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Discrete homotopies and the fundamental group

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Abstract

We generalize and strengthen the theorem of Gromov that the fundamental group of any compact Riemannian manifold of diameter at most D has a set of generators $g_1, \ldots, g_k$ of length at most $2D$ and relators of the form $g_i g_m = g_j$. In particular, we obtain an explicit bound for the number k of generators in terms of the number of "short loops" at every point and the number of balls required to cover a given semi-locally simply connected geodesic space. As a corollary we obtain a fundamental group finiteness theorem (new even for Riemannian manifolds) that replaces the curvature and volume conditions of Anderson and the 1-systole bound of Shen–Wei, by more general geometric hypothesis implied by these conditions. This theorem, in turn, is a special case of a theorem for arbitrary compact geodesic spaces, proved using the method of discrete homotopies introduced by the first author and V. N. Berestovskii. Central to the proof is the notion of "homotopy critical spectrum", introduced in this paper as a natural consequence of discrete homotopy methods. This spectrum is closely related to the Sormani–Wei covering spectrum which is a subset of the classical length spectrum studied by de Verdiere and Duistermaat–Guillemin. It is completely determined (including multiplicity) by special closed geodesics called "essential circles".

Keywords: Fundamental group; Finiteness theorem; Discrete homotopy; Gromov generators; Length spectrum; Covering spectrum; Homotopy critical spectrum

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1. Introduction

In [19,20], Gromov proved the following: If M is a compact Riemannian manifold of diameter D then $\pi_1(M)$ has a set of generators $g_1, \ldots, g_k$ represented by loops of length at most $2D$ and relations of the form $g_i g_m = g_j$. Among the uses for this theorem are fundamental group finiteness theorems: If $\mathcal{X}$ is any collection of spaces with a global bound N for the number of elements of $\pi_1(M)$ represented by loops of length at most $2D$ in any $X \in \mathcal{X}$ then $\pi_1(X)$ has at most N generators and N^3 relators for any $X \in \mathcal{X}$. Therefore there are only finitely many possible fundamental groups of spaces in $\mathcal{X}$. This strategy was employed by Michael Anderson [2] to show that compact n-manifolds with global lower bounds on volume and Ricci curvature, and diameter $\leq D$, have finitely many possible fundamental groups. Shen–Wei [31] reached the same conclusion, replacing the lower volume bound by a positive uniform lower bound on the 1-systole (the infimum of lengths of non-null closed geodesics). Note that the Anderson and Shen–Wei Theorems are independent [22].

In this paper we generalize Gromov's theorem by giving an explicit bound for the number k of generators in terms of the number of "short loops" and the number of balls required to cover a space, at a given scale. As a consequence we are able to prove a finiteness theorem (Corollary 2) for fundamental groups of compact geodesic spaces that generalizes and dissects some of the essential ideas of both of the previously mentioned finiteness theorems. Our most general theorem (Theorem 3) applies to certain deck groups $\pi_\varepsilon(X)$ of covering maps that measure the fundamental group at a given scale, refining a finiteness theorem of Sormani–Wei (Proposition 7.8, [29])—see also Remark 52.

We will first state our main result for fundamental groups, saving the more general theorem from which it follows until we provide a little background concerning the method of discrete homotopies.

For any path c in a metric space X, we define $|[c]|$ to be the infimum of the lengths of paths in the fixed endpoint homotopy class of c. For any $L > 0$, let $\Gamma(X, L)$ be the supremum, over all possible basepoints $*$, of the number of distinct elements $g \in \pi_1(X, *)$ such that $|g| \leq L$. For a compact geodesic space there may be no rectifiable curves in the homotopy class of a path (cf. [8]), and certainly $\Gamma(X, L)$ need not be finite (e.g. a geodesic Hawaiian Earring). If X is *semilocally simply connected* then $|g|$ and $\Gamma(X, L)$ are always both finite (Theorem 26, Corollary 51), and the 1-systole of X is positive if X is not simply connected (Corollary 43). We denote by $C(X, r, s)$ (resp. $C(X, s)$) the minimum number of open s-balls required to cover a closed r-ball in X (resp. X).

Theorem 1. *Suppose X is a semilocally simply connected, compact geodesic space of diameter D, and let $\varepsilon > 0$. Then for any choice of basepoint, $\pi_1(X)$ has a set of generators $g_1, \ldots, g_k$ of length at most $2D$ and relations of the form $g_i g_m = g_j$ with*

$$k \leq \frac{8(D+\varepsilon)}{\varepsilon} \cdot \Gamma(X, \varepsilon) \cdot C\left(X, \frac{\varepsilon}{4}\right)^{\frac{8(D+\varepsilon)}{\varepsilon}}.$$

In particular, if the 1-systole of X is $\sigma > 0$ then we may take

$$k \leq \frac{8(D+\sigma)}{\sigma} C\left(X, D, \frac{\sigma}{4}\right)^{\frac{8(D+\sigma)}{\sigma}}.$$

Here we have captured the key elements in Shen–Wei's finiteness theorem, because they observe that $C(X, r, s)$ is uniformly controlled in any collection of Riemannian manifolds which are precompact in the Gromov–Hausdorff topology and use the lower bound on Ricci curvature to provide that precompactness [20]. In so doing, we have replaced the lower bound on first systole with $\Gamma(X, L)$. On the other hand, Anderson showed (Remark 2.2(2), [2]) that if M has Ricci curvature $\geq -(n-1)k^2$, diameter $\leq D$ and volume $\geq v$ then for any basepoint, the subgroup of $\pi_1(M)$ generated by loops of length less than $\frac{Dv}{v_k(2D)}$ has order bounded by above by $\frac{v_k(2D)}{v}$ (here $v_k(2D)$ is the volume of the $2D$-ball in the space form of curvature $-k$ and dimension $n = \dim M$). In other words, for the class of spaces with these uniform bounds, with $\varepsilon := \frac{Dv}{v_k(2D)}$ one has $\Gamma(M, \varepsilon) \leq \frac{v_k(2D)}{v}$. Therefore Theorem 1 is also an extension of Anderson's finiteness theorem. In fact, we have the following.

Corollary 2. *Let $\mathcal{X}$ be any Gromov–Hausdorff precompact class of semilocally simply connected compact geodesic spaces. If there are numbers $\varepsilon > 0$ and N such that for every $X \in \mathcal{X}$, $\Gamma(X, \varepsilon) \leq N$, then there are finitely many possible fundamental groups for spaces in $\mathcal{X}$.*

We should point out a subtle but important difference between Anderson's final step (i.e. from Remark 2.2(2) to the finiteness theorem) and our proof. Anderson's final step depends on the fact that the universal covering space also has Ricci curvature $\geq -(n-1)k^2$ and hence one may use Bishop's volume comparison theorem in the universal covering space. One can "translate" his argument into one that relies instead on global control of the numbers $C(X, r, s)$ in the universal cover. However, this approach *requires that one know that the collection of all universal covers of all spaces in the class is (pointed) Gromov–Hausdorff precompact*. The Shen–Wei and Sormani–Wei theorems also rely on precompactness of the universal covering spaces, which they show is true with a lower bound on the 1-systole. But without a lower bound on the 1-systole it is in general impossible to conclude from precompactness of a class of spaces that the collection of their universal covers is precompact (Example 46), so we cannot use this strategy in our proof.

We now review the work of Berestovskii and the first author [5,6] as it applies in the special case where X is a metric space. For $\varepsilon > 0$, an *ε-chain* is a finite sequence $\alpha := \{x_0, \ldots, x_n\}$ such that

$$d(x_i, x_{i+1}) < \varepsilon \quad \text{for all } i. \tag{1}$$

We define the *length* of α to be

$$L(\alpha) = \sum_{i=1}^{n} d(x_i, x_{i-1}) \tag{2}$$

and define the size of α to be $\nu(\alpha) := n$. The *reversal* of α is the chain $\overline{\alpha} := \{x_n, \ldots, x_0\}$. A *basic move* on an ε-chain α consists of either adding or removing a single point, as long as the resulting chain is still an ε-chain. An ε-homotopy between ε-chains α and β with the same endpoints is a finite sequence of ε-chains $\langle \alpha = \eta_0, \eta_1, \ldots, \eta_k = \beta \rangle$ such that all η_i have the same endpoints and for all i, η_i and η_{i+1} differ by a basic move. The resulting equivalence classes are denoted $[\alpha]_\varepsilon$; for simplicity we will usually write $[x_0, \ldots, x_n]_\varepsilon$ rather than $[\{x_0, \ldots, x_n\}]_\varepsilon$. If $\alpha = \{x_0, \ldots, x_n\}$ and $\beta = \{x_n = y_0, \ldots, y_m\}$ are ε-chains then the *concatenation* $\alpha * \beta$ is the ε-chain $\{x_0, \ldots, x_n = y_0, \ldots, y_m\}$. It is easy to check that there is a well-defined operation induced by concatenation: $[\alpha]_\varepsilon * [\beta]_\varepsilon := [\alpha * \beta]_\varepsilon$. We define two ε-loops λ_1 and λ_2 to be *freely* ε-homotopic if there exist ε-chains α and β starting at a common point x_0, such that $\alpha * \lambda_1 * \overline{\alpha}$ is ε-homotopic to $\beta * \lambda_2 * \overline{\beta}$.

Fix a basepoint $*$ in X. (Change of basepoint is algebraically and geometrically immaterial for connected metric spaces. In fact, similar to the case of the traditional fundamental group, concatenation with an ε-chain joining basepoints $*$ and $*'$ induces a covering equivalence between the spaces $(X_\varepsilon, *)$ and $(X_\varepsilon, *')$ and an isomorphism between the groups $\pi_\varepsilon(X, *)$ and $\pi_\varepsilon(X, *')$ discussed below—see [6, Remark 18]. As in the proof of Proposition 14, the induced bijection is also an isometry with respect to the metric defined in Definition 12. Therefore we will generally avoid using notation involving basepoints and assume all maps are basepoint-preserving.) The set of all ε-homotopy classes $[\alpha]_\varepsilon$ of ε-loops starting at $*$ forms a group $\pi_\varepsilon(X)$ with operation induced by concatenation of ε-loops. The group $\pi_\varepsilon(X)$ can be regarded as a kind of fundamental group that measures only "holes at the scale of ε". An ε-loop $\alpha = \{x_0, \ldots, x_n = x_0\}$ that is ε-homotopic to the trivial loop $\{x_0\}$ is called *ε-null*.

The set of all ε-homotopy classes $[\alpha]_\varepsilon$ of ε-chains α in X starting at $*$ will be denoted by X_ε. The "endpoint mapping" will be denoted by $\phi_\varepsilon : X_\varepsilon \to X$. That is, if $\alpha = \{* = x_0, x_1, \ldots, x_n\}$ then $\phi_\varepsilon([\alpha]_\varepsilon) = x_n$. Since ε-homotopic ε-chains always have the same endpoints, the function ϕ_ε is well-defined. We choose $[*]_\varepsilon$ for the basepoint of X_ε so ϕ_ε is basepoint preserving. For any ε-chain α in X, let

$$|[\alpha]_\varepsilon| := \inf\{L(\gamma) : \gamma \in [\alpha]_\varepsilon\}. \tag{3}$$

The above definition allows us to define a metric on X_ε so that $\pi_\varepsilon(X)$ acts by isometries induced by concatenation (Definition 12). When X is connected, ϕ_ε is a regular covering map with deck group $\pi_\varepsilon(X)$, and when X is geodesic the metric coincides with the usual lifted length metric (Proposition 23). For any $\delta \geq \varepsilon > 0$ there is a natural mapping $\phi_{\delta\varepsilon} : X_\varepsilon \to X_\delta$ given by $\phi_{\delta\varepsilon}([\alpha]_\varepsilon) = [\alpha]_\delta$. This map is well defined because every ε-chain (resp. ε-homotopy) is a δ-chain (resp. δ-homotopy). Note that for a compact geodesic space, $\phi_\varepsilon(X)$ is naturally isometric to the "delta covering" of X defined in the work of Sormani–Wei, taking $\delta = \frac{3\varepsilon}{2}$.

