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Generalized filters, the low-pass condition, and connections to multiresolution analyses

Lawrence W. Baggett [a,*], Veronika Furst [b], Kathy D. Merrill [c], Judith A. Packer [a]

[a] *Department of Mathematics, University of Colorado, Boulder, CO 80309, USA*
[b] *Department of Mathematics, Fort Lewis College, Durango, CO 81301, USA*
[c] *Department of Mathematics, Colorado College, Colorado Springs, CO 80903, USA*

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Abstract

We study generalized filters that are associated to multiplicity functions and homomorphisms of the dual of an abelian group. These notions are based on the structure of generalized multiresolution analyses. We investigate when the Ruelle operator corresponding to such a filter is a pure isometry, and then use that characterization to study the problem of when a collection of closed subspaces, which satisfies all the conditions of a GMRA except the trivial intersection condition, must in fact have a trivial intersection. In this context, we obtain a generalization of a theorem of Bownik and Rzeszotnik.

Keywords: Wavelet; Generalized multiresolution analysis; Filter; Ruelle operator; Pure isometry

1. Introduction

Filters have historically been an essential tool used in both building and analyzing wavelets and multiresolution structures. In particular, filters traditionally called "low-pass" arise naturally from refinement equations for multiresolution analyses (MRAs) and generalized multiresolution analyses (GMRAs). Beginning with work of Mallat [14] and Meyer [15], the process of

* Corresponding author.
E-mail addresses: baggett@euclid.colorado.edu (L.W. Baggett), furst_v@fortlewis.edu (V. Furst), kmerrill@coloradocollege.edu (K.D. Merrill), packer@euclid.colorado.edu (J.A. Packer).

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defining filters from a multiresolution structure was also reversed; that is, functions that behave like low-pass filters have been used to build the structures. This construction technique has been remarkably fruitful, producing, for example, the smooth and well-localized wavelets of Daubechies [11]. In generalizing this procedure to allow less restrictive conditions on the filters as well as on the setting, for example in [2,4,10], properties of an operator associated with the filter, called a Ruelle operator, are used to justify this construction. The essential ingredient is that the Ruelle operator be a pure isometry. A theorem giving general conditions under which the Ruelle operator is a pure isometry in the case of an integer dilation in $L^2(\mathbb{T})$ appeared in [10].

In this paper we derive a similar theorem (Theorem 4 in Section 2) in a quite general context. We then exploit this theorem both in analyzing multiresolution structures and in building them. Our central result of the first type addresses the question of when a structure that satisfies all the properties of a GMRA except possibly the trivial intersection property, must satisfy that as well. This generalizes work of Bownik and collaborators [7,9]. Our main result of the second type is to show that very little in the way of a low-pass condition is needed when building GMRAs from filters using direct limits as in [4] and [5].

Our general context is as follows: Let Γ be a countable abelian group (written additively) with dual group $\widehat{\Gamma}$ (written multiplicatively), equipped with Haar measure μ (of total mass 1). Let α be an isomorphism of Γ into itself, and suppose that the index of $\alpha(\Gamma)$ in Γ equals $N > 1$. Assume further that $\bigcap_{n\geqslant 0}\alpha^n(\Gamma) = \{0\}$. Write α^* for the dual endomorphism of $\widehat{\Gamma}$ onto itself defined by $[\alpha^*(\omega)](\gamma) = \omega(\alpha(\gamma))$, and note that the kernel of α^* contains exactly N elements and that α^* is ergodic with respect to the Haar measure on $\widehat{\Gamma}$. Write $K = \bigcup_{n>0}\ker(\alpha^{*n})$, and note that, because $\bigcap_{n\geqslant 0}\alpha^n(\Gamma) = \{0\}$, K is dense in $\widehat{\Gamma}$.

Of course the standard example (e.g., from wavelet theory) of these ingredients is where $\Gamma = \mathbb{Z}$, $\widehat{\Gamma} = \mathbb{T}$, and $\alpha(k) = 2k$. Or, more generally, $\Gamma = \mathbb{Z}^d$, and $\alpha(\vec{x}) = A\vec{x}$, where A is a $d \times d$ integer dilation matrix of determinant N.

Let $m : \widehat{\Gamma} \to \{0, 1, 2, \ldots, \infty\}$ be a Borel map into the set of nonnegative integers union ∞, and for each $i \in \mathbb{N}$, write σ_i for $\{\omega \in \widehat{\Gamma}\colon m(\omega) \geqslant i\}$. Note that

$$m(\omega) = \sum_i \chi_{\sigma_i}(\omega).$$

We remark that such functions m arise, via Stone's Theorem on unitary representations of abelian groups, as multiplicity functions associated to such representations of Γ. In that context, we will make use of a unitary representation π of Γ, acting in a Hilbert space $\mathcal{H}$, and a unitary operator δ on $\mathcal{H}$ for which

$$\delta^{-1}\pi_\gamma\delta = \pi_{\alpha(\gamma)}$$

for all $\gamma \in \Gamma$.

In this general setting, we define a filter as follows:

Definition 1. Let $H = [h_{i,j}]_{i,j\in\mathbb{N}}$ be a matrix of Borel, complex-valued functions on $\widehat{\Gamma}$, with $h_{i,j}$ supported in σ_j, and such that for every i and almost all ω, $\sum_{j=1}^{\infty}|h_{i,j}(\omega)|^2 < \infty$. Then H is called a *filter* relative to m and α^* if the "filter equation"

$$\sum_{\alpha^*(\zeta)=1}\sum_j h_{i,j}(\omega\zeta)\overline{h_{i',j}(\omega\zeta)} = N\delta_{i,i'}\chi_{\sigma_i}\big(\alpha^*(\omega)\big) \tag{1}$$

is satisfied for almost all $\omega \in \widehat{\Gamma}$.

Remark 2. When the multiplicity function m is bounded, so that the rows and columns of the filter are all eventually identically 0, we can treat the infinite matrix H as a finite-dimensional matrix. Even when the multiplicity function takes on the value ∞, an elementary functional analysis argument shows that a filter H is a bounded operator on l^2, so that products of filter matrices are well defined. In the following sections, we will often need a uniformity of this bound that will require additional hypotheses.

In the standard situation described above, i.e., where $\Gamma = \mathbb{Z}$, $\alpha(k) = 2k$, and where m is the identically 1 function, a filter relative to m and α^* is just a 1×1 matrix (function) h, and the filter equation becomes

$$\left|h(z)\right|^2 + \left|h(-z)\right|^2 = 2,$$

for almost all $z \in \mathbb{T}$, which is the classical equation satisfied by a quadrature mirror filter. These are the filters that played a central role in the early theory of multiresolution analyses and wavelets in $L^2(\mathbb{R})$. Indeed, in the classical case, where ϕ is a scaling function for an MRA in $L^2(\mathbb{R})$, we know that the integral translates $T_n(\phi) = \phi(\cdot - n)$ form an orthonormal basis for the core subspace V_0, and we may define a unitary operator J from V_0 onto $L^2(\mathbb{T})$ by sending the basis vector $T_n(\phi)$ to the function z^n. This correspondence between an orthonormal basis of V_0 with the canonical Fourier basis for $L^2(\mathbb{T})$ is clearly a unitary operator. Furthermore, J sends the element $\phi(x/2)/\sqrt{2} = \sum c_n T_n(\phi)$ to the function $\sum_n c_n z^n = h(z)$, where h is the associated quadrature mirror filter. We notice that, in addition to the fact that h satisfies the quadrature mirror equation, it satisfies another condition. Namely, if δ denotes the dilation operator on $L^2(\mathbb{R})$ given by $[\delta(f)](x) = \sqrt{2} f(2x)$, then one can verify that the operator $J \circ \delta^{-1} \circ J^{-1}$ on $L^2(\mathbb{T})$ is given by

$$\left[\left[J \circ \delta^{-1} \circ J^{-1}\right](f)\right](z) = h(z) f\left(z^2\right) = \left[S_h(f)\right](z)$$

for every $f \in L^2(\mathbb{T})$. We will call such an operator S_h a *Ruelle operator*. Because δ^{-1} is an isometry on V_0, and $\bigcap \mathrm{Range}(\delta^{-n}) = \bigcap V_n = \{0\}$, it follows that the operator S_h has these same properties. That is, S_h is a "pure isometry."

In the next section we define a Ruelle operator S_H similarly associated with an abstract filter H as in Definition 1, and present our first main result, a characterization of when this Ruelle operator is a pure isometry. The final two sections contain the applications of this result.

