Integration in Finite Terms with Special Functions: the Error Function†

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A decision procedure for integrating a class of transcendental elementary functions in terms of elementary functions and error functions is described. The procedure consists of three mutually exclusive cases. In the first two cases a generalised procedure for completing squares is used to limit the error functions which can appear in the integral to a finite number. This reduces the problem to the solution of a differential equation and we use a result of Risch (1969) to solve it. The third case can be reduced to the determination of what we have termed E-decompositions. The result presented here is the key procedure to a more general algorithm which is described fully in Cherry (1983).

1. Introduction

In 1969 Risch published a decision procedure for determining whether an arbitrary element of a transcendental elementary field possesses an elementary integral. Since then there has been an interest in extending this result by expanding either the class of integrands or the class of allowable integrals. In the first case there is the work of Davenport (1981) and Trager (1979, 1984) on the integration of algebraic functions and the decision procedure by Mack (1976) which leads to an algorithm for integrating elements of finitely generated regular liouvillian fields† (see the appendix of Singer et al. (1983)). The second case is explored in Cherry (1983) where decision procedures are given for integrating transcendental elementary functions in terms of elementary functions and various special functions. In this paper we present a procedure for integrating a class of elementary functions in terms of elementary functions and error functions:§

$$\text{erf}(u) = \int u' e^{-u^2} \, dx.$$  

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‡ We shall assume the reader is familiar with the basic terminology of differential algebra (cf. Ritt, 1948; Kaplansky, 1957).

§ The usual error function differs from our definition by a multiplicative constant:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$$  

(Bateman, 1953). We shall ignore the constant $\frac{2}{\sqrt{\pi}}$ throughout the paper.

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Examples of integrations in terms of error functions are

\[ \int e^{\log^2(x)} \, dx = e^{a} \, i \, \text{erf} \left( i \, \log \left( x \right) + \frac{1}{2} \right) \]

and

\[ \int \sin \left( x^2 \right) \, dx = \left( \frac{\sqrt{2i + \sqrt{2}}}{4} \right) \, \text{erf} \left( \frac{\sqrt{2i + \sqrt{2}}}{2} \, x \right) + \left( \frac{\sqrt{2i - \sqrt{2}}}{4} \right) \, \text{erf} \left( \frac{\sqrt{2i - \sqrt{2}}}{2} \, x \right). \]

The result we present here is the key procedure to a more general algorithm for integrating elements of reduced transcendental elementary fields in terms of elementary functions and error functions. For a definition of reduced elementary fields and a statement of the more general result, see the Appendix. The complete work can be found in Cherry (1983).

Although the idea of writing integrals in terms of various special functions has been mentioned frequently in the literature (Moses, 1971, 1972; Norman & Davenport, 1979), the techniques employed in existing algebra systems rely entirely on heuristics and are quite limited. To begin designing decision procedures, however, requires a theory concerning the structure of the resulting integrals. The Risch decision procedure is based on Liouville’s theorem on integration in finite terms (cf. Ritt, 1948; Rosenlicht, 1976). Roughly stated this theorem says that an element, \( \gamma \), of a differential field, \( F \), will have an integral in an elementary extension of \( F \) if and only if \( \gamma \) can be written in the form

\[ \gamma = w_0 + \sum c_i w_i, \]

where \( w_0 \) is in \( F \), the \( c_i \) are algebraic constants, and the \( w_i \), \( 1 \leq i \), are in \( F(\xi_1, \ldots, \xi_n) \), where each \( \xi_j \) is an algebraic constant. (By algebraic constant we mean algebraic over the constant field of \( F \).) A special function extension of Liouville's theorem was published in Moses & Zippel (1979) and recently a more general result was announced in Singer et al. (1981). The decision procedure presented here is based on the Singer et al. (1981) (or Singer et al. (1983)) result. In fact, our theorem is a continuation of the work presented in part II of Singer et al. (1983) where a decision procedure is presented for integrating expressions involving only exponential functions in terms of error functions.

The paper is arranged as follows: section 2 contains preliminary definitions, a statement of the Liouville theorem, upon which our main theorem is based, and a statement of the main theorem. In sections 3 and 4 two problems, which are encountered in the proof of the main theorem, are examined. Section 5 contains a proof of the main theorem and section 6 concludes the paper with some examples.