One additional very important feature of geodesic spaces is that any ε-chain has a "midpoint refinement" obtained by adding a midpoint between each point in the chain and its successor (which is clearly an ε-homotopy), producing an $\frac{\varepsilon}{2}$-chain in the same ε-homotopy class. Refinement is often essential for arguments involving limits, since being an ε-chain is not a closed condition. For this reason, many arguments in this paper do not carry over to general metric spaces.

The main relationship between $\pi_\varepsilon(X)$ and the fundamental group $\pi_1(X)$ involves a function Λ defined as follows (see also Proposition 78, [6]). Given any continuous path $c : [0, 1] \to X$, choose $0 = t_0 < \cdots < t_n = 1$ fine enough that every image $c([t_i, t_{i+1}])$ is contained in the open ball $B(c(t_i), \varepsilon)$. Then $\Lambda([c]) := [c(t_0), \ldots, c(t_n)]_\varepsilon$ is well-defined by Corollary 21. Note that Λ is "length non-increasing" in the sense that $|\Lambda([c])| \leq |[c]|$. Restricting Λ to the fundamental group at any base point yields a homomorphism $\pi_1(X) \to \pi_\varepsilon(X)$. When X is geodesic, Λ is surjective since the successive points of an ε-loop λ may be joined by geodesics to obtain a path loop whose class goes to $[\alpha]_\varepsilon$. The kernel of Λ is precisely described in [25]; for the purposes of this paper we need only know that if X is a compact semilocally simply connected geodesic space then for small enough ε, Λ is a length-preserving isomorphism (Theorem 26). All of our theorems about the fundamental group are directly derived from the next theorem via Λ.

Theorem 3. *Let X be a compact geodesic space of diameter D, and $\varepsilon > 0$. Then*

1. *$\pi_\varepsilon(X)$ has a finite set of generators $G = \{[\gamma_1]_\varepsilon, \ldots, [\gamma_k]_\varepsilon\}$ such that $L(\gamma_i) \leq 2(D + \varepsilon)$ for all i, and relators of the form $[\gamma_i]_\varepsilon[\gamma_j]_\varepsilon = [\gamma_m]_\varepsilon$.*

2. *For any* $L > 0$ *there are at most* $C\left(X, \frac{\varepsilon}{4}\right)^{\frac{4L}{\varepsilon}}$ *distinct elements* $[\alpha]_\varepsilon$ *of* $\pi_\varepsilon(X)$ *such that* $|[\alpha]_\varepsilon| < L$*, and in particular we may take*
$$k \leq C\left(X, \frac{\varepsilon}{4}\right)^{\frac{8(D+\varepsilon)}{\varepsilon}}$$
in the first part.
3. *Suppose, in addition, that for any basepoint* $*$ *and* $0 < \delta < \varepsilon$ *there are at most* M *distinct non-trivial elements* $[\alpha]_\delta \in \pi_\delta(X)$ *such that* $|[\alpha]_\delta| < \varepsilon$*. Then the number of generators of* $\pi_\delta(X)$ *with relators as in the first part may be taken to be at most*
$$M\left[\frac{8(D+\varepsilon)}{\varepsilon}\right]\left[C\left(X, \frac{\varepsilon}{4}\right)\right]^{\frac{8(D+\varepsilon)}{\varepsilon}}.$$

The proof of the second part of the theorem is a nice illustration of the utility of discrete methods. Fix any covering $\mathcal{B}$ of X by $N := C(X, \frac{\varepsilon}{4})\frac{\varepsilon}{4}$-balls. Applying Lemma 17 and a midpoint refinement, we may represent any element of $\pi_\varepsilon(X)$ by an $\frac{\varepsilon}{2}$-loop α such that $\nu(\alpha) \leq \frac{4L}{\varepsilon} + 2$. We may choose one $B \in \mathcal{B}$ containing each point in the loop. Since the first and last balls may always be chosen to be the same (containing the basepoint), each α corresponds to a sequential choice of at most $\frac{4L}{\varepsilon}$ balls in $\mathcal{B}$. But Proposition 16 tells us that if any two loops share the same sequence of balls (so corresponding points are distance $<\frac{\varepsilon}{2}$ apart), they must be ε-homotopic. So there is at most one class $[\alpha]_\varepsilon$ for each such sequence of balls, and there are at most $N^{\frac{4L}{\varepsilon}}$ different sequences of balls.

The proof of the first part of the theorem requires the construction of a metric simplicial 2-complex called an (ε, δ)-chassis for a compact geodesic space X, which is described in the final section of this paper. For small enough $\delta > 0$, any (ε, δ)-chassis has edge group isomorphic to $\pi_\varepsilon(X)$ (although the two spaces may not have the same homotopy type!). In this way our proof of Theorem 1 is quite different from Gromov's proof of his theorem. However, it is interesting to note that he exploits the fact that the set of lengths of minimal loops representing fundamental group elements in a compact Riemannian manifold is discrete. Our proof depends on discreteness of what we call the "homotopy critical spectrum". This notion is closely related to the Sormani–Wei covering spectrum [28,29] and, like that spectrum, is (up to a multiplied constant) a subset of the classical length spectrum that captures topological information.

We will now describe this spectrum and related concepts (and questions) that are of independent interest.

Definition 4. An ε-loop λ in a metric space X is called ε-critical if λ is not ε-null, but is δ-null for all $\delta > \varepsilon$. (An ε-loop is δ-null if it is δ-homotopic to the trivial loop.) When an ε-critical ε-loop exists, ε is called a homotopy critical value; the collection of these values is called the homotopy critical spectrum.

When X is a geodesic space the functions $\phi_{\varepsilon\delta} : X_\delta \to X_\varepsilon$ are all covering maps, which are homeomorphisms precisely if there are no critical values σ with $\varepsilon > \sigma \geq \delta$ (Lemma 24). In a compact geodesic space, the homotopy critical spectrum is discrete in $(0, \infty)$ (more about this below) and therefore indicates the exact values $\varepsilon > 0$ where the equivalence type of the ε-covering maps changes.

In [29], Sormani–Wei first introduced the idea of a spectrum which captures the values at which equivalence type of a covering map changes. Their spectrum, called the covering spectrum, was defined using their notion of a delta cover defined in [28]. They utilized a classical construction of Spanier [32] for locally pathwise connected topological spaces that provides a covering map $\pi^\delta : \widetilde{X}^\delta \to X$ corresponding to the open cover of a geodesic space X by open δ-balls, which they called the δ-cover of X. As it turns out, despite the very different construction methods, when $\delta = \frac{3\varepsilon}{2}$ and X is a compact geodesic space, this covering map is isometrically equivalent to our covering map $\phi_\varepsilon : X_\varepsilon \to X$ [25]. In fact, $\ker \Lambda$ is precisely the Spanier group for the open cover of X by $\frac{3\varepsilon}{2}$-balls. It follows that in the compact case, the covering spectrum and homotopy critical spectrum differ precisely by a factor of $\frac{2}{3}$.

In this paper, rather than applying prior theorems about the covering spectrum, we will directly prove stronger results about the homotopy critical spectrum. Moreover, as should be clear from the present paper and [25], discrete methods have many advantages, including simplicity, amenability with the Gromov–Hausdorff metric, and applicability to non-geodesic spaces. For example, Sormani–Wei [29, Theorem 4.7] show that the covering spectrum is contained in $\frac{1}{2}$ times the length spectrum (set of lengths of closed geodesics) for geodesic spaces with a universal cover. We not only show that this statement is true (replacing $\frac{1}{2}$ by $\frac{1}{3}$ in our notation) without assuming a universal cover, we identify precisely the very special closed geodesics that contribute to the homotopy critical spectrum:

Definition 5. An essential ε-circle in a geodesic space consists of the image of an arclength parameterized (path) loop of length 3ε that contains an ε-loop that is not ε-null.

Being an essential circle is stronger than it may seem at first: an essential circle is the image of a closed geodesic that is not null-homotopic, which is also a *metrically embedded* circle in the sense that its metric as a subset of X is the same as the intrinsic metric of the circle (Theorem 39). As Example 44 shows, even in flat tori this is not always true for the image of a closed geodesic, even when it is the shortest path in its homotopy class. We prove the following.

Theorem 6. *If X is a compact geodesic space then $\varepsilon > 0$ is a homotopy critical value of X if and only if X contains an essential ε-circle.*

This theorem is connected to a problem with a long history in Riemannian geometry: to relate the spectrum of the Laplace–Beltrami operator and the length spectrum to one another and to topological and geometric properties of the underlying compact manifold. The relationship between the covering and Laplace spectra first emerged in [4,14,17], but many problems remain. For example, an important open question is whether the "weak" length spectrum (i.e. without multiplicity) is completely determined by the Laplace spectrum (see [16] for more discussion). To this mix one may add the covering/homotopy critical spectrum (with or without multiplicity, see below), which up to multiplied constant is a subset of the length spectrum. The analog of the main question has already been answered: de Smit, Gornet, and Sutton recently showed that the covering spectrum is not a spectral invariant [16]. Intriguingly they have extended Sunada's method [33] to determine when two manifolds share a common covering spectrum.

Sormani [27] stratified the length spectrum according to the length on which a given closed geodesic is a (minimizing) geodesic. In particular, the $1/2$ length spectrum $L_{\frac{1}{2}}(X)$ consists of the lengths of closed geodesics of length L that are minimizing on all segments of length $L/2$ (called $1/2$ geodesics). It is not hard to see that this condition is equivalent to being metrically embedded (Remark 40). We obtain the following corollary, which removes the requirement in Theorem 4.1, [27] that the space have a simply connected universal cover.

Corollary 7. *If X is a compact geodesic space then the covering spectrum is a subset of $\frac{1}{2}L_{\frac{1}{2}}(X)$.*

We also now have the additional precise information that any $1/2$ geodesic that does not contribute to the covering spectrum must be non-essential in our sense. Moreover, our work suggests that a natural definition of multiplicity for the $1/2$ length spectrum would use equivalence of essential circles, as in Definition 42, rather than free homotopies. As for additional connections to topology, in [25] we show that essential circles can be used to create a new set of generators for the fundamental group of a compact, semilocally simply connected space, which we conjecture has minimal cardinality.

Essential circles give a nice geometric picture, but their discrete analogs, which we will define now, are more useful for the type of problems we are presently considering.

Definition 8. An ε-triad in a geodesic space X is a triple $T := \{x_0, x_1, x_2\}$ such that $d(x_i, x_j) = \varepsilon$ for all $i \neq j$; when ε is not specified we will simply refer to a triad. We denote by α_T the loop $\{x_0, x_1, x_2, x_0\}$. We say that T is essential if some midpoint refinement of α_T is not ε-null. Essential ε-triads T_1 and T_2 are defined to be equivalent if a midpoint refinement of α_{T_1} is freely ε-homotopic to a midpoint refinement of either α_{T_1} or $\overline{\alpha_{T_1}}$.

Of course α_T is not an ε-chain (1); that is why we use a midpoint refinement. We show that if one joins the corners of an essential ε-triad by geodesics then the resulting geodesic triangle is an essential ε-circle (Proposition 37). Conversely, given an essential ε-circle, every triad on it is an essential ε-triad (Corollary 41). We may now define essential ε-circles to be equivalent if their corresponding essential ε-triads are equivalent, and Theorem 6 allows us to define the *multiplicity* of a homotopy critical value ε to be the number of non-equivalent essential ε-triads (or ε-circles).

We prove that "close" essential triads are equivalent:

Proposition 9. *Suppose $T = \{x_0, x_1, x_2\}$ is an essential ε-triad in a geodesic space X and $T' = \{x_0', x_1', x_2'\}$ is any set of three points such that $d(x_i, x_i') < \frac{\varepsilon}{3}$ for all i. If T' is an essential triad then T' is an ε-triad equivalent to T.*

Now suppose we cover a compact geodesic space X by N open metric balls of radius r. If T is an essential ε-triad with $\varepsilon \geq 3r$ then there are three distinct balls B_1, B_2, B_3 in the cover, each containing one of the points of the triad. By Proposition 9, any triad having one point in each of B_1, B_2, B_3 is either not essential or is an ε-triad equivalent to T. We obtain the following.