2. Filters and pure isometries

Let H be a filter relative to m and α^*. Whenever the formula

$$\begin{aligned}\left[S_H(f)\right](\omega) &= H^t(\omega) f\left(\alpha^*(\omega)\right) \\ &= \bigoplus_j \sum_i H_{i,j}(\omega) f_i\left(\alpha^*(\omega)\right)\end{aligned}$$

defines a bounded operator from $\bigoplus_i L^2(\sigma_i, \mu)$ into itself, we will call S_H the Ruelle operator associated to H. In the contexts we study in this paper, this will always be the case.

We now prove a generalization of the filter equation of Definition 1 that will provide a crucial step in determining when the Ruelle operator S_H is an isometry. If the function m associated to H is finite a.e., this proposition follows from the standard filter equation by induction (see Lemma 9 in [4]). However, without this restriction, it requires a more careful argument exploiting the fact that, in the situations we study, S_H is an isometry.

Proposition 3. *Let H be a filter relative to m and α^*, and assume that the associated Ruelle operator S_H is an isometry. Then*:

$$\frac{1}{N^n}\sum_{\alpha^{*n}(\zeta)=1}\sum_i\left[\prod_{k=0}^{n-1}H^t\big(\alpha^{*k}(\omega\zeta)\big)\right]_{i,j}\overline{\left[\prod_{k'=0}^{n-1}H^t\big(\alpha^{*k'}(\omega\zeta)\big)\right]_{i,j'}}=\delta_{j,j'}\chi_{\sigma_j}\big(\alpha^{*n}(\omega)\big).$$

Proof. Let f, g be elements of $\bigoplus_j L^2(\sigma_j,\mu)$. Relying on the fact that $S_H^n(f)\in\bigoplus_j L^2(\sigma_j,\mu)$ whenever $f\in\bigoplus_j L^2(\sigma_j,\mu)$, we may, by Fubini's Theorem, exchange the sum and integral in the following calculation:

$$\begin{aligned}
&\int_{\widehat{\Gamma}}\sum_i\sum_j\left[\prod_{k=0}^{n-1}H^t\big(\alpha^{*k}(\omega)\big)\right]_{i,j}f_j\big(\alpha^{*n}(\omega)\big)\overline{\sum_{j'}\left[\prod_{k'=0}^{n-1}H^t\big(\alpha^{*k'}(\omega)\big)\right]_{i,j'}g_{j'}\big(\alpha^{*n}(\omega)\big)}\,d\omega\\
&=\sum_i\int_{\widehat{\Gamma}}\big[S_H^n(f)\big]_i(\omega)\overline{\big[S_H^n(g)\big]_i(\omega)}\,d\omega\\
&=\big\langle S_H^n(f)\mid S_H^n(g)\big\rangle\\
&=\langle f\mid g\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\widehat{\Gamma}}\frac{1}{N^n}\sum_{\alpha^{*n}(\zeta)=1}\sum_i\left(\sum_j\left[\prod_{k=0}^{n-1}H^t\big(\alpha^{*k}(\omega\zeta)\big)\right]_{i,j}f_j\big(\alpha^{*n}(\omega)\big)\right)\\
&\qquad\cdot\overline{\left(\sum_{j'}\left[\prod_{k'=0}^{n-1}H^t\big(\alpha^{*k'}(\omega\zeta)\big)\right]_{i,j'}g_{j'}\big(\alpha^{*n}(\omega)\big)\right)}\,d\omega\\
&=\langle f\mid g\rangle.
\end{aligned}$$

Write C_j for the element of the direct sum space $\bigoplus_j L^2(\sigma_j,\mu)$ whose jth coordinate is χ_{σ_j} and whose other coordinates are 0. Set $f=\chi_E C_j$, for $E\subseteq\sigma_j$, and $g=\chi_{E'}C_{j'}$, for $E'\subseteq\sigma_{j'}$. We then have

$$\int_{\alpha^{*-n}(E\cap E')}\frac{1}{N^n}\sum_\zeta\sum_i\left[\prod_{k=0}^{n-1}H^t\big(\alpha^{*k}(\omega\zeta)\big)\right]_{i,j}\overline{\left[\prod_{k'=0}^{n-1}H^t\big(\alpha^{*k'}(\omega\zeta)\big)\right]_{i,j'}}\,d\omega$$

$$= \int_{\widehat{\Gamma}} \frac{1}{N^n} \sum_{\zeta} \sum_{i} \left[\prod_{k=0}^{n-1} H^t\big(\alpha^{*k}(\omega\zeta)\big) \right]_{i,j} \chi_E\big(\alpha^{*n}(\omega)\big) C_j\big(\alpha^{*n}(\omega)\big)$$
$$\cdot \overline{\left[\prod_{k'=0}^{n-1} H^t\big(\alpha^{*k'}(\omega\zeta)\big) \right]_{i,j'}} \chi_{E'}\big(\alpha^{*n}(\omega)\big) C_{j'}\big(\alpha^{*n}(\omega)\big)\, d\omega$$
$$= \langle \chi_E C_j \mid \chi_{E'} C_{j'} \rangle$$
$$= \delta_{j,j'} \int_{\widehat{\Gamma}} \chi_{E\cap E'}\big(\alpha^{*n}(\omega)\big)\, d\omega$$
$$= \delta_{j,j'} \int_{\alpha^{*-n}(E\cap E')} 1\, d\omega.$$

Since this is true for any Borel sets E and E', the proposition follows. □

We now prove our first main result, establishing conditions under which a Ruelle operator S_H that is an isometry must in fact be a pure isometry. This theorem generalizes Theorem 3.1 in [10], which finds a similar conclusion in the setting of integer dilations in $L^2(\mathbb{T})$.

Theorem 4. *Assume that m is finite on a set of positive measure. If S_H is an isometry on $\bigoplus_j L^2(\sigma_j, \mu)$, then S_H fails to be a pure isometry if and only if it has an eigenvector. Specifically, S_H fails to be a pure isometry if and only if there exists a nonzero element $f \in \bigoplus_j L^2(\sigma_j, \mu)$, and a scalar λ of absolute value 1, such that $S_H(f) = \lambda f$. Moreover, if f is a unit eigenvector for S_H, then $\|f(\omega)\| = 1$ a.e.*

Proof. Write R_n for the range of the isometry S_H^n, and write R_∞ for the intersection $\bigcap R_n$ of the R_n's. By definition, S_H is a *pure isometry* if and only if $R_\infty = \{0\}$.

If S_H has an eigenfunction f, say $S_H(f) = \lambda f$, with $\lambda \neq 0$, then clearly f belongs to the range of each operator S_H^n, and hence $f \in R_\infty$. Therefore, $R_\infty \neq \{0\}$, and S_H is not a pure isometry.

Conversely, suppose S_H is not a pure isometry. We now adapt an argument in [4] that was based on the reverse martingale convergence theorem. (See Theorem 10.6.1 in [12].) For each $n \geqslant 1$, let $\mathcal{M}_n$ be the σ-algebra of Borel subsets of $\widehat{\Gamma}$ that are invariant under multiplication by elements in the kernel of α^{*n}. Let f and g be two nonzero vectors in R_∞, and define a sequence of random variables $\{X_n\} \equiv \{X_n^{f,g}\}$ on $\widehat{\Gamma}$ by

$$X_n(\omega) = \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta)=1} \big\langle f(\omega\zeta) \bigm| g(\omega\zeta) \big\rangle.$$

Then it follows directly that X_n is $\mathcal{M}_n$-measurable, and the conditional expectation of X_n, given $\mathcal{M}_{n+1}$, equals X_{n+1}. Therefore, the sequence $\{X_n, \mathcal{M}_n\}$ is an integrable, reverse martingale. Hence, using the reverse martingale convergence theorem, we have that the sequence $\{X_n(\omega)\}$ converges almost everywhere and in L^1 norm to an integrable function L on $\widehat{\Gamma}$.

Clearly, $L(\omega\zeta) = L(\omega)$ for almost every ω and every $\zeta \in K = \bigcup_{n>0} \ker(\alpha^{*n})$. Hence, the Fourier coefficient $c_\gamma(L)$ satisfies $c_\gamma(L) = \gamma(\zeta) c_\gamma(L)$ for every $\zeta \in K$, implying that $c_\gamma(L) = 0$

unless $\gamma(\zeta) = 1$ for all $\zeta \in K$. Since K is dense in $\widehat{\Gamma}$, it then follows that $c_\gamma(L) = 0$ for all γ except $\gamma = 0$. Consequently, $L(\eta)$ is a constant function, and we have, from the L^1 convergence of the sequence $\{X_n\}$,

$$\begin{aligned} L(\eta) &= \int_{\widehat{\Gamma}} L(\omega)\, d\omega \\ &= \lim_{n\geqslant 1} \int_{\widehat{\Gamma}} X_n(\omega)\, d\omega \\ &= \lim_{n\geqslant 1} \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta)=1} \int_{\widehat{\Gamma}} \bigl\langle f(\omega\zeta) \bigm| g(\omega\zeta)\bigr\rangle\, d\omega \\ &= \int_{\widehat{\Gamma}} \bigl\langle f(\omega) \bigm| g(\omega)\bigr\rangle\, d\omega \\ &= \langle f \mid g\rangle. \end{aligned}$$

Therefore, the reverse martingale X_n converges almost everywhere to the constant $\langle f \mid g\rangle$.