### 2. Preliminaries and Statement of the Main Theorem

Let \( F \) be a differential field of characteristic zero with derivation ‘ and constants \( C \). We say that a differential extension \( E \) of \( F \) is a erf-elementary extension of \( F \) if \( F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = E \) such that \( F_i = F_{i-1}(\theta_i) \), where for each \( i \), \( 1 \leq i \leq n \), one of the following holds:

(i) \( \theta_i \) is algebraic over \( F_{i-1} \).
(ii) \( \theta_i = u \, \theta_i \) for some \( u \) in \( F_{i-1} \) (denoted \( \theta_i = \exp(u) \)).
(iii) \( \theta_i = u / u \) for some nonzero \( u \) in \( F_{i-1} \) (denoted \( \theta_i = \log(u) \)).
(iv) \( \theta_i = u \, v \) for some nonzero \( u \) and \( v \) in \( F_{i-1} \) such that \( v = (-u^2)^v \) (i.e. \( v = \exp(-u^2) \)). In this case we write \( \theta_i = \text{erf}(u) \).
Cases (i), (ii) and (iii) describe the elementary functions. Case (iv) extends the class of elementary functions by allowing the inclusion of nonelementary expressions of the form \( \int u \, e^{-x^2} \, dx \). By *erf-elementary field*, we mean any field which can be described as an erf-elementary extension of \( C(x) \), where \( C \) is a field of constants and \( x \) is a solution to \( y' - 1 = 0 \).

**Example 2.1.** Let \( R \) be the set of real numbers and let \( R(x) \) be the set of rational functions with coefficients in \( R \). Then \( R(x) \) is a differential field under the usual derivative: \( \frac{d}{dx} \). Adjoining \( \sqrt{x} \), where \( \sqrt{x} \) denotes a fixed root of \( y^2 - x = 0 \), and adjoining the real valued function \( e^x \), produces the elementary field \( R(x, \sqrt{x}, e^x) \). Consider next the real valued function

\[
\int_0^x \frac{1}{2\sqrt{t} \, e^t} \, dt.
\]

Denoting this function by \( \theta \) we have

\[
\theta' = \frac{-1}{2\sqrt{x} \, e^x} = u'v,
\]

where \( u = \sqrt{x} \) and \( v = 1/e^x \). Since \( \theta = (-u^2)'v \) we have \( \theta = \text{erf}(\sqrt{x}) \) and thus, \( R(x, \sqrt{x}, e^x, \theta) \) is an erf-elementary field.

The following theorem from Singer et al. (1981) extends Liouville's theorem on integration to erf-elementary extensions.

**Theorem 2.1.** Let \( F \) be a liouvillian extension of its field of constants \( C \). Assume \( C \) is of characteristic zero and algebraically closed and let \( y \) be an element of \( F \). If \( y \) has an antiderivative in some erf-elementary extension of \( F \), then there exist constants \( c_i \) and \( d_i \) in \( C \), elements \( w_i \) in \( F \), and elements \( u_i \) and \( v_i \) algebraic over \( F \), such that

\[
y = w_0 + \sum c_i \frac{w_i'}{w_i} + \sum d_i u_i^* v_i,
\]

where \( v_i = (-u_i^*)'v_i \) and where \( u_i^*, v_i^2 \) and \( u_i^* v_i \) are in \( F \).

Notice that this result is weaker than Liouville's theorem for elementary functions due to the possible introduction of new algebraic expressions. For a proof that such algebraic expressions are, in fact, necessary, see Singer et al. (1983), Example 1.2.

There is also a technicality implicit in this theorem concerning the constant terms \( d_i \). Consider the following example.

**Example 2.2.** Let \( \bar{Q} \) be the algebraic closure of the rational numbers, let \( F = \bar{Q}(x, e^{-x^2+1}) \), and let \( y = e^{-x^2+1} \). Singer et al. (1983, Example 2.1) showed that \( y \) cannot be written as

\[
y = w_0 + \sum c_i \frac{w_i'}{w_i} + \sum d_i u_i^* e^{-s_i^2}
\]

for any \( w_i, u_i \), algebraic over \( F \) and constants \( c_i \) and \( d_i \) in \( \bar{Q} \). However, equation (2.1) is satisfied by letting \( u_1 = x, \ v_1 = e^{-x^2+1}, \ b_1 = 1 \) and \( w_1 = c_1 = 0 \) (since \( e^{-x^2+1} \) is an

† We shall use the notation \( e^x \) for the particular solution of \( y' - y = 0, y(0) = 1 \) and reserve the notation \( \exp(x) \) for its meaning as defined in (ii) above.
exponential of \(-x^2\). Thus \(\gamma\) has an antiderivative in some erf-elementary extension of \(F\) which contains no new constants. Note that if one allows transcendental constants, then \(\gamma\) can be written as \(\tilde{b}_1 \tilde{u}^t e^{-\tilde{x}_1}\) where \(\tilde{b}_1 = e\) and \(\tilde{u}_1 = x\).