Corollary 10. *Let X be a compact metric space and $a > 0$. Then there are at most*

$$\binom{C\left(X, \frac{a}{3}\right)}{3}$$

non-equivalent essential triads that are ε-triads for some $\varepsilon \geq a$.

Naturally one wonders how optimal this estimate is and whether it can be improved (see also Example 47). From Gromov's Precompactness Theorem we immediately obtain the following.

Theorem 11. *Let $\mathcal{X}$ be a Gromov–Hausdorff precompact collection of compact geodesic spaces. For every $a > 0$, there is a number N such that for any $X \in \mathcal{X}$ the number of homotopy critical values of X greater than a, counted with multiplicity, is at most N.*

One consequence is that the homotopy critical spectrum of any compact geodesic space is discrete in $(0, \infty)$, which is essential for the proof of our main theorem. In [28], Sormani–Wei prove a version of Theorem 11 assuming that all spaces in question have a universal cover. The arguments there are indirect and without an explicit bound, since they first show that the set of corresponding covering spaces is itself Gromov–Hausdorff pointed precompact, then proceed by contradiction. Obtaining even better control over the distribution of critical values for specific classes of geodesic spaces is likely to be an interesting problem. For example, it was shown by Sormani–Wei in [30] that limits of compact manifolds with non-negative Ricci curvature have finite covering spectra. The proof depends on deep results concerning the local structure of spaces with non-negative curvature and their limits due to Abresch–Gromoll [1], Cheeger–Colding [11], and Colding [13]. That the limiting spaces have finite covering spectra implies that they have a universal cover in the categorical sense, but leaves open the interesting question of whether they are semilocally simply connected.

Gromov's Betti Numbers Theorem [18] inspires the following question: Is there a number $C(n)$ such that if M is a Riemannian n-manifold with nonnegative sectional curvature then M has at most $C(n)$ homotopy critical values, counted with multiplicity?

2. Basic discrete homotopy tools

As is typical for metric spaces, the term "geodesic" in this paper refers to an arclength parameterized length minimizing curve (and a geodesic space is one in which every pair of points is joined by a geodesic). This is distinguished from the traditional term "geodesic" in Riemannian geometry, which is only a local isometry; we will refer to such a path in this paper as "locally minimizing". The term "closed geodesic" will refer to a function from a standard circle into X such that the restriction to any sufficiently small arc is an isometry onto its image. We begin with a few results for metric spaces in general, including the definition of a natural metric on the space X_ε. While the lifting of a *geodesic* metric to a covering space is a well-known construction (see below), to our knowledge Definition 12 gives the first method to lift the metric of a general metric space to a covering space in such a way that the covering map is uniformly a local isometry and the deck group acts as isometries. In a metric space X we denote by $B(x, r)$ the open metric ball $\{y : d(x, y) < r\}$. For what follows, recall (1)–(3) from Section 1 of this paper.

Definition 12. Let X_ε be the family of ε-chains based at a given point $*$. For $[\alpha]_\varepsilon, [\beta]_\varepsilon \in X_\varepsilon$ we define

$$d([\alpha]_\varepsilon, [\beta]_\varepsilon) = \inf\{L(\kappa) : \alpha * \kappa * \overline{\beta} \text{ is } \varepsilon\text{-null}\} = \left|[\overline{\alpha} * \beta]_\varepsilon\right|.$$

The second equality above follows from the fact that $\alpha * (\overline{\alpha} * \beta) * \overline{\beta}$ is ε-null.

Proposition 13. *Let α, β be ε-chains such that the endpoint of α is the beginning point of β. Then*

1. *(Positive definite) $|[\alpha]_\varepsilon| \geq 0$ and $|[\alpha]_\varepsilon| = 0$ if and only if α is ε-null.*
2. *(Triangle inequality) $\left|[\alpha * \beta]_\varepsilon\right| \leq |[\alpha]_\varepsilon| + |[\beta]_\varepsilon|$.*

As a consequence, Definition 12 defines a metric on X_ε.

Proof. That $|[\alpha]_\varepsilon| \geq 0$ and that $|[\alpha]_\varepsilon| = 0$ when α is ε-null are both immediate consequences of definition (3). In general, if $|[\alpha]_\varepsilon| = 0$ then this means that for every $\delta > 0$ there is some ε-chain $\xi = \{y_0, \ldots, y_n\}$ such that $[\alpha]_\varepsilon = [\xi]_\varepsilon$ and $L(\xi) < \delta$. In particular we may take $\delta < \varepsilon$.

Now for any $i < j$ we have $d(y_i, y_j) \leq \sum_{k=i+1}^{j} d(y_k, y_{k-1}) \leq L(\xi) < \delta < \varepsilon$ and α is ε-homotopic to the ε-chain $\{y_0, y_n\}$. By the same argument, $d(y_0, y_n) < \delta$ for all $\delta > 0$ and therefore $d(y_0, y_n) = 0$ and $y_0 = y_n$. That is, α is ε-homotopic to $\{y_0\}$.

For the triangle inequality, simply note that if α' is ε-homotopic to α and β' is ε-homotopic to β, then $\alpha' * \beta'$ is ε-homotopic to $\alpha * \beta$. Therefore

$$\left|[\alpha * \beta]_\varepsilon\right| \leq L(\alpha' * \beta') = L(\alpha') + L(\beta'). \tag{4}$$

Passing to the infimum over α and over β we obtain the triangle inequality. □

We will always use the metric from Definition 12 for X_ε.

Proposition 14. *Let X be a metric space and $\varepsilon > 0$. Then*

1. *The function $\phi_\varepsilon : X_\varepsilon \to X$ preserves distances of length less than ε and is injective when restricted to any open ε-ball. In particular, ϕ_ε is an isometry onto its image when restricted to any open $\frac{\varepsilon}{2}$-ball.*
2. *For any ε-loop λ at $*$, the function $\tau_\lambda : X_\varepsilon \to X_\varepsilon$ defined by $\tau_\lambda([\alpha]_\varepsilon) = [\lambda * \alpha]_\varepsilon$ is an isometry such that $\tau_\lambda \circ \phi_\varepsilon = \phi_\varepsilon$.*

Proof. As in the proof of the positive definite property, if $d([\alpha]_\varepsilon, [\beta]_\varepsilon) < \varepsilon$ then $[\overline{\alpha} * \beta]_\varepsilon$ must contain the chain $\{y_0, y_1\}$ with $d(y_0, y_1) = d([\alpha]_\varepsilon, [\beta]_\varepsilon) < \varepsilon$, where y_0 and y_1 are the endpoints of α and β. That ϕ_ε is injective on any ε-ball was proved in greater generality in the work of the Berestovskii and the first author [6], but the argument is simple enough to repeat here. If $[\alpha]_\varepsilon, [\beta]_\varepsilon \in B([\gamma]_\varepsilon, \varepsilon)$ where $\gamma = \{* = x_0, \ldots, x_n\}$, then we may take $\alpha = \gamma * \{x_n, y_0\}$ and $\beta = \gamma * \{x_n, y_1\}$. Then $\phi_\varepsilon([\alpha]_\varepsilon) = \phi_\varepsilon([\beta]_\varepsilon)$ if and only if $y_0 = y_1$, which is true if and only if $[\alpha]_\varepsilon = [\beta]_\varepsilon$.

To prove the second part, note that for any $[\alpha]_\varepsilon \in X_\varepsilon$,

$$\tau_\lambda\left(\left[\overline{\lambda} * \alpha\right]_\varepsilon\right) = \left[\lambda * \overline{\lambda} * \alpha\right]_\varepsilon = [\alpha]_\varepsilon,$$

showing that τ_λ is onto. Next, for any $[\beta]_\varepsilon \in X_\varepsilon$ we have

$$\begin{aligned} d\left(\tau_\lambda([\alpha]_\varepsilon), \tau_\lambda\left([\beta]_\varepsilon\right)\right) &= d\left([\lambda * \alpha]_\varepsilon, [\lambda * \beta]_\varepsilon\right) = \left|\left[\overline{\lambda * \alpha} * \lambda * \beta\right]_\varepsilon\right| \\ &= \left|\left[\overline{\alpha} * \overline{\lambda} * \lambda * \beta\right]_\varepsilon\right| = \left|[\overline{\alpha} * \beta]_\varepsilon\right| = d([\alpha]_\varepsilon, [\beta]_\varepsilon). \end{aligned}$$

Since $\lambda * \alpha$ has the same endpoint as α, we also have that $\tau_\lambda \circ \phi_\varepsilon = \phi_\varepsilon$. □

The second part of the proposition shows that the group $\pi_\varepsilon(X)$ acts by isometries on X_ε. This action is *discrete* as described in prior work of the first author [24]; that is, if for any $[\alpha]_\varepsilon$ and λ we have that $d(\tau_\lambda([\alpha]_\varepsilon), [\alpha]_\varepsilon) < \varepsilon$ then τ_λ is the identity, i.e. λ is ε-null. Being discrete is stronger than being free and properly discontinuous, and hence when $\phi_\varepsilon : X_\varepsilon \to X$ is surjective, ϕ_ε is a regular covering map with covering group $\pi_\varepsilon(X)$ (via the faithful action $[\lambda]_\varepsilon \to \tau_\lambda$). Surjectivity of ϕ_ε for all ε is clearly equivalent to X being "chain connected" in the sense that every pair of points in X is joined by an ε-chain for all ε. Chain connected is equivalent to what is sometimes called "uniformly connected" and is in general weaker than connected (see [6] for more details).

For consistency, we observe that our metric on X_ε is compatible with the uniform structure defined on X_ε in work of Berestovskii and the first author [6]. Recall that a uniform space is a generalization of a metric space (cf. [21]). A basis for the uniform structure of a metric space consists of all sets (called *entourages*) E_δ^*, with $0 < \delta \leq \varepsilon$, where E_δ^* is defined as all ordered pairs $([\alpha]_\varepsilon, [\beta]_\varepsilon)$ such that $[\alpha * \overline{\beta}]_\varepsilon = [\{y, z\}]_\varepsilon$ for some y, z with $d(y, z) < \delta$. That is, $E^*_\delta = \{([\alpha]_\varepsilon, [\beta]_\varepsilon) : d([\alpha]_\varepsilon, [\beta]_\varepsilon) < \delta\}$, which is a basis entourage for the uniform structure of the metric defined in Definition 12. So the two uniform structures are identical.

We next consider a useful result showing that uniformly close ε-chains are ε-homotopic.

Definition 15. Let X be a metric space. Given $\alpha = \{x_0, \dots, x_n\}$ and $\beta = \{y_0, \dots, y_n\}$ with $x_i, y_i \in X$, define $\Delta(\alpha, \beta) := \max_i\{d(x_i, y_i)\}$. For any $\varepsilon > 0$, if α is an ε-chain we define $E_\varepsilon(\alpha) := \min_i\{\varepsilon - d(x_i, x_{i+1})\} > 0$. When no confusion will result we will eliminate the ε subscript.

Proposition 16. *Let X be a metric space and $\varepsilon > 0$. If $\alpha = \{x_0, \dots, x_n\}$ is an ε-chain and $\beta = \{x_0 = y_0, \dots, y_n = x_n\}$ is such that $\Delta(\alpha, \beta) < \frac{E(\alpha)}{2}$ then β is an ε-chain that is ε-homotopic to α.*

Proof. We will construct an ε-homotopy η from α to β. By definition of $E(\alpha)$ and the triangle inequality, each chain below is an ε-chain, and hence each step below is legal. Here and in the future we use the upper bracket to indicate that we are adding a point, and the lower bracket to indicate that we are removing a point in each basic step.

$$\begin{aligned}
\alpha = {} & \{x_0, x_1, \dots, x_n\} \to \{x_0, \overbrace{x_1}, x_1, \dots, x_n\} \to \{x_0, x_1, \overbrace{y_1}, x_1, \dots, x_n\} \\
& \to \{x_0, \underbrace{x_1}, y_1, x_1, \dots, x_n\} \to \{x_0, y_1, \underbrace{x_1}, x_2, \dots, x_n\} \\
& \to \{x_0, y_1, \overbrace{x_2}, x_2, \dots, x_n\} \to \{x_0, y_1, x_2, \overbrace{y_2}, x_2, \dots, x_n\} \\
& \to \{x_0, y_1, \underbrace{x_2}, y_2, x_2, \dots, x_n\} \to \{x_0, y_1, y_2, \underbrace{x_2}, x_3, \dots, x_n\} \to \cdots \to \beta. \qquad \square
\end{aligned}$$

In order to properly use Proposition 16, one needs chains of the same size, and the next lemma helps with this.