For each ω, write N_ω for the set of all natural numbers n for which $m(\alpha^{*n}(\omega)) < \infty$. Since m is finite on a set of positive measure, the ergodicity of α^* implies that N_ω is infinite for almost all ω. We show next that, for each $n \in N_\omega$, there is a different expression for $X_n(\omega)$. To wit, for each $n \in N_\omega$, define $f_n = {S_H^*}^n(f)$ and $g_n = {S_H^*}^n(g)$. Since S_H is a unitary operator on R_∞, we have

$$\begin{aligned} X_n(\omega) &= \frac{1}{N^n} \sum_{\alpha^{*n}(\zeta)=1} \bigl\langle f(\omega\zeta) \bigm| g(\omega\zeta)\bigr\rangle \\ &= \frac{1}{N^n} \sum_{\zeta} \left\langle \prod_{k=0}^{n-1} H^t\bigl(\alpha^{*k}(\omega\zeta)\bigr) f_n\bigl(\alpha^{*n}(\omega)\bigr) \,\middle|\, \prod_{k'=0}^{n-1} H^t\bigl(\alpha^{*k'}(\omega\zeta)\bigr) g_n\bigl(\alpha^{*n}(\omega)\bigr) \right\rangle \\ &= \frac{1}{N^n} \sum_{\zeta}\sum_{j}\sum_{i} \left[\prod_{k=0}^{n-1} H^t\bigl(\alpha^{*k}(\omega\zeta)\bigr)\right]_{j,i} \overline{\sum_{i'} \left[\prod_{k'=0}^{n-1} H^t\bigl(\alpha^{*k'}(\omega\zeta)\bigr)\right]_{j,i'}} \\ &\quad \cdot f_{n\,i}\bigl(\alpha^{*n}(\omega)\bigr) \overline{g_{n\,i'}\bigl(\alpha^{*n}(\omega)\bigr)}. \end{aligned}$$

When interchanging the sums in the previous expression is justified, we may continue this computation; then, using Proposition 3, we would obtain

$$\begin{aligned} &\frac{1}{N^n} \sum_{i}\sum_{i'} f_{n\,i}\bigl(\alpha^{*n}(\omega)\bigr) \overline{g_{n\,i'}\bigl(\alpha^{*n}(\omega)\bigr)} \sum_{\zeta}\sum_{j} \left[\prod_{k=0}^{n-1} H^t\bigl(\alpha^{*k}(\omega\zeta)\bigr)\right]_{j,i} \overline{\left[\prod_{k'=0}^{n-1} H^t\bigl(\alpha^{*k'}(\omega\zeta)\bigr)\right]_{j,i'}} \\ &\quad = \sum_{i} f_{n\,i}\bigl(\alpha^{*n}(\omega)\bigr) \overline{g_{n\,i}\bigl(\alpha^{*n}(\omega)\bigr)} \\ &\quad = \bigl\langle f_n\bigl(\alpha^{*n}(\omega)\bigr) \bigm| g_n\bigl(\alpha^{*n}(\omega)\bigr)\bigr\rangle. \end{aligned}$$

This gives the different expression for $X_n(\omega)$ that we want, whenever we can justify the interchanges of sums in the previous computations:

$$X_n(\omega) = \langle f_n(\alpha^{*n}(\omega)) \mid g_n(\alpha^{*n}(\omega))\rangle. \tag{2}$$

The following calculation, which again uses Proposition 3 and the Cauchy–Schwarz inequality, shows that the interchange of sums above is justified whenever the sums on i and i' are finite sums. Because of Proposition 3, the sums on i and i' will be finite if $m(\alpha^{*n}(\omega)) < \infty$, and this is the case when $n \in N_\omega$. Hence, the computation below will complete the derivation of Eq. (2). Note also that the sums on i and i' will be finite sums if the vectors $f_n(\alpha^{*n}(\omega))$ and $g_n(\alpha^{*n}(\omega))$ only have a finite number of nonzero coordinates. We will use this later on.

$$\begin{aligned}
&\frac{1}{N^n}\sum_{i=1}^{c}\sum_{i'=1}^{c'}\sum_{\zeta}\sum_{j}\left|\left[\prod_{k=0}^{n-1}H^t\big(\alpha^{*k}(\omega\zeta)\big)\right]_{j,i}\overline{\left[\prod_{k'=0}^{n-1}H^t\big(\alpha^{*k'}(\omega\zeta)\big)\right]_{j,i'}}f_{n\,i}\big(\alpha^{*n}(\omega)\big)\overline{g_{n\,i'}\big(\alpha^{*n}(\omega)\big)}\right| \\
&\quad\leqslant \frac{1}{N^n}\sum_{i=1}^{c}\sum_{i'=1}^{c'}\left|f_{n\,i}\big(\alpha^{*n}(\omega)\big)g_{n\,i'}\big(\alpha^{*n}(\omega)\big)\right| \\
&\qquad\cdot\left(\sum_{\zeta,j}\left|\left[\prod_{k=0}^{n-1}H^t\big(\alpha^{*k}(\omega\zeta)\big)\right]_{j,i}\right|^2\sum_{\tilde\zeta,\tilde j}\left|\left[\prod_{k'=0}^{n-1}H^t\big(\alpha^{*k'}(\omega\tilde\zeta)\big)\right]_{\tilde j,i'}\right|^2\right)^{1/2} \\
&\quad= \sum_{i=1}^{c}\left|f_{n\,i}\big(\alpha^{*n}(\omega)\big)\chi_{\sigma_i}\big(\alpha^{*n}(\omega)\big)\right|\sum_{i'=1}^{c'}\left|g_{n\,i'}\big(\alpha^{*n}(\omega)\big)\chi_{\sigma_{i'}}\big(\alpha^{*n}(\omega)\big)\right| \\
&\quad\leqslant \left(\sum_{i=1}^{c}\left|f_{n\,i}\big(\alpha^{*n}(\omega)\big)\right|^2\sum_{\tilde i=1}^{c}\left|\chi_{\sigma_{\tilde i}}\big(\alpha^{*n}(\omega)\big)\right|^2\right)^{1/2}\left(\sum_{i'=1}^{c'}\left|g_{n\,i'}\big(\alpha^{*n}(\omega)\big)\right|^2\sum_{\tilde i'=1}^{c'}\left|\chi_{\sigma_{\tilde i'}}\big(\alpha^{*n}(\omega)\big)\right|^2\right)^{1/2} \\
&\quad\leqslant \sqrt{cc'}\,\big\|f_n\big(\alpha^{*n}(\omega)\big)\big\|\,\big\|g_n\big(\alpha^{*n}(\omega)\big)\big\| \\
&\quad< \infty,
\end{aligned}$$

for almost every ω.

The first conclusion we can draw from Eq. (2) is that for almost all ω,

$$\begin{aligned}
\lim_{n\in N_\omega}\langle f_n(\alpha^{*n}(\omega)) \mid g_n(\alpha^{*n}(\omega))\rangle &= \lim_{n\to\infty}X_n(\omega) \\
&= \langle f \mid g\rangle,
\end{aligned}$$

or, setting $g = f$, for f a unit vector in R_∞,

$$\lim_{n\in N_\omega}\big\|f_n\big(\alpha^{*n}(\omega)\big)\big\| = \|f\| = 1.$$

A second conclusion we may draw is that we must have $\sigma_1 = \widehat{\Gamma}$, i.e., $m(\omega) \geqslant 1$ a.e. Indeed, if $m(\omega) = 0$ for all ω in a set F of positive Haar measure, then from the ergodicity of α^*, we must have $\alpha^{*n}(\omega) \in F$ infinitely often for almost all ω, so that $\| f_n(\alpha^{*n}(\omega))\| = 0$ infinitely often. But, since each such integer n belongs to N_ω, this contradicts the first claim above.