As this example indicates, the error functions appearing in equation \((2.1)\) are not necessarily the "classical" error functions. However, we can make the following observation: Let \(F\) be as in Theorem 2.1 and let \(\gamma\) be an element of \(F\) such that equation \((2.1)\) holds for some \(w_i, u_i, v_i\) algebraic over \(F\) and constants \(c_i\) and \(d_i\). Consider an element \(\bar{v}_i\) of a differential extension \(E\) of \(F\) where \(\bar{v}_i = (-u_i^2)\tilde{v}_i\). It is easy to show that \(\bar{v}_i = \lambda_i \tilde{v}_i\) for some nonzero \(\lambda_i\) in the constant field of \(E\). Moreover,

\[
\gamma = w_0' + \sum c_i \frac{w_i'}{w_i} + \sum \frac{d_i}{\lambda_i} u_i \bar{v}_i.
\]

Therefore, when integrating in terms of error functions, one can first determine if there exist \(w_0, c_i\) and \(d_i\) in \(F\) and \(u_i\) algebraic over \(F\) satisfying Theorem 2.1. Then, for each \(i\), substitute \(\tilde{v}_i\) for \(v_i\) and \(d_i/\lambda_i\) for \(d_i\) to satisfy the condition \(v_i = e^{w_i}\). For another example of this, see Example 6.4.

We next turn our attention to the structure of the integrand fields. Let \(F = C(x, \theta_1, \ldots, \theta_n)\) be a transcendental elementary extension of \(C(x)\), where \(C\) is the field of constants and \(x\) is a solution to \(x' = 1\). We shall call \(F\) factored if for each logarithmic monomial \(\theta_i = \log(a_i)\), \(a_i\) is an irreducible polynomial in \(C[x, \theta_1, \ldots, \theta_{i-1}]\). An easy induction shows that given any elementary field \(F = C(x, \theta_1, \ldots, \theta_n)\), one can construct a factored field \(F = C(x, \theta_1, \ldots, \theta_m)\) such that \(F\) is differentially isomorphic to a subfield of \(F\). We may assume, therefore, that the fields that define our integrands are factored.

Next, rearrange the \(\theta\)'s into a tower \(C(x) = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_r = F\), where \(F_i = F_{i-1}(\theta_{1i}, \ldots, \theta_{imi})\) for \(i = 1, \ldots, r\) and where one of the following holds for each \(\theta_{ij}\):

(i) \(\theta_{ij} = a_{ij}/a_{ij}\) for some nonzero \(a_{ij}\) in \(F_{i-1}\), where \(a_{ij}\) is not in \(F_{i-2}\).

(ii) \(\theta_{ij} = \theta_{ij}/a_{ij}\) for some \(a_{ij}\) in \(F_{i-1}\), where \(a_{ij}\) is not in \(F_{i-2}\).

Given an element \(a\) in \(F\), we define the rank of \(a\), denoted \(\text{rank}(a)\), to be \(k\) if \(a\) is an element of \(F_k\), but \(a\) is not an element of \(F_{k-1}\). Finally, let \(\theta_i = \exp(a_i)\) be an exponential monomial of rank \(k\). We say that \(\theta_i\) has false rank if \(a_i = \delta_i \theta_j + \cdots + \delta_s \theta_j + y\) where each \(\theta_j\) is a logarithmic monomial of rank \(k-1\), \(\delta_j \neq 0\) is a rational number and \(\text{rank}(y) < k-1\). If \(\theta\) does not have false rank, we say that it is reduced. An example of an exponential with false rank is \(\theta = e^{1/2 \log(x)}\) which has rank 2 in the field \(C(x, \log(x), e^{1/2 \log(x)})\). The terminology false rank suggests that \(\theta\) is algebraic over \(C(x, \log(x), e^x)\) in which all elements are of rank 1.