Lemma 17. *Let $L, \varepsilon > 0$ and α be an ε-chain in a metric space X with $L(\alpha) \le L$. Then there is some $\alpha' \in [\alpha]_\varepsilon$ such that $L(\alpha') \le L(\alpha)$ and $\nu(\alpha') = \left\lfloor \frac{2L}{\varepsilon} + 1 \right\rfloor$.*

Proof. If α has one or two points then we may simply repeat x_0, if necessary, (which does not increase length) to obtain α' with $\nu(\alpha') = \left\lfloor \frac{2L}{\varepsilon} + 1 \right\rfloor$. Otherwise, let $\alpha := \{x_0, \dots, x_n\}$ with $n \ge 2$. Suppose that for some i, $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) < \varepsilon$. Then $d(x_{i-1}, x_{i+1}) < \varepsilon$ and the point x_i may be removed to form a new ε-chain α_1 that is ε-homotopic to α with $L(\alpha_1) \le L(\alpha)$. After finitely many such steps, we have a chain α_0 that is ε-homotopic to α and not longer, which either has two points (then proceed as above), or α_0 has the property that for every i, $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \ge \varepsilon$. By pairing off terms we see that $L(\alpha_0) \ge \left\lfloor \frac{\nu(\alpha_0)}{2} \right\rfloor \varepsilon$ and hence

$$\nu(\alpha_0) \le \left\lfloor \frac{2L(\alpha_0)}{\varepsilon} + 1 \right\rfloor \le \left\lfloor \frac{2L}{\varepsilon} + 1 \right\rfloor.$$

As before, repeat x_0 enough times to make α' with $\nu(\alpha') = \left\lfloor \frac{2L}{\varepsilon} + 1 \right\rfloor$. $\square$

The above lemma can be used like a discrete version of the Arzela–Ascoli theorem. That is, if one has a sequence of ε-chains of length at most L (i.e., "equicontinuous") then one can assume that all the chains have the same finite size n. In a compact space, one can then choose a subsequence so that the ith elements in each chain form a convergent sequence for all $0 \leq i \leq n$. For example, one may use this method in conjunction with Proposition 16 to obtain the following:

Corollary 18. *If X is a compact metric space, $\varepsilon > 0$, and α is an ε-chain then there is some $\beta \in [\alpha]_\varepsilon$ such that $L(\beta) = |[\alpha]_\varepsilon|$.*

We next move onto the relationship between paths and chains. The notion of a "stringing" formalizes a construction used by Berestovskii and the first author in [6, Proposition 78].

Definition 19. Let $\alpha := \{x_0, \ldots, x_n\}$ be an ε-chain in a metric space X, where $\varepsilon > 0$. A stringing of α consists of a path $\widehat{\alpha}$ formed by concatenating paths γ_i from x_i to x_{i+1} where each path γ_i lies entirely in $B(x_i, \varepsilon)$. If each γ_i is a geodesic then we call $\widehat{\alpha}$ a chording of α.

Note that by uniform continuity, any path c defined on a compact interval may be subdivided into an ε-chain α such that c is a stringing of α, and in any geodesic space every ε-chain has a chording.

Proposition 20. *If α is an ε-chain in a chain connected metric space X then the unique lift of any stringing $\widehat{\alpha}$ starting at the basepoint $[*]_\varepsilon$ in X_ε has $[\alpha]_\varepsilon$ as its endpoint.*

Proof. Let $\alpha_i := \{x_0, \ldots, x_i\}$, with $\alpha_n = \alpha$. We will prove by induction that the endpoint of the lift of a stringing $\widehat{\alpha_i}$ is $[\alpha_i]_\varepsilon$. The case $i = 0$ is trivial; suppose the statement is true for some $i < n$ and consider some stringing $\widehat{\alpha_{i+1}}$. Then the restriction to a segment of $\widehat{\alpha_{i+1}}$ is a stringing $\widehat{\alpha_i}$ and by the inductive step the lift of $\widehat{\alpha_i}$ ends at $[\alpha_i]_\varepsilon$. By definition of stringing, $\widehat{\alpha_{i+1}}$ is obtained from $\widehat{\alpha_i}$ by adding some path c from x_i to x_{i+1} that lies entirely within $B(x_i, \varepsilon)$. By Proposition 14, ϕ_ε is bijective from the set $B([\alpha_i]_\varepsilon, \varepsilon)$ onto $B(x_i, \varepsilon)$. Therefore the lift $\widetilde{c}$ of c starting at $[\alpha_i]_\varepsilon$, must be contained entirely in $B([\alpha_i]_\varepsilon, \varepsilon)$. By uniqueness of lifts, the endpoint of the lift of $\widehat{\alpha_{i+1}}$ must be the endpoint $[\beta]_\varepsilon$ of $\widetilde{c}$. Note that $\phi_\varepsilon([\beta]_\varepsilon) = x_{i+1}$; i.e. the endpoint of β is x_{i+1}. Next, $[\beta]_\varepsilon \in B([\alpha_i]_\varepsilon, \varepsilon)$ means that there is some ε-chain $\sigma = \{y_0, \ldots, y_m\}$ such that $[\alpha_i]_\varepsilon = [y_0, \ldots, y_m, x_i]_\varepsilon$ and $[\beta]_\varepsilon = [y_0, \ldots, y_m, x_{i+1}]_\varepsilon$. Since $[\alpha_{i+1}]_\varepsilon$ is also clearly in $B([\alpha_i]_\varepsilon, \varepsilon)$ (just take $\sigma = \alpha_i$) and $\phi_\varepsilon([\alpha_{i+1}]_\varepsilon) = x_{i+1} = \phi_\varepsilon([\beta]_\varepsilon)$, the injectivity of ϕ_ε on $B([\alpha_i]_\varepsilon, \varepsilon)$ shows that $[\beta]_\varepsilon = [\alpha_{i+1}]_\varepsilon$. $\square$

Corollary 21. *If α and β are ε-chains in a chain connected metric space X such that there exist stringings $\widehat{\alpha}$ and $\widehat{\beta}$ that are path homotopic then α and β are ε-homotopic.*

Proof. Choose the basepoint to be the common starting point of α and β. Since $\widehat{\alpha}$ and $\widehat{\beta}$ are path homotopic, the endpoints $[\alpha]_\varepsilon$ and $[\beta]_\varepsilon$ of their lifts must be equal. $\square$

Note that two path loops c_1 and c_2 in a topological space X are freely homotopic if and only if for some paths p_i from some particular point q to the start/endpoint of c_i, $\overline{p_1} * c_1 * p_1$ is fixed-endpoint homotopic to $\overline{p_2} * c_2 * p_2$. For one direction, suppose that $h : [0, 1] \times [0, 1] \to X$ is a homotopy from c_1 to c_2. Then we may take $q := h(0, 0)$, p_1 to be the constant path at q and $p_2(t) := h(t, 0)$. For the converse, simply note that the loops $\overline{p_i} * c_i * p_i$ are freely homotopic to the loops c_i, respectively, by retracting along the paths p_i.

Hence free ε-homotopy, as defined in the Introduction, is the correct discrete analog of continuous free homotopy. We have used this form because imitating the standard continuous version of free homotopy is notationally tricky for chains. The following lemma will be used later, and is the discrete analog of "rotation" of a path loop in itself.

Lemma 22. *Let $\alpha := \{x_0, \ldots, x_n = x_0\}$ be an ε-loop in a metric space. Then α is freely ε-homotopic to $\alpha^P := \{x_{P(0)}, x_{P(1)}, \ldots, x_{P(n-1)}, x_{P(0)}\}$, where P is any cyclic permutation of $\{0, 1, \ldots, n-1\}$.*

Proof. It suffices to consider the cycle P that adds one to each index, mod(n). Let $\beta = \{* = y_0, \ldots, y_m = x_0\}$ be an ε-chain. Here, and in the future, we will denote ε-homotopies in the following form, where bracket on top denotes insertion and bracket on the bottom denotes deletion:

$$\begin{aligned}\beta * \alpha * \overline{\beta} = &\ \{y_0, \ldots, y_m, x_1, \ldots, x_0, y_{m-1}, \ldots, y_0\} \\ \to &\ \{y_0, \ldots, y_m, x_1, \ldots, x_0, \overbrace{x_0}, y_{m-1}, \ldots, y_0\} \\ \to &\ \{y_0, \ldots, y_m, x_1, \ldots, x_0, \overbrace{x_1}, x_0, y_{m-1}, \ldots, y_0\} \\ \to &\ \{y_0, \ldots, y_m, x_1, \ldots, x_0, \overbrace{x_1}, x_1, x_0, y_{m-1}, \ldots, y_0\}.\end{aligned}$$

That is, $\beta * \alpha * \overline{\beta}$ is ε-homotopic to $\eta * \alpha^P * \eta$, where $\eta = \{y_0, \ldots, y_m = x_0, x_1\}$. □

The rest of this section is devoted to basic results that are true for geodesic spaces. The situation for metric spaces in general is much more complicated—for example, as part of an REU project, Jim Conant, Victoria Curnutte, Corey Jones, Kristen Pueschel, Maria Walpole and the authors showed that the homotopy critical spectrum of a compact metric space may not be discrete [15].

The following statement is easy to check: Let $f : X \to Y$ be a bijection between geodesic spaces X and Y. Then the following are equivalent: (1) f is an isometry. (2) f is a local isometry (i.e. for each $x \in X$ the restriction of f to some $B(x, \varepsilon)$ is an isometry onto $B(f(x), \varepsilon)$). (3) f is a length-preserving homeomorphism (i.e. if c is a rectifiable path in X then $f \circ c$ is rectifiable and $L(f \circ c) = L(c)$).

Recall that if $f : X \to Y$ is a covering map, Y is a geodesic space, and X is a connected topological space, then the *lifted length metric* on X is defined by $d(x, y) = \inf\{L(f \circ c)\}$, where c is a path joining x and y. When Y is proper (i.e. its closed metric balls are compact) then X, being uniformly locally isometric to Y, is locally compact and complete. Hence by a classical result of Cohn-Vossen [12], *X with the lifted length metric* is also a proper geodesic space. In this case it also follows from what was stated previously that the lifted metric is the unique geodesic metric on X such that f is a local isometry. In particular, if $g : Z \to Y$ is a covering map, where Z is geodesic, and $h : X \to Z$ is a covering equivalence then h is an isometry.

Proposition 23. *If X is a geodesic space then the metric on X_ε given in Definition 12 is the lifted length metric. In particular if X is proper then X_ε is a proper geodesic space.*

Proof. Let $[\alpha]_\varepsilon, [\beta]_\varepsilon \in X_\varepsilon$. Then

$$d([\alpha]_\varepsilon, [\beta]_\varepsilon) = \inf\{L(\kappa) : [\alpha * \kappa * \overline{\beta}]_\varepsilon \text{ is } \varepsilon\text{-null}\}.$$

Let $\alpha = \{x_0, \ldots, x_n\}$, $\beta = \{y_0, \ldots, y_k\}$ and $\kappa = \{x_n = z_0, \ldots, z_m = y_k\}$. Let $\widehat{\alpha}$ and $\widehat{\kappa}$ be chordings of α and κ, and note that the length of the chain κ is the same as the length of the curve $\widehat{\kappa}$. Moreover, when $[\alpha * \kappa * \overline{\beta}]_\varepsilon$ is ε-null, $[\alpha * \kappa]_\varepsilon = [\beta]_\varepsilon$. We will now apply Proposition 20 a couple of times. First, the lift of $\widehat{\alpha}$ starting at $[*]_\varepsilon$ ends at $[\alpha]_\varepsilon$ and the lift of $\widehat{\alpha} * \widehat{\kappa}$ (which is a chording of $\alpha * \kappa$) starting at $[*]_\varepsilon$ ends at $[\alpha * \kappa]_\varepsilon = [\beta]_\varepsilon$. By uniqueness, the lift of $\widehat{\alpha} * \widehat{\kappa}$ starting at $[*]_\varepsilon$ must be the concatenation of the lift of $\widehat{\alpha}$ starting at $[*]_\varepsilon$ with the lift $\widetilde{\kappa}$ of $\widehat{\kappa}$ starting at $[\alpha]_\varepsilon$. That is, $\widetilde{\kappa}$ is a path in X_ε starting at $[\alpha]_\varepsilon$ and ending at $[\beta]_\varepsilon$, with $L(\phi_\varepsilon \circ \widetilde{\kappa}) = L(\widehat{\kappa}) = L(\kappa)$.