Now let i_0 satisfy $\sigma_{i_0} = \widehat{\Gamma}$ and σ_{i_0+1} be a proper subset of $\widehat{\Gamma}$ of measure strictly less than 1. (Of course σ_{i_0+1} could be the empty set, if $m(\omega) \equiv i_0$.) Then, by a similar kind of ergodicity argument as was used above, we know that for almost all ω, and for infinitely many values of n, $[f(\alpha^{*n}(\omega))]_i = 0$ for all $i > i_0$ and all $f \in R_\infty$. Indeed, this is true whenever $\alpha^{*n}(\omega) \notin \sigma_{i_0+1}$, and this occurs infinitely often for almost all ω. Moreover, each such n belongs to N_ω.

Let $f^1, \ldots, f^k$ be orthonormal vectors in R_∞. Then, for infinitely many sufficiently large n, we must have that the k i_0-dimensional vectors

$$\bigl\{\bigl[f_n^p\bigl(\alpha^{*n}(\omega)\bigr)\bigr]_1, \ldots, \bigl[f_n^p\bigl(\alpha^{*n}(\omega)\bigr)\bigr]_{i_0}\bigr\}$$

are nearly orthogonal and nearly of unit length. Consequently, k must be $\leqslant i_0$. Hence R_∞ is finite-dimensional, and therefore S_H (a unitary operator on R_∞) must have an eigenvector.

To prove the final part of the proposition, let f be a unit vector in R_∞. From the second claim above, we know that the coordinates f_i of f are all 0 for $i > i_0$. Therefore, the interchanges of summations in the calculations above are justified, and we obtain

$$X_n^{f,f}(\omega) = \bigl\| f_n\bigl(\alpha^{*n}(\omega)\bigr)\bigr\|^2,$$

so that

$$\lim_{n\to\infty} \bigl\| f_n\bigl(\alpha^{*n}(\omega)\bigr)\bigr\|^2 = \|f\|^2 = 1$$

for almost all ω.

Finally, let f be a unit eigenvector for S_H. We have then that

$$\begin{aligned}\lim_{n\to\infty} \bigl\| f\bigl(\alpha^{*n}(\omega)\bigr)\bigr\|^2 &= \lim_{n\to\infty} \bigl\|\bigl[S_H^n(f_n)\bigr]\bigl(\alpha^{*n}(\omega)\bigr)\bigr\|^2 \\ &= \lim_{n\to\infty} \bigl\| f_n\bigl(\alpha^{*n}(\omega)\bigr)\bigr\|^2 \\ &= 1.\end{aligned}$$

By the ergodicity of α^*, it follows that $\|f(\omega)\| = 1$ almost everywhere. □

3. Pure isometries and the low pass condition

In this section, we use Theorem 4 to eliminate the need for a restrictive low-pass condition when building GMRAs from filters via the direct limit construction of [4] and [5]. First we recall the definition:

Definition 5. A collection $\{V_j\}_{-\infty}^{\infty}$ of closed subspaces of $\mathcal{H}$ is called a *g*eneralized multiresolution analysis (GMRA) relative to π and δ if:

(1) $V_j \subseteq V_{j+1}$ for all j.
(2) $V_{j+1} = \delta(V_j)$ for all j.

(3) $\bigcap V_j = \{0\}$, and $\bigcup V_j$ is dense in $\mathcal{H}$.
(4) V_0 is invariant under the representation π.

The subspace V_0 is called the *core subspace* of the GMRA $\{V_j\}$.

In order to use the theorems from the previous section to build GMRAs from filters, we need to know that associated Ruelle operators are isometries. The proof requires the additional assumption that the multiplicity function m is finite a.e. This hypothesis is standard in much of the literature.

Proposition 6. *Assume $m(\omega) < \infty$ for almost all ω, and let H be a filter relative to m and α^*. Then the Ruelle operator S_H is an isometry of $\bigoplus_i L^2(\sigma_i, \mu)$ into itself.*

Proof. Note that, because $m(\omega) < \infty$ almost everywhere, the filter equation, together with the very definition of $h_{i,j}$, implies that $h_{i,j}(\omega) = 0$ if $j > m(\omega)$ or $i > m(\alpha^*(\omega))$. Therefore, all the sums in the following calculation, that are inside integrals, are finite, so that interchanges of these sums is allowed.

$$\begin{aligned}
\|S_H(f)\|^2 &= \sum_j \int_{\sigma_j} \big|\big[S_H(f)\big]_j(\omega)\big|^2\, d\omega \\
&= \sum_j \int_{\sigma_j} \big|\big[H^t(\omega) f\big(\alpha^*(\omega)\big)\big]_j\big|^2\, d\omega \\
&= \sum_j \int_{\sigma_j} \Big|\sum_i H_{i,j}(\omega) f_i\big(\alpha^*(\omega)\big)\Big|^2\, d\omega \\
&= \sum_j \int_{\widehat{\Gamma}} \Big|\sum_i h_{i,j}(\omega) f_i\big(\alpha^*(\omega)\big)\Big|^2\, d\omega \\
&= \int_{\widehat{\Gamma}} \sum_j \Big[\sum_i h_{i,j}(\omega) f_i\big(\alpha^*(\omega)\big)\Big] \overline{\Big[\sum_{i'} h_{i',j}(\omega) f_{i'}\big(\alpha^*(\omega)\big)\Big]}\, d\omega \\
&= \frac{1}{N} \sum_{\alpha^*(\zeta)=1} \int_{\widehat{\Gamma}} \sum_j \Big[\sum_i h_{i,j}(\omega\zeta) f_i\big(\alpha^*(\omega)\big)\Big] \overline{\Big[\sum_{i'} h_{i',j}(\omega\zeta) f_{i'}\big(\alpha^*(\omega)\big)\Big]}\, d\omega \\
&= \frac{1}{N} \int_{\widehat{\Gamma}} \sum_i \sum_{i'} \sum_\zeta \sum_j h_{i,j}(\omega\zeta) \overline{h_{i',j}(\omega\zeta)} f_i\big(\alpha^*(\omega)\big) \overline{f_{i'}\big(\alpha^*(\omega)\big)}\, d\omega \\
&= \int_{\widehat{\Gamma}} \sum_i \chi_{\sigma_i}\big(\alpha^*(\omega)\big) \big|f_i\big(\alpha^*(\omega)\big)\big|^2\, d\omega \\
&= \sum_i \int_{\sigma_i} \big|f_i(\omega)\big|^2 \\
&= \|f\|^2,
\end{aligned}$$

as claimed. □

In earlier works (e.g., [2] and [4]), a so-called "low-pass condition" on the filter H was used to guarantee that S_H was a pure isometry. This condition had various forms, but they all required something like H being continuous at the identity 1 in $\widehat{\Gamma}$ and the matrix $H(1)$ being diagonal with $\sqrt{N}$'s at the top of the diagonal and 0's at the bottom. Results from [16] and more recently [9] loosened these assumptions somewhat by separating out a phase factor. We will show below that such assumptions on H imply that S_H can have no eigenvector, and so by Theorem 4, S_H must be a pure isometry. The following theorem gives quite general conditions on H under which S_H is a pure isometry, subsuming the conditions on $H(1)$ mentioned above as well as the results for a 1×1 filter H given in [5]. In particular, note that this theorem blurs the distinction between classical low-pass and high-pass filters by not requiring H to take on specific values near the identity.

Theorem 7. *Let H be a filter relative to m and α^*. Suppose there exists a positive number δ and a set $F \subseteq \widehat{\Gamma}$ of positive measure, such that for all $\omega \in F$ the matrix $H(\omega)$ is in block form*

$$H(\omega) = \begin{pmatrix} A(\omega) & B(\omega) \\ C(\omega) & D(\omega) \end{pmatrix},$$

where the four blocks satisfy the following:

(1) *$A(\omega)$ is an $a \times a$ expansive matrix with the property that $\|A(\omega)^{-1}\| \leqslant \frac{1}{1+\delta}$.*
(2) $\max(\|B(\omega)\|, \|C(\omega)\|, \|D(\omega)\|) < \epsilon = \min(\frac{1}{8}, \frac{\delta}{8})$. (*The norm here can either be the operator norm of a matrix or the Euclidean norm.*)

Finally, assume that $F \cap \alpha^*(F)$ also has positive measure. Then S_H is a pure isometry, i.e., S_H has no eigenvector.

Remark 8. The hypothesis of this theorem clearly covers the previously cited cases where $H(\omega)$ is continuous and has the relevant diagonal at $\omega = 1$.