We are now ready to state the main theorem.

**Theorem 2.2.** Let \(E = C(x, \theta_1, \ldots, \theta_n)\) be a differential field of characteristic zero with algebraically closed subfield of constants \(C\). Assume \(x\) is transcendental over \(C\) and a solution to \(x' = 1\), \(\theta_i\) is a monomial over \(C(x, \theta_1, \ldots, \theta_{i-1})\) for each \(i\), and \(E = C(x, \theta_1, \ldots, \theta_n)\) is factored. Further, assume that \(\theta_i\) is a reduced exponential over \(C(x, \theta_1, \ldots, \theta_{i-1})\), \(\text{rank}(\theta_i) = r\) and \(\text{rank}(\theta_i) < r\) for \(1 \leq i < n\). Then, given \(A\) in \(C(x, \theta_1, \ldots, \theta_{n-1})\), one can determine in a finite number of steps if \(A\theta_i\) has an antiderivative in some erf-elementary extension of \(E\) and if so find \(c_i, d_i, w_i, u_i\) and \(v_i\) satisfying equation \((2.1)\).
An example of an integrand which does not satisfy the hypothesis of this theorem is

\[ \int -\frac{1}{2} e^{-1/2 \log(x)} \log(x) \, dx \]

which is erf(√x). However, there are many integrals which, although they do not satisfy the hypothesis directly, can be presented in a form which does satisfy the hypothesis of the theorem. Consider, for instance, the integral

\[ \int \frac{(e^{-\log^2(x)})^4}{x^9} \, dx \]

over the field \( C(x, \log^2(x), e^{-\log^2(x)}) \). This integral does not satisfy the hypothesis of the theorem due to the exponent 4. However, the integral can be presented in the form

\[ \int \frac{e^{-4 \log^2(x)}}{x^9} \, dx \]

over the field \( C(x, \log^2(x), e^{-4 \log^2(x)}) \) which does satisfy the hypothesis of Theorem 2.2.

3. Completing Squares

One of the basic strategies to be employed in section 5 is to compute, by examining the integrand, the set of all possible error functions (i.e. the \( u_i \) and \( v_i \) pairs from Theorem 2.1), which could appear in the integral. If this set is finite, then we can use a result of Risch's to compute the constants \( d_i \) and the elementary part of the integral. In this context the following problem arises: let \( K \) be a field of characteristic zero,† and let \( p, q \) and \( f \) be elements of \( K[x] \) where \( p \) and \( q \) are relatively prime. Determine all values, \( \alpha \) and \( \beta \), such that

\[ f \left( \frac{p(x)}{q(x)} + \alpha x + \beta \right) \]

is a perfect square in \( K(x) \). Also, under what conditions is the solution set finite?

**Theorem 3.1.** Let \( K \) be a field of characteristic zero, let \( p \) and \( q \) be relatively prime elements of \( K[x] \) and let \( n \) be a nonnegative integer. Let \( S \) be the set of all \( d \) in \( K[x] \) such that \( \deg (d) \leq n \) and \( p + dq = r^2 \) for some \( r \) in \( K[x] \). If \( S \) is an infinite set, then \( \deg (q) \leq \deg (r) \leq n \).

**Proof.** Setting \( d(x) = d_n x^n + \cdots + d_0 \) and \( r(x) = r_m x^m + \cdots + r_0 \), substituting these expressions into \( p + dq = r^2 \) and comparing coefficients yields a system of polynomial equations in the variables \( d_n, \ldots, d_0 \). Using the Kronecker method of elimination one can eliminate variables and thus solve this system of equations. Moreover, if one has an infinite solution set, then in the course of this elimination, an indeterminate, say \( t \), may be introduced to parameterize an infinite set of solutions (Van der Waerden, 1950, Chapter 11, pages 1-4). In this way we denote a one parameter family of solutions by \( d(t) \) and \( r(t) \). These will, in general, be polynomials in \( x \) with coefficients which are algebraic over \( K(t) \). We next define the derivation \( \partial / \partial t \) for the field \( K(t) \) to be the usual partial derivative with respect to \( t \) and extend this derivation to an algebraic extension of \( K(t) \) which contains