This shows that the metric of Definition 12 is a geodesic metric, and since we already know that ϕ_ε is a local isometry, by our previous comments on uniqueness, it must be the lifted length metric. □

One consequence of Proposition 23 is that each X_ε is path connected. Then it follows from the results of Berestovskii and the first author [6] that the maps $\phi_{\varepsilon\delta} : X_\delta \to X_\varepsilon$ are also regular covering maps (in general surjectivity is the only question, and this requires X_ε to be chain connected).

Lemma 24. *If X is a geodesic space then the covering map $\phi_{\varepsilon\delta} : X_\delta \to X_\varepsilon$ is injective if and only if there are no homotopy critical values σ with $\delta \le \sigma < \varepsilon$.*

Proof. If there is such a critical value σ then there is a σ-loop λ that is not σ-null but is ε-null. That is, $[\lambda]_\sigma \ne [*]_\sigma$ but $[\lambda]_\varepsilon = [*]_\varepsilon$, i.e. $\phi_{\varepsilon\sigma}$ is not injective. But since $\phi_{\sigma\delta}$ is surjective, $\phi_{\varepsilon\delta} = \phi_{\varepsilon\sigma} \circ \phi_{\sigma\delta}$ is not injective. Conversely, if $\phi_{\varepsilon\delta}$ is not injective then there is some δ-loop λ that is not δ-null but is ε-null. Let $\sigma := \sup\{\tau : [\lambda]_\tau \ne [*]_\tau\}$; so $\delta \le \sigma < \varepsilon$. If λ were σ-null then any σ-null homotopy would also be a τ-homotopy for $\tau < \sigma$ sufficiently close to σ. So λ is not σ-null; hence σ is a homotopy critical value and $\sigma < \varepsilon$. □

Remark 25. Unfortunately the false statement that every free homotopy class in a compact geodesic space has a shortest path, and this path is a closed geodesic, is present in both editions of [19], [20, Remarque/Remark 1.13], despite the intermediate publication of two kinds of counterexamples by Berestovskii, the first author, and Stallman in [8]. One of these counterexamples is a metric space formed by a circle with line segments connecting any $\left(\cos\frac{\pi n}{2^m}, \sin\frac{\pi n}{2^m}\right)$ to $\left(\cos\frac{\pi(n+1)}{2^m}, \sin\frac{\pi(n+1)}{2^m}\right)$, with the induced length metric. The circle itself is the shortest curve in its homotopy class but it is not a closed geodesic. Note that by our Theorem 39, the circle is not an essential circle because it is not metrically embedded.

The other counterexample is an infinite metric product of circles,

$$X := S^1_{D_1} \times S^1_{D_2} \times \cdots \times S^1_{D_j} \times \cdots$$

with square summable diameters $\sum_{j=1}^{\infty} D_j^2 < \infty$. Berestovskii, the first author and Stallman proved that the curve $C(t) = (D_1e^{it}, D_2e^{2it}, D_3e^{4it}, \ldots)$ in X is not rectifiable and does not even have a rectifiable curve in its homotopy class. See Section 4 of [8] for more details.

The next theorem clarifies that when a geodesic space is assumed to be semilocally simply connected, Gromov's statement is true (and which, according to private communication between him and Berestovskii and the first author, was likely his intended statement).

Theorem 26. *If X is a compact semilocally simply connected geodesic space then the homotopy critical spectrum has a positive lower bound. If $\varepsilon > 0$ is any such lower bound then*

1. *$\phi_\varepsilon : X_\varepsilon \to X$ is the universal covering map of X.*
2. *The function Λ is length preserving and hence the restriction to $\pi_1(X) \to \pi_\varepsilon(X)$ is an isomorphism.*
3. *Every path has a shortest path in its fixed-endpoint homotopy class, which is either constant or locally minimizing.*
4. *Every path loop has a shortest path in its free homotopy class, which is either constant or a closed geodesic.*

Proof. Berestovskii and the first author have proven in Proposition 69 and Theorem 77 of [6] that for all sufficiently small $\sigma > 0$, ϕ_σ is the simply connected covering map of X. Lemma 24 shows that ϕ_ε is equivalent to ϕ_σ since there are no homotopy critical values between ε and σ, proving the first part. Now let $c : [0, a] \to X$ be a path; take $* = c(0)$ to be the basepoint. By definition, $\Lambda([c]) = [\alpha]_\varepsilon$, where $\alpha := \{c(t_0), \dots, c(t_n)\}$ for a partition $\{0 = t_0, \dots, t_n = a\}$ that is sufficiently fine. By Corollary 18 there is some $\beta \in [\alpha]_\varepsilon$ such that $|[\alpha]_\varepsilon| = L(\beta)$. Let c' be any chording of β; so $|[\alpha]_\varepsilon| = L(\beta) = L(c')$. Now according to Proposition 20, the unique lifts of c and c' at the basepoint of X_ε have the same endpoint, and therefore form a loop. But since X_ε is simply connected, this means that c and c' are homotopic, and we have that $|[c]| \le L(c') = |[\alpha]_\varepsilon|$. Since we already have the other inequality, the second part is finished. Moreover, we have shown that $|[c]|$ is actually realized by any chording of a shortest ε-loop in $\Lambda([c])$. If any segment σ of such a shortest loop having length at most $\frac{\varepsilon}{2}$ were not a geodesic then the endpoints of σ could be joined by a shorter geodesic σ'. But then the loop formed by these two paths would lie in a ball of radius $\frac{\varepsilon}{2}$, and hence would lift as a loop. That is, we could replace σ by σ' while staying in the same homotopy class, a contradiction. This proves the third part, and the proof of the fourth part is similar. $\square$

Remark 27. In the previous theorem, if X is already simply connected then the proof shows that the homotopy critical spectrum is empty (the statement of the theorem is still correct in this case, since any real number is a lower bound for the empty set). Conversely, if X is compact and semilocally simply connected with empty covering spectrum, then X is simply connected. The latter implication is not true without the assumption that X is semilocally simply connected (see Example 45).

Definition 28. Let $c : [0, L] \to X$ be an arclength parameterized path in a metric space. A subdivision ε-chain of c is an ε-chain $\{x_0, \dots, x_n\}$ of the form $x_i := c(t_i)$ for some subdivision $t_0 = 0 < \dots < t_n = L$ such that for all t_i, $t_{i+1} - t_i < \varepsilon$ (we will refer to this condition as ε-fine). If X is a geodesic space and α is a chain in X then a refinement of α consists of a chain β formed by inserting between each x_i and x_{i+1} some subdivision chain of a geodesic joining x_i and x_{i+1}. If β is an ε-chain we will call β an ε-refinement of α.

Since c is 1-Lipschitz, any subdivision ε-chain is indeed an ε-chain. Obviously a refinement of an ε-chain α is ε-homotopic to α (just add the points one at a time) and hence any two refinements of α are ε-homotopic. A special case is the midpoint refinement defined in the Introduction.

Definition 29. If X is a metric space and $\varepsilon > 0$, an ε-loop of the form $\lambda = \alpha * \tau * \overline{\alpha}$, where $\nu(\tau) = 3$, will be called ε-small. Note that this notation includes the case when α consists of a single point, i.e. $\lambda = \tau$.

Note that any ε-small loop is ε-null, although it may or may not be δ-null for smaller δ.

Proposition 30. *Let X be a geodesic space and $0 < \varepsilon < \delta$. Suppose α, β are ε-chains and $\langle \gamma_0, \dots, \gamma_n \rangle$ is a δ-homotopy such that $\gamma_0 = \alpha$ and $\gamma_n = \beta$. Then $[\beta]_\varepsilon = [\lambda_1 * \dots * \lambda_r * \alpha * \lambda_{r+1} * \dots * \lambda_n]_\varepsilon$, where each λ_i is an ε-refinement of a δ-small loop.*

Proof. We will prove by induction that for every $k \le n$, an ε-refinement γ_k' of γ_k is ε-homotopic to $\lambda_1 * \dots * \alpha * \dots * \lambda_k$, where each λ_i is an ε-refinement of a δ-small loop. The case $k = 0$ is trivial. Suppose the statement is true for some $0 \le k < n$. The points required to ε-refine γ_k to γ_k' will be denoted by m_i. Suppose that γ_{k+1} is obtained from γ_k by adding a point x between

x_i and x_{i+1}. Let $\{x_i, a_1, \ldots, a_k, x\}$ be an ε-refinement of $\{x_i, x\}$ and $\{x, b_1, \ldots, b_m, x_{i+1}\}$ an ε-refinement of $\{x, x_{i+1}\}$, so

$$\gamma'_{k+1} = \{x_0, m_0, \ldots, x_i, a_1, \ldots, a_k, x, b_1, \ldots, b_m, x_{i+1}, m_r, \ldots, x_j\}$$

is an ε-refinement of γ_{k+1}. Defining $\mu_{k+1} := \{x_0, m_0, \ldots, x_i\}$ and

$$\kappa_{k+1} = \{x_i, a_1, \ldots, a_k, x, b_1, \ldots, b_m, x_{i+1}, m_r, \ldots, x_i\}$$

we have

$$\left[\gamma'_{k+1}\right]_\varepsilon = \left[\mu_{k+1} * \kappa_{k+1} * \overline{\mu_{k+1}} * \gamma'_k\right]_\varepsilon$$

and since the homotopy is a δ-homotopy, $\lambda_{k+1} := \mu_{k+1} * \kappa_{k+1} * \overline{\mu_{k+1}}$ is a refinement of a δ-small loop. The case when a point is removed from γ_k is similar, except that the δ-small loop is multiplied on the right. $\square$

Example 31. Since circles play an important role in this paper, we will conclude this section with a discussion of this simple example. Let C be the standard circle of circumference 1 in the plane with the induced length (also Riemannian) metric. If $\varepsilon > \frac{1}{2}$ then since all points in C are of (intrinsic!) distance at most $\frac{1}{2}$, every ε-loop is ε-null: just remove the points in the loop (except the endpoints) one by one. The group $\pi_\varepsilon(C)$ is trivial and $\phi_\varepsilon : C_\varepsilon \to C$ is an isometry. On the other hand, if $\varepsilon > 0$ is fairly small, it should be intuitively clear that it is impossible to "cross the hole" with an ε-homotopy, since any basic move "spans a triangle" with side lengths smaller than ε; therefore $\pi_\varepsilon(C)$ should be non-trivial (and in fact Theorem 26 tells us that it will be $\pi_1(C) = \mathbb{Z}$). One can check that in fact the homotopy critical spectrum of C is $\{\frac{1}{3}\}$. This example was explored in the work of REU students Byrd et al. [9], where they developed a nice argument involving "discrete winding numbers". That the homotopy critical spectrum is $\{\frac{1}{3}\}$ also follows from results in the next section.

3. Essential triads and circles

Definition 32. If c is an arclength parameterized loop, we say that c is ε-null if every (or equivalently, some) ε-subdivision chain of c is ε-null.

Lemma 33. *Every arclength parameterized loop of length less than 3ε in a geodesic space X is ε-null.*

Proof. Let $c : [0, L] \to X$ be arclength parameterized with $c(0) = c(L) = p$ and $0 < L < 3\varepsilon$. Then there exists an ε-fine subdivision $\{0 = t_0, t_1, t_2, t_3 = L\}$. Since $d(c(t_1), c(t_3)) = d(c(t_1), c(t_0)) < \varepsilon$, we may simply remove $c(t_2)$ and then $c(t_1)$ to get an ε-null homotopy. $\square$

The next corollary is proved by simply joining the points in the loop by geodesics and concatenating them to obtain an arclength parameterized loop of length less than 3ε.