Proof. Suppose, by way of contradiction, that f is a unit eigenvector for S_H with eigenvalue λ, $|\lambda| = 1$. For each ω, write the vector $f(\omega)$ in the form $f(\omega) = (f^1(\omega), f^2(\omega))$, where $f^1(\omega)$ is a-dimensional. Because f is an eigenvector for S_H, we have

$$\lambda f(\omega) = \big[S_H(f)\big](\omega) = H^t(\omega) f\big(\alpha^*(\omega)\big).$$

It follows from this, and the fact that $\|f(\omega)\| = 1$ by the final conclusion of Theorem 4, that for $\omega \in F$ we must have

$$\big\|f^2(\omega)\big\| = \big\|B^t(\omega) f^1\big(\alpha^*(\omega)\big) + D^t(\omega) f^2\big(\alpha^*(\omega)\big)\big\| < 2\epsilon.$$

Hence, again because $\|f(\omega)\| = 1$ for $\omega \in F$, we must have $\|f^1(\omega)\| > 1 - 2\epsilon$. Since condition (1) on the matrix $A(\omega)$ implies that $\|A(\omega)v\| \geqslant (1+\delta)\|v\|$ for every a-dimensional vector v, we must have, for ω and $\alpha^*(\omega)$ both in F,

$$\begin{aligned} 1 &\geqslant \big\|f^1(\omega)\big\| \\ &= \big\|A^t(\omega) f^1\big(\alpha^*(\omega)\big) + C^t(\omega) f^2\big(\alpha^*(\omega)\big)\big\| \end{aligned}$$

$$
\begin{aligned}
&> (1+\delta)\big\| f^1\big(\alpha^*(\omega)\big)\big\| - 2\epsilon^2 \\
&> (1+\delta)(1-2\epsilon) - 2\epsilon \\
&\geqslant 1+\delta - \frac{\delta}{4} - \frac{\delta}{4} - \frac{\delta}{4} \\
&= 1 + \frac{\delta}{4}.
\end{aligned}
$$

We have arrived at a contradiction, and the theorem is proved. □

Theorem 7 can be used to build generalized multiresolution analyses with more general filters, using approaches that do not require an infinite product construction, such as the direct limit construction in [4] and [5].

Example 9. For another application of Theorem 7, consider the Journé filter system given by

$$H(\omega) = \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix}.$$

In the classical Journé example described by Baggett, Courter, and Merrill in [1],

$$
\begin{aligned}
h_{1,1}\big(e^{2\pi i x}\big) &= \sqrt{2}\chi_{E_1}(x), \qquad h_{1,2} = 0, \\
h_{2,1}\big(e^{2\pi i x}\big) &= \sqrt{2}\chi_{E_2}(x), \qquad h_{2,2} = 0,
\end{aligned}
$$

where E_1 and E_2 are the periodizations of the sets $[-\frac{2}{7}, -\frac{1}{4}) \cup [-\frac{1}{7}, \frac{1}{7}) \cup [\frac{1}{4}, \frac{2}{7})$ and $[-\frac{1}{2}, -\frac{3}{7}) \cup [\frac{3}{7}, \frac{1}{2})$, respectively.

These are the classical filters in $L^2(\mathbb{R})$ for dilation by 2 that can be associated to the GMRA in $L^2(\mathbb{R})$ coming from the Journé wavelet.

We now apply a device very similar to that first used on pp. 259–260 of [3]. Choose a very small $\delta > 0$, and an even smaller $\varepsilon > 0$. Define $q : \mathbb{T} \to \mathbb{R}$ by

$$
q\big(e^{2\pi i x}\big) = \begin{cases}
\sqrt{2}\sqrt{1-r^2}, & \text{if } x = 0, \\
0, & \text{if } \frac{1}{7} - \varepsilon < x < \frac{3}{14} + \varepsilon, \\
C^\infty \text{ monotone decreasing}, & \text{if } 0 < x < \frac{1}{7} - \varepsilon, \\
\sqrt{2}, & \text{if } \frac{2}{7} - \varepsilon < x < \frac{5}{14} + \varepsilon, \\
C^\infty \text{ monotone increasing}, & \text{if } \frac{3}{14} + \varepsilon < x < \frac{2}{7} - \varepsilon, \\
0, & \text{if } \frac{3}{7} - \varepsilon < x < \frac{3}{7} + \varepsilon, \\
\sqrt{2} \cdot r, & \text{if } x = \frac{1}{2}, \\
C^\infty \text{ monotone increasing}, & \text{if } \frac{3}{7} + \varepsilon < x < \frac{1}{2}, \\
\sqrt{2 - [q(e^{2\pi i (x + \frac{1}{2})})]^2}, & \text{if } -\frac{1}{2} < x < 0.
\end{cases}
$$

Here $r \in (0, 1)$ is a number yet to be determined. We now define generalized filters $\{h_{i,j}^q\}$ by:

$$h^q_{1,1}\left(e^{2\pi ix}\right) = q\left(e^{2\pi ix}\right)\chi_{[-\frac{2}{7},\frac{2}{7})}(x),$$

$$h^q_{2,1}\left(e^{2\pi ix}\right) = \sqrt{2}\chi_{[-\frac{1}{2},-\frac{3}{7})\cup[\frac{3}{7},\frac{1}{2})}(x),$$

$$h^q_{1,2}\left(e^{2\pi ix}\right) = q\left(e^{2\pi i(x+\frac{1}{2})}\right)\chi_{[-\frac{1}{7},\frac{1}{7})}(x),$$

$$h^q_{2,2}\left(e^{2\pi ix}\right) = 0.$$

Denote the matrix $[h^q_{i,j}(z)]$ by H^q. A routine calculation shows that the filter equations of Baggett, Courter and Merrill are satisfied, i.e.

$$\sum_{j=1}^{2}\sum_{k=0}^{1} h_{i,j}\left(e^{2\pi i\frac{x+k}{2}}\right)\overline{h_{i',j}\left(e^{2\pi i\frac{x+k}{2}}\right)} = \delta_{i,i'}2\chi_{\sigma_i}(x), \quad i = 1, 2,$$

where $\sigma_1 = [-\frac{1}{2}, -\frac{3}{7}) \cup [-\frac{2}{7}, \frac{2}{7}) \cup [-\frac{3}{7}, \frac{1}{2})$, and $\sigma_2 = [-\frac{1}{7}, \frac{1}{7})$.

We now take $A(z) = h^q_{1,1}(z)$, $B(z) = h^q_{1,2}(z)$, $C(z) = h^q_{2,1}(z)$, and $D(z) = h^q_{2,2}(z)$ in the matrix $H = H^q$. We want to determine a specific value $r \in (0, 1)$ and a set $F = \{e^{2\pi ix}\colon x \in \mathcal{F}\}$, where $\mathcal{F} \subset [-\frac{1}{2}, \frac{1}{2})$ such that the hypotheses of Theorem 7 are satisfied. We let $\mathcal{F} = [-\frac{1}{n}, \frac{1}{n}]$, where $n \in \mathbb{N}$ is chosen so that $n \geqslant 7$ and $q(e^{2\pi ix}) > \sqrt{2}\sqrt{1-2r^2}$, for all $x \in \mathcal{F}$. This can be done by applications of the Intermediate Value Theorem, since q is continuous. It's clear that $F \cap \alpha^*(F) = F$ has positive measure. Also we want to find $\delta > 0$ such that $|h^q_{1,1}(e^{2\pi ix})| \geqslant 1 + \delta$ and $\max(|h^q_{1,2}(e^{2\pi ix})|, |h^q_{2,1}(e^{2\pi ix})|, |h^q_{2,2}(e^{2\pi ix})|) < \epsilon = \min(\frac{1}{8}, \frac{\delta}{8})$, $\forall x \in \mathcal{F}$. Note that $h^q_{2,1}(e^{2\pi ix})$ and $h^q_{2,2}(e^{2\pi ix})$ are identically 0 on $\mathcal{F}$, so we need only show that $|h^q_{1,2}(e^{2\pi ix})| < \epsilon$ on $\mathcal{F}$. Since $h^q_{1,2}(e^{2\pi ix})$ is continuous at $x = 0$, where its value is equal to $\sqrt{2} \cdot r$, we choose $\mathcal{F} = [-\frac{1}{n}, \frac{1}{n}]$ so that $\sqrt{2} \cdot r \leqslant h^q_{1,2}(e^{2\pi ix}) < 2 \cdot r$, $\forall x \in \mathcal{F}$.

Having chosen $\delta > 0$, we thus must choose r so that $\frac{1}{\sqrt{2}\sqrt{1-2r^2}} \leqslant \frac{1}{1+\delta}$ and $2r < \epsilon = \min(\frac{1}{8}, \frac{\delta}{8})$. So we first choose $r_1 < \min(\frac{1}{16}, \frac{\delta}{16})$.