† We shall assume, in this section and the next, that \( K \) is computable. That is, the field operations can be effectively carried out in \( K \).
the coefficients of \( d(t) \) and \( r(t) \). Now differentiating both sides of \( p + d(t)q = (r(t))^2 \) with respect to \( t \) yields

\[
\frac{\partial d}{\partial t} q = 2r \frac{\partial r}{\partial t}.
\]

This implies, since \( \gcd(q, r) = 1 \), that \( q \left| \frac{\partial r}{\partial t} \right. \) and \( r \left| \frac{\partial d}{\partial t} \right. \). Thus

\[
\deg_x(q) \leq \deg_x \left( \frac{\partial r}{\partial t} \right) \leq \deg_x(r) \leq \deg_x \left( \frac{\partial d}{\partial t} \right) \leq \deg_x(d).
\]

This theorem yields the two results we shall need in section 5.

**Corollary 3.2.** Let \( p \) and \( q \) be as above where \( q \) is monic and let \( S \) be the set of all pairs \((\alpha, \beta)\), \( \alpha \) and \( \beta \) in \( K \), such that

\[
p \cdot \frac{\alpha x + \beta}{q} = r^2
\]

for some \( r \) and \( s \) in \( K[x] \). If \( S \) is an infinite set, then \( q = 1 \) and \( \deg(p) \leq 2 \).

**Proof.** We may assume that \( r \) and \( s \) share no common factors and that \( s \) is monic. Then, since \( p \) and \( q \) are relatively prime and \( q \) is monic we have \( p + (\alpha x + \beta)q = r^2 \) and \( q = s^2 \). If \( S \) is infinite, then Theorem 3.1 yields \( \deg(q) \leq \deg(r) \leq 1 \). But then \( q = s^2 \) implies \( q = 1 \) and \( p + (\alpha x + \beta) = r^2 \). Since \( \deg(r) \leq 1 \) we have \( \deg(p) \leq 2 \).

**Corollary 3.3.** Let \( p \) and \( q \) be as in Corollary 3.2 and let \( f \neq 0 \) be in \( K[x] \). Suppose that \( p \) and \( q \) are not both in \( K \). Then the set of \( \alpha \) in \( K \) such that

\[
f \left( \frac{p}{q} + \alpha \right) = \frac{r^2}{s^2}
\]

for some \( r \) and \( s \) in \( K[x] \) is finite.

**Proof.** Letting \( f = f/gcd(f, q) \) and \( \bar{q} = q/gcd(f, q) \) we have

\[
\bar{f} \left( \frac{p + \alpha q}{\bar{q}} \right) = \frac{r^2}{s^2}.
\]

First assume that \( \bar{f} \) is not in \( K \) and suppose there are two solutions \( \alpha_1 \) and \( \alpha_2 \). We may assume without loss of generality that \( \bar{f} \) is squarefree and therefore \( \bar{f} \mid p + \alpha q \). Then there exists \( g_1 \) and \( g_2 \) so that \( \bar{f}g_1 = p + \alpha_1 q \) and \( \bar{f}g_2 = p + \alpha_2 q \). Subtracting yields \( \bar{f}(g_1 - g_2) = q(\alpha_1 - \alpha_2) \). Since \( \gcd(f, q) = 1 \), we have \( \bar{f} \mid \alpha_1 - \alpha_2 \), contradicting the assumption that \( \bar{f} \) is not in \( K \). Now assume \( \bar{f} \) is in \( K \) and suppose there is an infinite set of \( \alpha \) such that \( \bar{f} \mid p + \alpha q \). But then by Theorem 3.1 (replacing \( p \) with \( \bar{f}p \) and \( q \) with \( \bar{f}q \)), we have \( q = 1 \) and \( r \) in \( K \). This implies that \( p \) is in \( K \), contradicting the hypothesis of the theorem.

We note finally that when a finite solution set is assured we can, of course, solve the corresponding system of polynomial equations to explicitly construct it.†

† There are also many cases where more efficient techniques are possible. One such method was published recently by Zwillinger (1984).
4. $\Sigma$-Decompositions

Let $K$ be a field of characteristic zero and let $\Sigma = (f_1, \ldots, f_m)$ be a sequence of distinct and irreducible elements of $K[x]$, where no $f_i$ is in $K$. Given $\Phi$ in $K(x)$ we say that $\Phi$ has a $\Sigma$-decomposition over $K$ if there exist $b_i$ in $K$, integers $a_{ij}$, and a natural number $n$ so that

$$\Phi = \sum_{i=1}^n b_i \prod_{j=1}^m f_j^{a_{ij}}.$$ 

In this section we shall define a specialised type of $\Sigma$-decomposition, called a restricted decomposition, and present two results concerning their uniqueness and construction. The proofs of these theorems can be found in Cherry (1983, 1984).