Corollary 34. *If λ is an ε-loop in a geodesic space X of length less than 3ε then λ is ε-null.*

Remark 35. If C is the image of a rectifiable loop of length L in a metric space X then by the basic theory of curves in metric spaces, for every point x on C there are precisely two possible arclength parameterizations $c : [0, L] \to X$ of C such that $c(0) = c(L) = x$ (cf. [26]).

Proposition 36. *The image C of a rectifiable path loop of length $L = 3\varepsilon$ in a geodesic space X is an essential ε-circle if and only if either arclength parameterization of it is not ε-null.*

Proof. Let $c : [0, L] \to C$ be an arclength parameterization of C. If C is not an essential ε-circle then by definition, every ε-chain in it is ε-null. But then any ε-subdivision of c, being an ε-chain, must be ε-null. Hence c is by definition ε-null. Conversely, suppose that C is essential, and so contains an ε-loop $\alpha = \{x_0, \ldots, x_n = x_0\}$ that is not ε-null, with $x_i := c(t_i)$. We will show that α is ε-homotopic to a concatenation of chains that are subdivision ε-chains of c or reversals of c. Then at least one of those subdivision chains must be not ε-null, finishing the proof. Form a path as follows: choose a shortest segment σ_i of c between x_{i-1} and x_i. By "segment" we mean the restriction of c to a closed interval, or a path of the form $c\,|_{[t,L]} * c\,|_{[0,s]}$ (i.e. when it is shorter to go through x_0). Let $\widetilde{c} := \sigma_1 * \cdots * \sigma_n$. Since each σ_i has length at most $\frac{L}{2}$, by adding points b_i that bisect each segment σ_i we see that α is ε-homotopic to a subdivision ε-chain $\widetilde{\alpha} := \{x_0, b_1, x_1, \ldots, b_n, x_n\}$ of $\widetilde{c}$. On the other hand, $\widetilde{c}$ is path homotopic (in the image of c, in fact) to its "cancelled concatenation" $\sigma_1 \star \cdots \star \sigma_n$. Recall that the cancelled concatenation $c_1 \star c_2$ is formed by starting with the concatenation $c_1 * c_2$ and removing the maximal final segment of c_1 that is equal to an initial segment of c_2 with reversed orientation (see the Berestovskii–Plaut paper [7, p. 1771], for more details). It is not hard to check by induction that $\sigma_1 \star \cdots \star \sigma_i$ is of the form $(k_1 * \cdots * k_m) * d$, where the following are true: $k_i = c$ or $k_i = \overline{c}$ for all i (and it is possible that $m = 0$, meaning there are no k_i factors), and for some $0 \leq s < L$, d is of the form $c\,|_{[0,s]}$ or $\overline{c\,|_{[s,L]}}$. Since α is a loop, $\sigma_1 \star \cdots \star \sigma_n$ has no nontrivial term d, and hence consists of concatenations of c or $\overline{c}$. Since $\widetilde{c}$ is a stringing of $\widetilde{\alpha}$, Corollary 21 implies that $\widetilde{\alpha}$, hence α, is ε-homotopic to any subdivision ε-chain of $\sigma_1 \star \cdots \star \sigma_n$. □

A *geodesic triangle* consists of three geodesics $\gamma_1, \gamma_2, \gamma_3$ such that for some three points v_1, v_2, v_3, γ_i goes from v_i to v_{i+1}, with addition of vertices (mod 3). A geodesic triangle may be considered as a loop by taking the arclength parameterization of the concatenation of the geodesics; as far as being ε-null is concerned, the specific orientation clearly does not matter. We say the triangle is ε-null if such a parameterization ε-null.

Proposition 37. *Let T be an ε-triad in a geodesic space. Then any two ε-refinements of α_T are ε-homotopic. Moreover, the following are equivalent:*

1. *T is essential.*
2. *No ε-refinement of α_T is ε-null.*
3. *Every geodesic triangle having T as a vertex set is an essential ε-circle.*

Proof. Let $T := \{x_0, x_1, x_2\}$ and $\beta = \{x_0, m_0, x_1, m_1, x_2, m_2, x_0\}$ be a midpoint refinement of α_T. If m_0' is another midpoint between x_0 and x_1 then the ε-chain $\{x_0, m_0, x_1, m_0', x_0\}$ has length at most $2\varepsilon < 3\varepsilon$ and is ε-null by Corollary 34. Therefore β is ε-homotopic to $\{x_0, m_0', x_1, m_1, x_2, m_2, x_0\}$. A similar argument shows that the other two midpoints may be replaced, up to ε-homotopy. In other words, any two midpoint refinements of α_T are ε-homotopic. But any ε-refinement of α_T has a common refinement with a midpoint refinement, so by the comments after Definition 28, any two ε-refinements of α_T are ε-homotopic.

$1 \Rightarrow 2$. If T is essential then by definition some midpoint refinement of α_T is not ε-null. By the very first statement of this proposition, any other ε-refinement of α_T is not ε-null. $2 \Rightarrow 3$. Suppose $C := (\gamma_0, \gamma_1, \gamma_2)$ is any geodesic triangle having T as a vertex set. Then the subdivision chain of C consisting of the vertices and midpoints of the geodesics is an ε-refinement of α_T and is not ε-null by assumption. Since C also has length 3ε, by definition C is an essential ε-circle, and 3 is proved. 3 $\Rightarrow$ 1. Form a geodesic triangle, hence an essential ε-circle C, having the points of T as vertices. Then any midpoint refinement of T is an ε-subdivision of C, which by Proposition 36 is not ε-null. By definition, T is essential. $\square$

An immediate consequence of Proposition 37 is the following:

Corollary 38. *The following statements are equivalent for two essential ε-triads T_1, T_2 in a geodesic space:*

1. *T_1 is equivalent to T_2.*
2. *Every ε-refinement of α_{T_1} is freely ε-homotopic to every ε-refinement of either α_{T_2} or $\overline{\alpha_{T_2}}$.*
3. *Some ε-refinement of α_{T_1} is freely ε-homotopic to some ε-refinement of either α_{T_2} or $\overline{\alpha_{T_2}}$.*

Proof of Proposition 9. Note that by Corollary 38 we may use any δ-refinement in the arguments that follow. Suppose that T' is a δ-triad; by the triangle inequality, $\delta < \frac{5}{3}\varepsilon$. Suppose first that $\delta \geq \frac{4}{3}\varepsilon$. By the triangle inequality, $L(\{x_0, x_0', x_1', x_1, x_0\}) < \frac{10}{\varepsilon}\varepsilon < 3\delta$, and therefore any δ-refinement of this chain is δ-null by Corollary 34. Since a similar statement applies to the loops $\{x_1, x_1', x_2', x_2, x_1\}$ and $\{x_0, x_2, x_2', x_0', x_0\}$, it follows that any δ-refinement of $\alpha_{T'}$ is freely δ-homotopic to a δ-refinement of α_T. Since T is an essential ε-triad and $\varepsilon < \delta$, any midpoint refinement of α_T, and hence any midpoint refinement of $\alpha_{T'}$, is δ-null. That is, T' is not essential.

Now suppose that $\delta < \frac{4}{3}\varepsilon$. By the triangle inequality, $L(\{x_0, x_0', x_1', x_1, x_0\}) < 3\varepsilon$ and therefore any ε-refinement of this chain is ε-null by Corollary 34. Since a similar statement applies to the loops $\{x_1, x_1', x_2', x_2, x_1\}$ and $\{x_0, x_2, x_2', x_0', x_0\}$, it follows that any ε-refinement of $\alpha_{T'}$ is freely ε-homotopic to an ε-refinement of α_T. Since no ε-refinement of α_T is ε-null, neither is any ε-refinement of $\alpha_{T'}$. On the other hand, if $\sigma > \varepsilon, \alpha_T$ is σ-null and hence $\alpha_{T'}$ is also σ-null. Therefore if T' is an essential triad then T' cannot be a σ-triad for any $\sigma > \varepsilon$. On the other hand, if T' were an essential σ-triad for some $\sigma < \varepsilon$ then any midpoint refinement of $\alpha_{T'}$ would have to be ε-null, a contradiction. $\square$

Theorem 39. *Let X be a geodesic space, $\varepsilon > 0, L = 3\varepsilon$ and $c : [0, L] \to X$ be arclength parameterized. If the image of c is an essential ε-circle C then c is not null-homotopic and C is metrically embedded.*

Proof. That c is not null-homotopic is immediate from Corollary 21. For the second part we will start by showing that the restriction of c to the interval $\left[\frac{L}{4}, \frac{3L}{4}\right]$ is a geodesic, hence a metric embedding. If not then $d\left(c\left(\frac{L}{4}\right), c\left(\frac{3L}{4}\right)\right) < \frac{L}{2}$. We will get a contradiction to Proposition 36 by proving that the ε-loop $\alpha = \{x_0, x_1, x_2, x_3, x_0\}$ for the subdivision $\{0, \frac{L}{4}, \frac{L}{2}, \frac{3L}{4}, L\}$ is ε-null. Let m be a midpoint between x_1 and x_3. By our assumption (and since c is arclength parameterized), $\xi := \{x_1, x_2, x_3, m, x_1\}$ is an ε-chain and has length strictly less than L and hence by Corollary 34, is ε-null. By adding points one at a time we have α is ε-homotopic to $\{x_0, x_1, x_2, x_3, m, x_3, x_0\}$, which is ε-homotopic to

$$\{x_0, x_1, x_2, x_3, m, x_1, m, x_3, x_0\} = \{x_0, x_1\} * \xi * \{x_1, m, x_3, x_0\}$$

which is ε-homotopic to $\beta = \{x_0, x_1, m, x_3, x_0\}$. But once again, since $d(x_1, x_3) < \frac{L}{2}, \beta$ is ε-null.

Now for any $s_0 \in [0, L]$ we may "shift" the parameterization of c to a new curve c_{s_0} : $[0, L] \to X$ that is the unique arclength monotone reparameterization of the concatenation $c|_{[s_0,L]} * c|_{[0,s_0]}$. Applying the above argument for arbitrary s_0 we obtain the following. For every

$x = c(s), y = c(t) \in C$, with $s < t, d(x, y)$ is the minimum of the lengths of the two curves $c\mid_{[s,t]}$ and $c\mid_{[t,L]} * c\mid_{[0,s]}$.

Define $r := \frac{L}{2\pi}$, and let K be the standard Euclidean circle of radius r (with the geodesic metric). Now we may define $f : C \to K$ by $f(c(t)) = (r\cos\frac{t}{r}, r\sin\frac{t}{r})$. Given that c is arclength parameterized, and what we proved above, it is straightforward to check that f is a well-defined isometry. □

Remark 40. In [27], Sormani defines a $1/k$ geodesic to be a closed curve of length L such that each segment of length L/k is a (length minimizing) geodesic. Clearly a metrically embedded circle has a unit parameterization with this property. In the above proof we essentially show the converse: that in a geodesic space, an arclength parameterized curve is a $1/2$ geodesic that has an image that is a metrically embedded circle.

Corollary 41. *Every ε-triad on an essential ε-circle is essential. Moreover, if C_1, C_2 are essential ε-circles in a geodesic space then the following are equivalent:*

1. *C_1 and C_2 have arclength parameterizations with subdivision ε-chains that are freely ε-homotopic.*
2. *For some triads T_i on C_i, T_1 is equivalent to T_2.*
3. *For any triads T_i on C_i, T_1 is equivalent to T_2.*
4. *For any arclength parameterizations c_i of C_i, any subdivision ε-chain of c_1 is freely ε-homotopic to any subdivision ε-chain of either c_2 or $\overline{c_2}$.*

Proof. A triad T on C must be an ε-triad since by Theorem 39, C is metrically embedded—in fact from the same theorem it follows that the segments of C between the points of T must be geodesics. Therefore the midpoints of these geodesics give a midpoint refinement of α_T that is also an ε-subdivision of a parameterization of C, and hence is not ε-null. That is, T is essential.