For r_2, as long as $\delta < \sqrt{2} - 1$, if we choose $r_2 \leqslant \sqrt{\frac{\sqrt{2}-(1+\delta)}{1+\delta}}$, one can verify that

$$\frac{1}{\sqrt{2}\sqrt{1-2r_2^2}} \leqslant \frac{1}{1+\delta}.$$

Finally, we choose $r = \min(r_1, r_2)$. Then,

$$2r < \min\left(\frac{1}{8}, \frac{\delta}{8}\right),$$

and

$$\frac{1}{\sqrt{2}\sqrt{1-2r^2}} \leqslant \frac{1}{1+\delta},$$

so that the conditions of Theorem 7 are satisfied, and S_H is a pure isometry acting on $L^2(\sigma_1) \oplus L^2(\sigma_2)$. In fact, letting $r \to 0+$, we can construct a one-parameter family of filter systems giving rise to pure isometries; when $r = 0$, we obtain exactly the filter system constructed in [3].

In Theorem 5 of [4], it is shown that given a pure isometry S on a Hilbert space $\mathcal{K}$ together with a representation ρ of a countable abelian group Γ, such that $\delta^{-1}\rho_\gamma\delta = \rho_{\alpha(\gamma)}$ for all $\gamma \in \Gamma$, then it was possible to construct a generalized multiresolution analysis via a direct limit process. Taking $S = S_H$, $\mathcal{K} = L^2(\sigma_1) \oplus L^2(\sigma_2)$, and $\Gamma = \mathbb{Z}$, the desired hypotheses will be satisfied, and it follows that a GMRA can be constructed from the above filter system in the direct limit Hilbert space. In a paper in preparation, the authors will present a more constructive approach to making the GMRA under the same hypotheses as in Theorem 5 of [4].

4. Pure isometries and the trivial intersection property

The following "problem" was first noticed by Baggett, Bownik and Rzeszotnik. Suppose $\{\psi_k\}$ is a Parseval multiwavelet in $L^2(\mathbb{R}^d)$; i.e., the functions $\{\psi_{j,n,k}(x)\} \equiv \{2^{\frac{jd}{2}}\psi_k(2^j x + n)\}$ form a Parseval frame for all of $L^2(\mathbb{R}^d)$. If V_j is defined to be the closed linear span of the functions $\{\psi_{l,n,k}\}$ for $l < j$, then these subspaces can be shown to satisfy all of the properties of a GMRA except for the condition $\bigcap V_j = \{0\}$. Bownik and Rzeszotnik demonstrated the delicacy of this condition in [8] by constructing, for any $\delta > 0$, a frame wavelet in $L^2(\mathbb{R})$, with frame bounds of 1 and $1 + \delta$, that has a negative dilate space V_0 equal to all of $L^2(\mathbb{R})$. They showed in [9], however, that a Parseval multiwavelet in $L^2(\mathbb{R}^d)$ generates a GMRA (that is, the trivial intersection property does hold) whenever the multiplicity function of the negative dilate space V_0 is finite on a set of positive measure. In fact, Bownik proved in [7] that the condition that m is not identically ∞ a.e. implies $\bigcap_{j=1}^{\infty} D_j(V_0) = \{0\}$ in the more general setting where $D_j f(x) = f(A_j x)$ for a sequence $\{A_j\}$ of invertible $n \times n$ real matrices that satisfy $\|A_j\| \to 0$ as $j \to \infty$. For a history of the intersection problem in $L^2(\mathbb{R}^d)$, see [6].

This question about subspaces of $L^2(\mathbb{R}^d)$ obviously generalizes to a collection $\{V_j\}$ of subspaces of a Hilbert space that satisfy all the conditions for a GMRA except the trivial intersection condition. Below, we apply the results from Section 2 to show that this intersection is $\{0\}$ if certain extra assumptions hold. In doing so, we extend some of the results mentioned in the previous paragraph.

Let Γ, α, π, and δ be as in the previous sections. We recall some implications of Stone's Theorem, whereby certain GMRAs give rise to an associated filter. Let $\{V_j\}$ be a GMRA in a Hilbert space $\mathcal{H}$, relative to the representation π and the operator δ. Then, according to Stone's Theorem on unitary representations of abelian groups, there exists a finite, Borel measure μ (unique up to equivalence of measures) on $\widehat{\Gamma}$, unique (up to sets of μ measure 0) Borel subsets $\sigma_1 \supseteq \sigma_2 \supseteq \cdots$ of $\widehat{\Gamma}$, and a (not necessarily unique) unitary operator $J : V_0 \to \bigoplus_i L^2(\sigma_i, \mu)$ satisfying

$$\big[J\big(\pi_\gamma(f)\big)\big](\omega) = \omega(\gamma)\big[J(f)\big](\omega)$$

for all $\gamma \in \Gamma$, all $f \in V_0$, and μ almost all $\omega \in \widehat{\Gamma}$. In this paper, we assume the measure μ is absolutely continuous with respect to Haar measure, in which case we may assume that μ is the restriction of Haar measure to the subset σ_1.

Write C_i for the element of the direct sum space $\bigoplus_j L^2(\sigma_j, \mu)$ whose ith coordinate is χ_{σ_i} and whose other coordinates are 0. Write $\bigoplus_j h_{i,j}$ for the element $J(\delta^{-1}(J^{-1}(C_i)))$. The following theorem displays a connection between GMRA structures and filters and will allow us to apply Proposition 3 and Theorem 4.

Theorem 10. *Let the functions $\{h_{i,j}\}$ be as in the preceding paragraph. Then the matrix $H = [h_{i,j}]$ is a filter relative to m and α^*. Moreover, the operator $J \circ \delta^{-1} \circ J^{-1}$ on $\bigoplus_i L^2(\sigma_i, \mu)$ is the corresponding Ruelle operator S_H:*

$$\big[J \circ \delta^{-1} \circ J^{-1}(f)\big](\omega) = H^t(\omega) f\big(\alpha^*(\omega)\big).$$

Proof. By definition we have

$$\begin{aligned}\mu(\sigma_i) &= \|C_i\|^2 \\ &= \big\| J\big(\delta^{-1}\big(J^{-1}(C_i)\big)\big)\big\|^2 \\ &= \sum_j \int_{\widehat{\Gamma}} \big|h_{i,j}(\omega)\big|^2 \, d\omega,\end{aligned}$$

which implies that $\sum_j |h_{i,j}(\omega)|^2$ is finite for almost all ω. Write

$$F_{i,i'}(\omega) = \sum_{\alpha^*(\zeta)=1} \sum_j h_{i,j}(\omega\zeta)\overline{h_{i',j}(\omega\zeta)},$$

and note, by the Cauchy–Schwarz inequality, that $F_{i,i'} \in L^1(\mu)$, and that the Fourier coefficient $c_\gamma(F_{i,i'}) = 0$ unless γ belongs to the range of α. We have that

$$\begin{aligned}c_{\alpha(\gamma)}(F_{i,i'}) &= \int_{\widehat{\Gamma}} F_{i,i'}(\omega)\omega\big(-\alpha(\gamma)\big)\, d\mu(\omega) \\ &= \sum_\zeta \int_{\widehat{\Gamma}} \sum_j h_{i,j}(\omega\zeta)\overline{h_{i',j}(\omega\zeta)}\omega\big(-\alpha(\gamma)\big)\, d\omega \\ &= N \sum_j \int_{\widehat{\Gamma}} h_{i,j}(\omega)\overline{h_{i',j}(\omega)}\omega\big(-\alpha(\gamma)\big)\, d\omega \\ &= N\big\langle J\big(\delta^{-1}\big(J^{-1}(C_i)\big)\big) \bigm| J\big(\pi_{\alpha(\gamma)}\big(\delta^{-1}\big(J^{-1}(C_{i'})\big)\big)\big)\big\rangle \\ &= N\big\langle J^{-1}(C_i) \bigm| \pi_\gamma\big(J^{-1}(C_{i'})\big)\big\rangle \\ &= N\langle C_i \mid \gamma C_{i'}\rangle \\ &= N\delta_{i,i'}\langle C_i \mid \gamma C_{i'}\rangle \\ &= N\delta_{i,i'} \int_{\widehat{\Gamma}} \chi_{\sigma_i}(\omega)\omega(-\gamma)\, d\omega \\ &= N\delta_{i,i'} \int_{\widehat{\Gamma}} \chi_{\sigma_i}\big(\alpha^*(\omega)\big)\alpha^*(\omega)(-\gamma)\, d\omega,\end{aligned}$$

showing that the two L^1 functions $F_{i,i'}(\omega)$ and $N\delta_{i,i'}\chi_{\sigma_i}(\alpha^*(\omega))$ have the same Fourier coefficients, and hence are equal almost everywhere. This verifies Eq. (1). It follows from the filter equation that $h_{i,j}$ is supported on $\alpha^{*-1}(\sigma_i)$. That is, $h_{i,j}(\omega) = 0$ unless both $\omega \in \sigma_j$ and $\alpha^*(\omega) \in \sigma_i$.

