(x) : D(\phi) \subseteq \mathbb{R}^n \to R$ be a lower semi-continuous convex function. The sub-differential $\partial \phi(u)$ at $u$ is defined by

$$\partial \phi(u) = \{f \in \mathbb{R}^n : \phi(x) - \phi(u) \geq (f, x - u), \quad \text{for all } x \in D(\phi)\}.$$  

Note that $\partial \phi$ is a maximal monotone mapping. For $\lambda > 0$, let

$$\psi_{\lambda}(x) = \inf_{y \in \mathbb{R}^n} \left[ \phi(y) + \frac{1}{2\lambda} \|x - y\|^2 \right]$$

be the so called Yosida–Moreau regularization of $\phi$. It is well known that $\psi_{\lambda}(x) = \phi(R_{\lambda}x) + \frac{1}{2\lambda} \|x - R_{\lambda}x\|^2$, where $R_{\lambda}$ is the resolvent with respect to $(\partial \phi)_{\lambda}$, $\lim_{\lambda \to 0^+} \psi_{\lambda}(x) = \phi(x)$ for all $x \in D(\phi)$, and $(\partial \phi)_{\lambda} = \partial \psi_{\lambda} = \Delta \phi_{\lambda}$.\hspace{1cm}
Theorem 2.4. Let \( \phi(t, x) : [0, 1] \times \mathbb{R}^n \to \mathbb{R} \) be a continuous function such that \( \phi(t, x) \) is convex in \( x \) for each \( t \in [0, 1] \), and \( \Omega \subset \mathbb{R}^n \) an open bounded set. Suppose \( \phi(t, \cdot) \) does not obtain a minimum on \( \partial \Omega \) for each \( t \in [0, 1] \). Then \( \deg(\partial \phi(t, \cdot), \Omega, 0) \) does not depend on \( t \in [0, 1] \).

Proof. Since \( \phi(t, \cdot) \) does not obtain a minimum on \( \partial \Omega \) for each \( t \in [0, 1] \), \( 0 \not\in \partial \phi(t, \partial \Omega) \) for \( t \in [0, 1] \). We claim that there exists \( \lambda_0 > 0 \) such that

\[
0 \not\in (\partial \phi(t, \cdot)_j(\partial \Omega)), \quad \text{for all } \lambda \in (0, \lambda_0), \ t \in [0, 1].
\]

Assume this is false. Then there exist \( \lambda_j \to 0^+, t_j \to t, x_j \in \partial \Omega \) with \( x_j \to x_0 \) such that

\[
(\partial \phi(t_j, \cdot))_j x_j = 0, \quad j = 1, 2, \ldots.
\]

Since \( \phi_j(t, \cdot)x - \phi_j(t, \cdot)x_j \geq (\partial \phi_j(t, \cdot)x_j, x - x_j) \), for all \( x \in \mathbb{R}^n, j = 1, 2, \ldots \), letting \( j \to \infty \) we get

\[
\phi(t_0, x) - \phi(t_0, x_0) \geq 0, \quad \text{for all } x \in \mathbb{R}^n.
\]

Therefore \( 0 \not\in \partial \phi(t_0, x_0) \), which is a contradiction. From [18], we know that \( \partial \phi_\lambda = \Delta \phi_\lambda : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous, so the homotopy property of the Brouwer degree implies that \( \deg(\partial \phi_\lambda(t, \cdot), \Omega, 0) \) does not depend on \( t \in [0, 1] \). Therefore \( \deg(\partial \phi(t, \cdot), \Omega, 0) \) does not depend on \( t \in [0, 1] \). \( \square \)

Let \( \Omega \subset \mathbb{R}^n \) be an open bounded set, \( p \in \mathbb{R}^n \) and \( T : D(T) \subset \mathbb{R}^n \to 2^{\mathbb{R}^n} \) a maximal monotone mapping. By the homotopy property we have that \( \deg(T, \Omega \cap D(T), p) \) has the same value as \( p \) ranges through the same connected component \( U \) of \( \mathbb{R}^n \setminus (\partial \Omega \cap D(T)). \) We denote this value by \( \deg(T, \Omega \cap D(T), U)_p \).

\[
\deg(f(T), \Omega \cap D(T), p) = \Sigma_i \deg(f, U_i, p) \deg(T, \Omega \cap D(T), U_i), \quad (2.4)
\]

where the right hand side only has finitely many nonzero terms, and \( U_i \) are bounded connected components of \( \mathbb{R}^n \setminus (\partial \Omega \cap D(T)). \)

Proof. We first prove that (2.4) only has finitely many nonzero terms. Now \( f^{-1}(p) \cap B(0, r) \) is compact, \( f^{-1}(p) \cap B(0, r) \subset \mathbb{R}^n \setminus (\partial \Omega \cap D(T)) = \bigcup_{i=1}^{k} U_i \) where \( U_i \) are connected components (note that the maximal monotonicity of \( T \) implies that \( T(\partial \Omega \cap D(T)) \) is closed in \( \mathbb{R}^n \), so \( \mathbb{R}^n \setminus (\partial \Omega \cap D(T)) \) is open). Therefore there exist finitely many \( i, \) say \( i = 1, 2, \ldots, k, k+1 \), such that

\[
f^{-1}(p) \cap B(0, r) \subset \bigcup_{i=1}^{k+1} U_i \quad \text{and } f^{-1}(p) \cap U_i = \emptyset \quad \text{for } i \geq k + 2
\]

and \( U_{k+1} = U_{\infty} \cap B(0, r + 1) \) where \( U_{\infty} \) is the unbounded connected component of \( \mathbb{R}^n \setminus (\partial \Omega \cap D(T)) \). Now \( \deg(f, U_i, p) = 0 \) (since \( f^{-1}(p) \cap U_j = \emptyset \) for \( j \geq k + 2 \) and \( \deg(T, \Omega \cap D(T), U_{k+1}) = 0 \) (since \( T(\partial \Omega \cap D(T)) \subset B(0, r) \)). Thus the right hand side of (2.4) only has finitely many nonzero terms.

Finally from the Brouwer degree (see [12, Theorem 1.2.13]) we have

\[
\Sigma_{i=1}^{k+1} \deg(f, U_i, p) \deg(T, \Omega \cap D(T), U_i) = \Sigma_{i=1}^{k+1} \deg(f, U_i, p) \lim_{\lambda \to 0^+} \deg(T, \lambda, \Omega, U_i)
\]

\[
= \lim_{\lambda \to 0^+} \deg(f(T, \lambda), \Omega, p) \deg(f(T), \Omega \cap D(T), p). \quad \square
\]

A proof similar to that of Theorem 2.5 yields the following result.

Theorem 2.6. Let \( f, T, \Omega \) be as above, and \( p \in \mathbb{R}^n \) such that \( f^{-1}(p) \subset B(0, r) \), where \( r > 0 \) is a constant and \( B(0, r) \) is the open ball centered at zero with radius \( r \), and \( p \not\in f(T(\partial \Omega \cap D(T))). \) Then

\[
\deg(f(T), \Omega \cap D(T), p) = \Sigma_i \deg(f, U_i, p) \deg(T, \Omega \cap D(T), U_i),
\]

where the right hand side only has finitely many nonzero terms, and \( U_i \) are bounded connected components of \( \mathbb{R}^n \setminus (T(\partial \Omega \cap D(T)) \subset B(0, r)). \)

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