We next show that any two triads $T = \{x_0, x_1, x_2\}$ and $T' = \{x'_0, x'_1, x'_2\}$ on C are equivalent. First note that T is equivalent to any reordering of its points. In fact, any reordering may be obtained by a cyclic permutation (which is covered by Lemma 22 applied to any midpoint refinement of α_T) and/or a swap of x_1 and x_2 (which by definition does not affect equivalence since it simply reverses α_T). Now applying some reordering of T we may suppose that the points are arranged around the circle in the following order: $\{x_0, x'_0, x_1, x'_1, x_2, x'_2, x_0\}$, which is an ε-refinement of α_T. By Lemma 22, this ε-chain is freely ε-homotopic to $\{x'_0, x_1, x'_1, x_2, x'_2, x_0, x'_0\}$, which is an ε-refinement of $\alpha_{T'}$. So the first part of the corollary is finished by Corollary 38.

$1 \Rightarrow 2$. Choose arclength parameterizations c_i of C_i with subdivision ε-chains λ_i starting at points z_i that are freely ε-homotopic. Choose one of the two triads, call it T_i, in each C_i starting at z_i, that is also a subdivision chain of c_i. By the comments after Definition 28 we see that λ_i and the midpoint refinement of α_{T_i} on C_i are ε-homotopic. Hence midpoint refinements of α_{T_1} and α_{T_2} are freely ε-homotopic, so T_1 is equivalent to T_2. $2 \Rightarrow 3$ is an immediate consequence of the first part of this corollary. $3 \Rightarrow 4$. Consider the triads $T_i = \{c_i(0), c_i(\varepsilon), c_i(2\varepsilon)\}$. By reversing one of the parameterizations, if necessary, we may suppose that T_1 is freely ε-homotopic to T_2. But then midpoint refinements of T_i are subdivision ε-chains of c_i that are freely ε-homotopic. $4 \Rightarrow 1$ simply follows from the definition. □

Definition 42. An essential ε-circle C_1 and an essential δ-circle C_2 are said to be equivalent if $\varepsilon = \delta$ and the four equivalent conditions in the previous corollary hold. When ε is not determined we will just refer to C as an essential circle.

Proof of Theorem 6. If there is an essential ε-circle C then there is an arclength parameterization $c : [0, 3\varepsilon] \to C$. Since c is not ε-null, by definition a subdivision of $[0, 3\varepsilon]$ into fourths results in an ε-loop α that is not ε-null. But for any $\delta > \varepsilon$, Lemma 33 (applied to δ) shows that α must be δ-null for all $\delta > \varepsilon$ and hence has ε as its critical value.

For the converse, suppose that λ is ε-critical. We will start by showing that for all $\varepsilon < \delta < 2\varepsilon$ there is a midpoint refinement of a δ-small loop that is not ε-null. In fact, since λ is ε-critical, it is δ-null and therefore by Proposition 30 can be written as a product of midpoint refinements of δ-small loops. If all of these loops were ε-null, then λ would also be ε-null, a contradiction. Now for every i we may find $\left(\varepsilon + \frac{1}{i}\right)$-small loops $\lambda_i = \mu_i * \{x_i, y_i, z_i, x_i\} * \overline{\mu_i}$ such that midpoint subdivisions $\theta_i = \{x_i, m_i, y_i, n_i, z_i, p_i, x_i\}$ are not ε-null. By choosing a subsequence if necessary, we may suppose that all six sequences converge to a limiting midpoint subdivision chain $\mu = \{x, m, y, n, z, p, x\}$ of length at most 3ε. But according to Proposition 16, for large enough i, μ is ε-homotopic to μ_i, which means that μ is not ε-null. This means that the chain $\{x, y, z, x\}$ must have length equal to 3ε. Since $d(x, y), d(y, z), d(x, z) \leq \varepsilon$ it follows that $\{x, y, z\}$ is a triad and hence is essential. By Proposition 37, any geodesic triangle having corners $\{x, y, z\}$ is an essential ε-circle. $\square$

Corollary 43. *Suppose X is a compact geodesic space with 1-systole σ_1. Then*

1. *$\frac{\sigma_1}{3}$ is a lower bound for the homotopy critical spectrum of X.*
2. *If X is semilocally simply connected and not simply connected then $\sigma_1 > 0$ and $\varepsilon := \frac{\sigma_1}{3}$ is the smallest homotopy critical value of X.*

Proof. Every parameterized essential circle is a closed geodesic that is not null-homotopic by Theorem 39; the first part is immediate. If X is semilocally simply connected and not simply connected, Theorem 26 implies that for some $\varepsilon > 0, \phi_\varepsilon : X_\varepsilon \to X$ is the simply connected covering map of X and ε is the smallest homotopy critical value of X. By Theorem 6, X contains an essential circle, which is the image of a closed geodesic γ of length 3ε. If γ were null-homotopic then γ would lift as a loop, contradicting Proposition 20 and the fact that any subdivision ε-chain of it is not ε-null. This implies that $\sigma_1 \leq 3\varepsilon$. Now X can be covered by open sets with the property that every loop in the set is null-homotopic in X. Therefore any loop of diameter smaller than the Lebesgue number of this cover is by definition contained in a set in the cover, hence null-homotopic, which implies $\sigma_1 > 0$. Now suppose that $\delta := \frac{\sigma_1}{3} < \varepsilon$. If γ were a non-null homotopic closed geodesic of length σ_1, then γ could not lift as a loop to the simply connected space X_ε. Hence by Proposition 20, any ε-subdivision chain α of γ has the property that $[\alpha]_\varepsilon \neq [*]_\varepsilon$. This contradicts Corollary 34. $\square$

Example 44. Let Y denote the flat torus obtained by identifying the sides of a rectangle of dimensions $0 < 3a \leq 3b$. When $a < b, a$ and b are distinct homotopy critical values: For $\varepsilon > b, Y_\varepsilon = Y$, for $a < \varepsilon \leq b, Y_\varepsilon$ is a flat metric cylinder over a circle of length $3a$, and for $\varepsilon \leq a, Y_\varepsilon$ is the plane. There are infinitely many essential a-circles and b-circles, but all essential a-circles are equivalent and all essential b-circles are equivalent (Corollary 21). When $a = b, a$ is the only homotopy critical value; both circles "unroll" simultaneously and the covers go directly from trivial to universal. There are still two equivalence classes of essential circles, but since the circles have the same length, a is a homotopy critical value of multiplicity 2. Now fix $a = b = \frac{1}{3}$ (i.e. Y comes from a unit square). The closed geodesic determined by a straight path starting at the bottom left corner of the square having a slope of $\frac{1}{2}$ is a Riemannian isometric embedding of a circle of length $\sqrt{5}$, which is the shortest path in its homotopy class. However, the distance between the images of any two antipodal points is only $\frac{1}{2}$, so this closed geodesic is not metrically embedded, hence not an essential circle. The diagonal of the square produces an ε-circle C with $\varepsilon = \frac{\sqrt{2}}{3}$, which is the shortest path in its homotopy class, is metrically embedded and not null-homotopic, but is not essential. In fact, C can be homotoped to the concatenation of the two circles of which the torus is a product. Hence any ε-loop λ on C can be ε-homotoped to a loop λ' in those circles. But each of these circles is not ε-essential ($\varepsilon = \frac{\sqrt{2}}{3} > \frac{1}{3}$) so λ', hence λ, is ε-null.

Note that if one adds a thin handle to the torus it will obstruct standard homotopies between some essential circles, but not ε-homotopies. This shows that using traditional homotopies rather than ε-homotopies in the definition of equivalence can "overcount" multiplicity. In the Sormani–Wei paper [29], the multiplicity of a number δ in the covering spectrum is defined for compact spaces with a universal cover (in the categorical sense, not necessarily simply connected) as the minimum number of generators of a certain type in a certain subgroup of the "revised fundamental group" (Definition 6.1, [29]). We will not recall the definition of these groups here because they require a universal cover and this assumption is unnecessary for our work.

Example 45. We will now recall the construction of a space V that is known to contain a path loop L that is homotopic to arbitrarily small loops but is not null-homotopic due to Zastrow and Cannon–Conner (see [34] or [10]), giving it a geodesic metric in the process. The Hawaiian Earring H consists of all circles of radius $\frac{1}{i}$ in the plane centered at $(0, \frac{1}{i})$, $i \in \mathbb{N}$, with the subspace topology. The induced geodesic metric on H measures the distance between any two points in H as the length of the shortest path in H joining them. It is easy to check that this metric is compatible with the subspace topology. Now take the cone on H, which also has a geodesic metric compatible with the topology of the cone (see, for example, the survey article of the first author [23] for details about geodesic metrics on glued spaces and cones). Glue two copies of this space together at the point $(0, 0)$ in H. One can check that every ε-loop is ε-null for every ε, so the homotopy critical spectrum is empty even though the space is not simply connected. This example is related to Corollary 43 in the following way: one wonders if the requirement that X be semilocally simply connected in the second part is required. If the path loop L mentioned above had a closed geodesic in its homotopy class then we would have a counterexample to the second part of Corollary 43 with the weaker hypothesis. However, such a thing is not guaranteed—see Remark 25.

Example 46. Let X_n be the geodesic space consisting of circles of radii $\frac{1}{i}$ for $1 \leq i \leq n$ joined at a point. These spaces are Gromov–Hausdorff convergent to a geodesic Hawaiian Earring, but their universal covers consist of infinite trees with valencies tending to infinity, and hence are not Gromov–Hausdorff (pointed) precompact. One can "thicken" these examples into a family of Riemannian 2-manifolds with same property. It seems like an interesting question to characterize when precompactness of a class of geodesic spaces (even a single space!) implies precompactness of the collection of all covering spaces.

The following example makes one wonder whether Corollary 10 is optimal.

Example 47. Let S_n denote the space consisting of two points joined by n edges of length $\frac{3}{2}$, with the geodesic metric. Each pair of edges determines a circle of length 3, so there is a single critical value 1 of multiplicity $\binom{n}{2} = \frac{1}{2}(n^2 + n)$. On the other hand, we can cover the space using one open $\frac{1}{3}$-ball at each of the two vertices and 2 additional $\frac{1}{3}$-balls on each edge for a total of $2(n+1)$. The estimate from Corollary 10 is $\frac{4}{3}n^3+2n^2+\frac{2}{3}n$ and at any rate each edge requires at least one ball, so one cannot do better than a degree 3 polynomial. Another example that can be checked in a similar fashion is the 1-skeleton of a regular n-simplex with every edge length equal to 1, with the geodesic metric. In this example each boundary of a 2-face is isometric to a standard circle of circumference 3. There is a single critical value 1 of multiplicity $\binom{n+1}{3}=\frac{1}{6}(n^3-n)$. But any cover by open $\frac{1}{3}$-balls will again require at least one ball for each of the $\binom{n+1}{2}$ edges and therefore the best that Corollary 10 can provide is a polynomial of order 6 in n.

4. (ε,δ)-chassis

In this section, X will be a compact geodesic space of diameter D, $\varepsilon > 0$ is fixed, and $0<\delta<\sigma$ will be positive numbers with $\sigma \le \varepsilon$, on which we will place additional requirements to reach stronger conclusions. We define an (ε,δ)-chassis to be a simplicial 2-complex that has for its vertex set a δ-dense set $V := \{v_0,\dots,v_m\}$ (i.e. for every $x\in X$ there is some v_i such that $d(x,v_i)<\delta$). We let v_i and v_j be joined by an edge if and only if $d(v_i,v_j)<\varepsilon$ and let v_i, v_j, v_k span a 2-simplex if and only if all three pairs of vertices are joined by an edge. Next, let K be the 1-skeleton of C and denote the edge joining v_i and v_j by e_{ij}, $i<j$. Define the length of e_{ij} to be $d(v_i,v_j)$ (distance in X), the length of an edge path to be the sum of the lengths of its edges, and the simplicial distance $d_S(v_i,v_j)$ between vertices $v_i \ne v_j$ to be the length of a shortest edge path joining them.