Next, for any γ, we have

$$\begin{aligned}
\big[J\big(\delta^{-1}\big(J^{-1}(\gamma C_i)\big)\big)\big](\omega) &= \big[J\big(\delta^{-1}\big(\pi_\gamma\big(J^{-1}(C_i)\big)\big)\big)\big](\omega) \\
&= \big[J\big(\pi_{\alpha(\gamma)}\big(\delta^{-1}\big(J^{-1}(C_i)\big)\big)\big)\big](\omega) \\
&= \omega\big(\alpha(\gamma)\big) \bigoplus_j h_{i,j}(\omega) \\
&= \omega\big(\alpha(\gamma)\big) \chi_{\sigma_i}\big(\alpha^*(\omega)\big) \bigoplus_j h_{i,j}(\omega) \\
&= H^t(\omega)[\gamma C_i]\big(\alpha^*(\omega)\big).
\end{aligned}$$

Then, by the Stone–Weierstrass Theorem, we must have

$$\big[J\big(\delta^{-1}\big(J^{-1}(f C_i)\big)\big)\big](\omega) = H^t(\omega)[f C_i]\big(\alpha^*(\omega)\big)$$

for every continuous function f on $\widehat{\Gamma}$. Then, by standard integration methods, this equality holds for all L^2 functions f. Finally, if $F = \bigoplus_i f_i$, then

$$\begin{aligned}
\big[J\big(\delta^{-1}\big(J^{-1}(F)\big)\big)\big](\omega) &= \left[J\left(\delta^{-1}\left(J^{-1}\left(\sum_i f_i C_i\right)\right)\right)\right](\omega) \\
&= \sum_i \big[J\big(\delta^{-1}\big(J^{-1}(f_i C_i)\big)\big)\big](\omega) \\
&= \sum_i H^t(\omega) f_i\big(\alpha^*(\omega)\big) C_i\big(\alpha^*(\omega)\big) \\
&= H^t(\omega) F\big(\alpha^*(\omega)\big) \\
&= \big[S_H(F)\big](\omega),
\end{aligned}$$

proving the second assertion. □

Remark 11. The preceding proof works in a more general setting. That is, we do not use all of the GMRA structure, particularly the property that $\bigcap V_j = \{0\}$. In particular, $\bigcap V_j = \{0\}$ if and only if $S_H = J \circ \delta^{-1} \circ J^{-1}$ is a pure isometry.

We introduce two more groups. Let D be the direct limit group determined by Γ and the monomorphism α of Γ into itself. (See for example [13].) For clarity, we make this construction explicit as follows.

Let $\widetilde{\Gamma}$ be the set of all pairs (γ, j) for $\gamma \in \Gamma$ and j a nonnegative integer. Define an equivalence relation on $\widetilde{\Gamma}$ by $(\gamma, k) \equiv (\gamma', k')$ if and only if $\alpha^{k'}(\gamma) = \alpha^k(\gamma')$, and let D be the set of equivalence classes $[\gamma, k]$ of this relation. Define addition on D by

$$[\gamma_1, k_1] + [\gamma_2, k_2] = \big[\alpha^{k_2}(\gamma_1) + \alpha^{k_1}(\gamma_2), k_1 + k_2\big].$$

One verifies directly that this addition is well defined, and that D is an abelian group.

Next, define a map $\widetilde{\alpha}$ on D by $\widetilde{\alpha}([\gamma, k]) = [\alpha(\gamma), k]$. Again, one verifies directly that $\widetilde{\alpha}$ is well defined and that it is an isomorphism of D onto itself. Indeed, the inverse $\widetilde{\alpha}^{-1}$ is given by $\widetilde{\alpha}^{-1}([\gamma, k]) = [\gamma, k+1]$.

Define G to be the semidirect product $D \rtimes \mathbb{Z}$, where the integer j acts on the element d by $j \cdot d = \widetilde{\alpha}^j(d)$. Explicitly, the multiplication in G is given by

$$(d_1, j_1) \times (d_2, j_2) = \bigl(d_1 + \widetilde{\alpha}^{j_1}(d_2), j_1 + j_2\bigr).$$

As before, π is a unitary representation of Γ, acting in a Hilbert space $\mathcal{H}$, and δ a unitary operator on $\mathcal{H}$ for which

$$\delta^{-1}\pi_\gamma \delta = \pi_{\alpha(\gamma)}$$

for all $\gamma \in \Gamma$. Define a representation $\widetilde{\pi}$ on G by

$$\widetilde{\pi}_{(d,j)} = \widetilde{\pi}_{([\gamma,k],j)} = \delta^k \pi_\gamma \delta^{-k-j}.$$

One verifies directly that this is a representation of G. Note also that $\widetilde{\pi}_{\widetilde{\alpha}(d)} = \delta^{-1}\widetilde{\pi}_d\delta$ and $\widetilde{\pi}_{(d,j)} = \widetilde{\pi}_d\delta^{-j}$.

Finally, for $|\lambda| = 1$, define the (irreducible) unitary representation P^λ of G acting in the Hilbert space $l^2(D)$ by

$$\bigl[P^\lambda_{(d,j)}(f)\bigr](d') = \lambda^j f\bigl(\widetilde{\alpha}^{-j}(d' - d)\bigr).$$

Remark 12. The representation P^λ is equivalent to the induced representation $\mathrm{Ind}_{\mathbb{Z}}^G \chi_\lambda$, where χ_λ is the character of the subgroup $\mathbb{Z}$ determined by λ.

Theorem 13. *Suppose $\{V_j\}$ is a collection of closed subspaces of $\mathcal{H}$ that satisfy all the conditions for a GMRA, relative to π and δ, except possibly the condition that $\bigcap V_j = \{0\}$. Assume that the measure μ associated to the representation π restricted to V_0 is Haar measure, and that the multiplicity function m is finite on a set of positive measure. Then the following conditions are equivalent*:

(1) $\bigcap V_j \neq \{0\}$.
(2) *δ has an eigenvector.*
(3) *The representation $\widetilde{\pi}$ of G contains a subrepresentation equivalent to the representation P^λ for some $|\lambda| = 1$.*

Proof. We define the functions $\{h_{i,j}\}$ as in Theorem 10 and reiterate that the matrix-valued function H is a filter relative to the space V_0 and that the operator $J \circ \delta^{-1} \circ J^{-1}$ is the Ruelle operator. Then S_H is a composition of isometries, and thus clearly an isometry from $\bigoplus_j L^2(\sigma_j, \mu)$ into itself.

Assume (1), and thus that S_H is not pure. From Theorem 4, we know that S_H has a unit eigenvector f: $S_H(f) = \lambda f$, and $|\lambda| = 1$. (We are using here the hypothesis that $m(\omega) < \infty$ on a set of positive measure.) For such an eigenfunction f, $v = J^{-1}(f)$ is an eigenvector for δ^{-1}. This proves (1) implies (2).

Assume (2), and let v be a unit eigenvector for δ with eigenvalue λ. Because π is equivalent to a subrepresentation of some multiple of the regular representation of Γ, we must have, from the Riemann–Lebesgue Lemma, that the function $\langle \pi_\gamma(w) \mid w\rangle$ vanishes at infinity on Γ for every $w \in \mathcal{H}$. But, for each γ, we have

$$\begin{aligned}\left|\left\langle \pi_{\alpha^j(\gamma)}(v) \mid v\right\rangle\right| &= \left|\left\langle \delta^{-j}\pi_\gamma\delta^j(v) \mid v\right\rangle\right| \\ &= \left|\left\langle \pi_\gamma(v) \mid v\right\rangle\right|,\end{aligned}$$

implying then that $\langle \pi_\gamma(v) \mid v\rangle = 0$ for all $\gamma \neq 0$, or equivalently that $\langle \pi_\gamma(v) \mid \pi_{\gamma'}(v)\rangle = 0$ unless $\gamma = \gamma'$. But now, for $d = [\gamma, k] \in D$, we have

$$\begin{aligned}\left\langle \widetilde{\pi}_d(v) \mid v\right\rangle &= \left\langle \widetilde{\pi}_{[\gamma,k]}(v) \mid v\right\rangle \\ &= \left\langle \delta^k\pi_\gamma\delta^{-k}(v) \mid v\right\rangle \\ &= \left\langle \pi_\gamma(v) \mid v\right\rangle \\ &= 0\end{aligned}$$

unless $\gamma = 0$. It follows then that the vectors $\{\widetilde{\pi}_d(v)\}$ form an orthonormal set. The span X of these vectors is obviously an invariant subspace for the representation $\widetilde{\pi}$ of G. Moreover, we claim that the restriction of $\widetilde{\pi}$ to X is equivalent to the representation P^λ. Thus, let U be the unitary operator from X onto $l^2(D)$ that sends the basis vector $\widetilde{\pi}_d(v)$ to the point mass basis vector ϵ_d in $l^2(D)$. We have