Every edge path in C starting at v_0 (which we take for the basepoint) is equivalent to a chain of vertices $\{v_0=v_{1_0},\dots,v_{i_k}\}$, which has a corresponding ε-chain $\{v_0=v_{1_0},\dots,v_{i_k}\}$ in X. Now the basic moves in an edge homotopy in C (replacing one side of a simplex by the concatenation of the other two, removal of an edge followed by its reversal, or vice versa) correspond precisely to the basic moves in an ε-homotopy. In other words, the function that takes the edge-homotopy class $[v_0=v_{1_0},\dots,v_{i_k}=v_0]$ of a loop to the ε-homotopy class $[v_0=v_{1_0},\dots,v_{i_k}=v_0]_\varepsilon$ is a well-defined homomorphism E from the group of edge homotopy classes of edge loops (i.e. the edge group) $\pi_E(C)$ of C into $\pi_\varepsilon(X)$. We denote by D_S the diameter of C with the simplicial metric.

Lemma 48. *If $\delta<\frac{\sigma}{4}$ then C is connected and E is surjective. In fact, if $\beta=\{v_a,y_1,\dots,y_{n-1},v_b\}$ is an ε-chain joining points in V in X, then $[\beta]_\varepsilon$ contains a "simplicial" σ-chain α (i.e. a chain having all points in the vertex set V) such that*

$$L(\alpha)\le L(\beta)+2\left(\frac{8L(\beta)}{\sigma}\right)\delta.$$

Proof. Given any $v_a,v_b\in V$, let c be a geodesic joining them in X. We may subdivide c into segments with endpoints x_k,x_{k+1}, $0\le k\le N$, of length at most $\frac{\varepsilon}{6}$. For each m we may choose a point $v_{i_m}\in V$ such that $d(x_m,v_{i_m})<\delta$. Since $\delta<\frac{\varepsilon}{4}$, the triangle inequality implies that v_{i_m} and $v_{i_{m+1}}$ are joined by an edge in C, and hence v_a, v_b are joined by an edge path in C. Surjectivity will follow from the last statement, since we may take $v_a=v_b=v_0$ and then resulting α is an ε-loop with $[\alpha]_\varepsilon$ in the image of E. By refinement we may suppose β is a $\frac{\sigma}{4}$-chain, and applying Lemma 17 we may assume that $n=\left\lfloor\frac{8L(\beta)}{\sigma}+1\right\rfloor$. For each i we may choose some v_{j_i} such that $d(v_{j_i},x_i)<\delta$ (letting $v_{j_0}=v_a$ and $v_{j_n}=v_b$). Since

$\delta < \frac{\sigma}{4}$, Proposition 16 now implies that β is σ-homotopic to the σ-chain $\alpha := \{v_{j_0}, \ldots, v_{j_n}\}$ and hence $[\beta]_\varepsilon = E([v_{j_0}, \ldots, v_{j_n}]) = [v_{j_0}, \ldots, v_{j_n}]_\varepsilon$. Moreover, the triangle inequality implies that $L(\alpha) \le L(\beta) + 2n\delta$, completing the proof. □

Lemma 49. *If $\delta < \min\{\frac{\varepsilon}{4}, \frac{\varepsilon^2}{32D}\}$ then for any $v_a, v_b \in V$, $d(v_a, v_b) \le d_S(v_a, v_b) \le d(v_a, v_b) + \frac{\varepsilon}{2}$.*

Proof. The left inequality is obvious. Subdivide a geodesic in X joining v_a, v_b to produce an ε-chain β of length equal to $d(v_a, v_b)$. Taking $\sigma = \varepsilon$ in Lemma 48 produces a simplicial chain α of length at most $L(\beta) + \frac{\varepsilon}{2}$ joining v_a and v_b. □

Lemma 50. *If $\phi_{\varepsilon\sigma}$ is a bijection and $\delta < \min\left\{\frac{\varepsilon-\sigma}{2}, \frac{\sigma}{16}\right\}$ then E is injective.*

Proof. Suppose $[v_0 = v_{1_0}, \ldots, v_{i_k} = v_0] \in \ker E$. This means that the ε-chain $\alpha := \{v_0 = v_{1_0}, \ldots, v_{i_k} = v_0\}$ is ε-null in X. The problem, of course, is that the ε-null-homotopy may not involve only simplicial ε-chains and hence does not correspond to a simplicial null-homotopy in C. However, by Lemma 48, we may assume that α is in fact an ε-null simplicial σ-chain. By our choice of σ, α is in fact σ-null. Let $\langle \alpha := \eta_0, \ldots, \eta_m = \{v_0\}\rangle$ be a σ-homotopy and A be the set of all points a such that a is in some chain η_i. For each $a \in A$ let $a' \in V$ be such that $d(a, a') < \delta < \frac{\varepsilon-\sigma}{2}$, provided that if a is already in V then $a' := a$. Finally, define $\eta'_k := \{v_0 = x'_{k1}, \ldots, x'_{kr_k} = v_0\}$ whenever $\eta_k := \{v_0 = x_{k1}, \ldots, x_{kr_k} = v_0\}$; by definition, η'_k is a simplicial chain and since α is already simplicial $\eta'_0 = \eta_0 = \alpha$. Moreover, $d(x'_{ki}, x'_{k(i+1)}) < \sigma + 2(\frac{\varepsilon-\sigma}{2}) = \varepsilon$. That is, $\left\langle \alpha := \eta'_0, \ldots, \eta'_m = \{v_0\}\right\rangle$ is an ε-homotopy via simplicial chains, and so is equivalent to a simplicial homotopy in C. □

We will now recall the well-known method of choosing generators and relations for $\pi_E(C)$, while adding a geometric twist (see for example [3, Section 6.4]). First, we obtain a maximal subtree T of the 1-skeleton K as follows. Choose some v_k of maximal simplicial distance from v_0 and connect v_k to v_0 by a shortest simplicial path Γ_1; Γ_1 is the starting point in the construction of T. Since Γ_1 is minimal it must be simply connected, hence a tree; if it is maximal then we are done. Otherwise there is at least one vertex not in Γ_1, and we choose one, v_j, of maximal simplicial distance from v_0. Let Γ_2 be a minimal simplicial path from v_j to v_0. If at some point Γ_2 meets (for the first time) any vertex w already in T, then we replace the segment of Γ_2 from w to v_0 by the unique shortest segment of Γ_1 from w to v_0. In doing so we do not change the length of Γ_2 and ensure that the union of Γ_1 and Γ_2 is still a tree. We iterate this process until all vertices are in the tree. The resulting maximal tree T has the property that every vertex v_j in K is connected to v_0 by a unique simplicial path contained in T having length at most the simplicial diameter D_S of C.

Now $\pi_E(C)$ has generators and relators defined as follows [3, Section 6.4]: The generators are concatenations of the form $[g_{ij}] = [p * e_{ij} * q]$, where e_{ij} is an edge that is in K but not in T and p (resp. q) is the unique shortest simplicial path in T from v_j to v_0 (resp. v_0 to v_i). The relations are of the form $[g_{ij}][g_{jk}] = [g_{ik}]$, provided v_i, v_j, v_k span a 2-simplex in K with $i < j < k$. Note that the simplicial length of g_{ij} is at most $2D_S + \varepsilon$.

Proof of Theorem 3. Since the homotopy critical values are discrete, we may always choose $\sigma < \varepsilon$ so that $\phi_{\varepsilon\sigma}$ is injective. We may then choose δ so that all of the requirements of the above lemmas all hold. Then the resulting generators of $\pi_E(C)$ correspond under the isomorphism E to classes $[\gamma_{ij}]_\varepsilon$ in X such that the length of each γ_{ij} is at most $2D_S + \varepsilon + \frac{\varepsilon}{2} < 2(D + \varepsilon)$. This proves the first part of the theorem, and the second part was proved in the Introduction.

For the third part, we begin by choosing an $\frac{\varepsilon}{4}$-dense set $W = \{w_1, \ldots, w_s\}$ in X and an arbitrary δ-chain μ_{ij} from w_i to w_j with $\mu_{ji} = \overline{\mu_{ij}}$. Given any δ-loop $\lambda = \{v_0 = x_0, \ldots, x_n = v_0\}$ of length at most $2(D + \varepsilon)$, choose a subchain $\mu = \{y_0 = v_0, \ldots, y_r = v_0\}$ (i.e. $y_j = x_{i_j}$ for some increasing i_j) with the following property: If λ_j denotes the δ-chain $\{y_j = x_{i_j}, x_{i_j+1}, \ldots, x_{i_{j+1}} = y_{j+1}\}$ (i.e. the "segment" of λ from y_i to y_{i+1}) then for any j, $L(\lambda_j) < \frac{\varepsilon}{4}$ and $L(\lambda_j) + L(\gamma_{j+1}) \geq \frac{\varepsilon}{4}$. This can be accomplished by iteratively removing points to form the subsequence, in a way similar to what was done in the proof of Lemma 17. The same counting argument as in that proof gives us $r \leq \frac{2L(\lambda)}{\varepsilon} \leq \frac{8(D+\varepsilon)}{\varepsilon}$. For each y_j, choose some $y'_j \in W$ such that $d(y_j, y'_j) < \frac{\varepsilon}{4}$. There is now a corresponding δ-chain λ' that is a concatenation of paths $\mu_{i_k j_k}$, where $y'_k = w_{i_k}$ and $y_{k+1} = w_{j_k}$. Next, let γ_j be a δ-chain from y'_j to y_j of length at most $\frac{\varepsilon}{4}$. It is not hard to check that λ is δ-homotopic to $\beta_r * \cdots \beta_0 * \lambda'$, where $\beta_0 := \overline{\lambda_0} * \gamma_1 * \mu_{i_0 j_0}$ and for $k > 0$,

$$\beta_k := \overline{\mu_{i_0 j_0}} * \cdots \overline{\mu_{i_k j_k}} * \overline{\gamma_k} * \overline{\lambda_k} * \gamma_{k+1} * \mu_{i_{k+1} j_{k+1}} * \cdots * \mu_{i_0 j_0}.$$

Let us count the ways to obtain λ. First, λ' corresponds to a sequential choice of r elements of W, so there are at most s^r possibilities. Next, λ is obtained from λ' by r concatenations, each of which involves a choice of the element $[\overline{\gamma_k} * \overline{\lambda_k} * \gamma_{k+1} * \mu_{i_{k+1} j_{k+1}}]_\delta \in \pi_\delta(X, w)$ for some $w \in W$ with $L(\overline{\gamma_k} * \overline{\lambda_k} * \gamma_{k+1} * \mu_{i_{k+1} j_{k+1}}) < \varepsilon$. So there are at most $r \cdot M$ distinct choices to change from λ' to λ. $\square$

From the second part of Theorems 3 and 26 we may immediately derive the following corollary:

Corollary 51. *Let X be a compact, semilocally simply connected geodesic space. If $\varepsilon > 0$ is a lower bound for the homotopy critical spectrum of X then for any $L > 0$, $\Gamma(X, L) \leq C\left(X, \frac{\varepsilon}{4}\right)^{\frac{4L}{\varepsilon}}$.*

Proof of Theorem 1. By Theorem 26, if $\delta < \varepsilon$ is sufficiently small, the function $\Lambda : \pi_1(X) \to \pi_\delta(X)$ is a length-preserving isomorphism. Then the desired generators are those corresponding to the generators of $\pi_\delta(X)$ given by the third part of Theorem 3, except that, *a priori* those generators have length $2(D+\delta)$. However, since X is compact and semilocally simply connected, the proof is finished by a standard application of Ascoli's Theorem. The statement about the 1-systole follows from Theorem 26. $\square$

Remark 52. Sormani and Wei generalize the Shen–Wei finiteness theorem to their notion of "revised fundamental groups" in Proposition 7.8 in [29]. Their argument consists of showing that the collection of universal covers is precompact when the 1-systole is uniformly bounded below and then referring vaguely to the proof of the Shen–Wei theorem. Theorem 3 provides a new more detailed proof of their result, since under their assumptions the revised fundamental group is $\pi_\varepsilon(X)$ when ε is $\frac{1}{3}$ of the 1-systole.

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