$$\begin{aligned}U\big(\widetilde{\pi}_{(d,j)}\big(\widetilde{\pi}_{d'}(v)\big)\big) &= U\big(\widetilde{\pi}_d\delta^{-j}\widetilde{\pi}_{d'}(v)\big) \\ &= U\big(\widetilde{\pi}_d\widetilde{\pi}_{\widetilde{\alpha}^j(d')}\delta^{-j}(v)\big) \\ &= \lambda^{-j}U\big(\widetilde{\pi}_{d+\widetilde{\alpha}^j(d')}(v)\big) \\ &= \lambda^{-j}\epsilon_{d+\widetilde{\alpha}^j(d')} \\ &= P^{\bar{\lambda}}_{(d,j)}(\epsilon_{d'}) \\ &= P^{\bar{\lambda}}_{(d,j)}\big(U\big(\widetilde{\pi}_{d'}(v)\big)\big),\end{aligned}$$

showing the equivalence of the restriction of $\widetilde{\pi}$ to the subspace X and the representation $P^{\bar{\lambda}}$. This proves (2) implies (3).

Assume (3). Because the operator $\widetilde{\pi}_{(0,1)}$ is δ^{-1}, it follows that there is a nonzero vector $v \in \mathcal{H}$ for which $\delta^{-1}(v) = \widetilde{\pi}_{(0,1)}(v) = \lambda v$. (Just note that $P^\lambda_{(0,1)}$ has an eigenvector with eigenvalue λ.) This shows (3) implies (2).

Finally, assume (2), and let v be an eigenvector for δ with eigenvalue λ. Write W_j for the orthogonal complement of V_j in V_{j+1}. Then

$$\mathcal{H} = \bigoplus_{j=-\infty}^{\infty} W_j \oplus R_\infty,$$

where $R_\infty = \bigcap V_j$, so we may write

$$v = \sum_{j=-\infty}^{\infty} v_j + v_\infty,$$

where v_j is the projection of v onto the subspace W_j, and v_∞ is the projection of v onto the subspace R_∞. Applying the operator δ gives

$$\sum_j v_j + v_\infty = v = \bar{\lambda} \sum_j \delta(v_j) + \bar{\lambda}\delta(v_\infty),$$

implying that $v_{j+1} = \bar{\lambda}\delta(v_j)$, whence $\|v_{j+1}\| = \|v_j\|$ for all j. Therefore $v_j = 0$ for all j, and hence $v = v_\infty$. So, $R_\infty \neq \{0\}$, and (2) implies (1). (This part of the proof uses neither the hypothesis on m nor the one on π.) This completes the proof of the theorem. □

Remark 14. We note that the proofs of Theorems 4 and 13 imply that δ has an eigenvector whenever $\bigcap V_j$ is nontrivial but finite-dimensional. Therefore, if δ is known to have no eigenvectors, $\bigcap V_j$ must be infinite-dimensional whenever it is nontrivial.

Example 15. Let $\mathcal{H} = L^2(\mathbb{R})$, let $\Gamma = \mathbb{Z}$ and π be the representation of Γ determined by translation. Let $\alpha(k) = 2k$, and define δ on $\mathcal{H}$ by $[\delta(f)](x) = \sqrt{2} f(2x)$. Then $\delta^{-1}\pi_k\delta = \pi_{2k}$. Set V_j equal to the subspace of $\mathcal{H}$ comprising those functions f whose Fourier transform is supported in the interval $(-\infty, 2^j)$. Then the V_j's satisfy all the conditions for a GMRA relative to π and δ except the trivial intersection condition. Indeed, the intersection is nontrivial, because $\bigcap V_j$ is the subspace of functions whose Fourier transform is supported in the interval $(-\infty, 0]$. The subspace V_0 comprises the functions whose transform is supported in the interval $(-\infty, 1)$, and it follows that the multiplicity function associated to the restriction of π to this subspace is infinite everywhere. Hence, since δ has no eigenvector, we see that we cannot drop the hypothesis that $m < \infty$ on a set of positive measure from the theorem above.

Example 16. Now let $\Gamma = \mathbb{Z}^2$, let $\alpha(n,k) = (2n, 2k)$, let $\mathcal{H} = L^2(\mathbb{R}^2)$, let π be the representation of Γ on $\mathcal{H}$ determined by translation, and let $[\delta(f)](x,y) = 2f(2x,2y)$. Then, $\delta^{-1}\pi_\gamma\delta = \pi_{\alpha(\gamma)}$. Let V_j be the subspace of $\mathcal{H}$ comprising the functions whose Fourier transform is supported in the rectangle $(-\infty, 2^j) \times (-2^j, 2^j)$. Then, the multiplicity function associated to the restriction of π to V_0 is infinite everywhere, but this time $\bigcap V_j = \{0\}$. Indeed, if $f \in \bigcap V_j$, then the support of the Fourier transform of f is supported on the negative x-axis, and such a function is the 0 vector in $L^2(\mathbb{R}^2)$. Hence, the finiteness assumption of Theorem 13 on m is not necessary for the intersection to be trivial.

The following example shows that for a Hilbert space $\mathcal{H} \neq L^2(\mathbb{R}^d)$, a collection of subspaces $\{V_j\}$ can satisfy all of the properties of a GMRA except the trivial intersection property, even though the multiplicity function is finite almost everywhere. As Theorem 13 shows, this is made possible by the presence of an eigenvector for the dilation δ. The implication (3) implies (2) in Theorem 13 suggests a method of construction of such examples.

Example 17. Let $\mathcal{H} = l^2(D)$, where D is the group of dyadic rationals. The group $\Gamma = \mathbb{Z}$ acts on this space by $\pi_\gamma(f)(x) = f(x-\gamma)$, and if we define $\alpha(\gamma) = 2\gamma$ and $\delta f(x) = f(2x)$, we see that $\delta^{-1}\pi_\gamma\delta = \pi_{\alpha(\gamma)}$. Note that $f = \chi_0$ is a fixed vector for this dilation. Now take V_0 to be the subspace $l^2(\mathbb{Z})$, and $V_j = \delta^j V_0$. Since π is the regular representation of Γ on V_0, the multiplicity function m is identically 1. We see that the fixed vector χ_0 is a nonzero element of the intersection of the V_j.

Finally, to clarify how Theorem 13 fits in with previous results about the intersection problem, let $\mathcal{H}$, π, and δ be as in the beginning of this section. If $\{\psi_i\}$ is a set of vectors in $\mathcal{H}$ for which the collection $\{\delta^j(\pi_\gamma(\psi_i))\}$ forms a frame for $\mathcal{H}$, then δ can have no eigenvector. Indeed, if v were an eigenvector for δ, then the numbers $|\langle v \mid \delta^j(\pi_\gamma(\psi_i))\rangle|$ are constant independent of j. Hence, they must all be 0, contradicting the frame assumption. Therefore, in the original context of the trivial intersection problem, i.e., where the subspaces $V_j = \overline{\operatorname{span}}\{\delta^k(\pi_\gamma(\psi_i))\colon\ k < j,\ \gamma,\ i\}$ are constructed from a Parseval wavelet $\{\psi_i\}$, there is no eigenvector for δ. Hence, if the intersection is nontrivial, then it must be infinite-dimensional, and the multiplicity function m is infinite almost everywhere. Theorem 13 therefore extends Theorem 6.1 of [9] from the classical case of $\mathcal{H} = L^2(\mathbb{R}^d)$ to an abstract Hilbert space.

In the case when $\mathcal{H} = L^2(\mathbb{R}^d)$, the group $\Gamma = \mathbb{Z}^d$ acts by translation, and $\delta(f) = \delta_A(f) = |\det A|^{1/2} f(A\cdot)$ for an expansive integer matrix A, we need not require the subspaces $\{V_j\}$ to be constructed from a Parseval wavelet, as in the previous paragraph. For any bounded neighborhood $E \subseteq \mathbb{R}^d$ of 0, there exists a positive integer n such that $E \subset A^n E$. Since

$$\int\limits_E \left|f(x)\right|^2 dx = \int\limits_{A^k E} \left|f(x)\right|^2 dx$$

for any f that satisfies $\bar{\lambda} f(x) = |\det A|^{1/2} f(Ax)$, we see that δ_A can have no eigenvector, and, by Theorem 13, the assumption that m is not identically ∞ a.e. implies the trivial intersection property. Thus, Theorem 13 provides an alternative proof for the classical scenario of Theorem 1.1 of [7].

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