Let $T$ be a subset of $\mathbb{Z}$ and let $g: T \rightarrow \mathbb{Z}^m$. We say that $\Phi$ in $K(x)$ has a $\Sigma$-decomposition restricted by $g$ if

$$\Phi = \sum_{i=1}^n b_i \prod_{j=1}^m f_j^{g(i)}.$$ 

where for all $i$, $a_{ij}$ is in $T$, $g(a_{ij}) = (a_{i1}, a_{i2}, \ldots, a_{im})$ and $b_i \neq 0$ except in the trivial case where $\Phi = 0$ and $n = 1$.

**Theorem 4.1.** Let $\Sigma$ and $g$ be as above and let $\Phi$ be an element of $K(x)$. If $\Phi$ has a $\Sigma$-decomposition restricted by $g$, then the $\Sigma$-decomposition is unique except for the possible reordering of terms and the combining of like terms. Furthermore, assume that for any $a$ in $T$ we can calculate $g(a)$ in a finite number of steps. Then we can calculate the $\Sigma$-decomposition of $\Phi$ in a finite number of steps.

Notice that this theorem says nothing about determining the existence of a $\Sigma$-decomposition for given element of $K(x)$. To do this we must restrict matters further. Let

$$\Phi = \sum_{i=1}^n b_i \prod_{j=1}^m f_j^{a_{ij}}$$ 

be a $\Sigma$-decomposition of $\Phi$ and, for each $j$, let $p_j(x)$ be the unique polynomial of degree less than $n$ for which $p_j(x_{a_{ij}}) = a_{ij}$, $i = 1, 2, \ldots, n$. We then define the degree of this decomposition to be the maximum of the degrees of the $p_j$.

**Theorem 4.2.** Let $\Sigma$ and $g$ be as above and let $d$ be a positive integer. Suppose that for any integer, $\alpha$, we can determine in a finite number of steps if $\alpha$ is in $T$ and, if so, can calculate $g(\alpha)$. Then given $\Phi$ in $K(x)$, we can determine in a finite number of steps if $\Phi$ has a $\Sigma$-decomposition restricted by $g$ with degree less than $d$.

These results generalise easily to the multivariate case. That is, let $\Sigma = (f_1, \ldots, f_m)$ be a sequence of pairwise relatively prime, irreducible elements of $K[x_1, \ldots, x_k]$, where $f_i$ is not in $K$ for each $i$. $\Sigma$-decompositions, restricted $\Sigma$-decompositions, and the degree of a decomposition are all defined as before.

In order to determine if $\Phi$ in $K(x_1, \ldots, x_k)$ has a $\Sigma$-decomposition restricted by $g$ of degree $d$ over $K$, we first single out $x_i$ such that $f_i$ is not in $K[x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_k]$. Then form $\Sigma' = (f_1, f_{k_1}, \ldots, f_{k_2})$ where $f_{k_j}$ is not in $K[x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_j]$ for each $j$. Now use Theorem 4.2 to calculate (if they exist) elements $B_i$ in $K(x_1, \ldots, x_{k-1},$
If $\Phi$ has a $\Sigma$-decomposition restricted by $g$ of degree $d$ over $K$, then equation (4.1) will be a $\Sigma'$-decomposition restricted by a projection of $g$ also of degree $d$ over $K(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_r)$. At this point we need only to factor each $B_i$ over $K(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_r)$ and examine these factors and multiplicities to determine if $\Phi$ has a decomposition over $K$. Also this decomposition is unique since equation (4.1) is unique and each factorisation of $B_i$ is unique.

5. Proof of the Main Theorem

We shall use the following notation throughout this section. Let $F$ denote the subfield $C(x, \theta_1, \ldots, \theta_{n-1}) = C(x, \theta_1, \ldots, \theta_{n-1})$ of $E$. Thus, $E = F(\theta_n)$. Let

$e = \{ i \mid \theta_i \text{ is an exponential monomial in } F \}$

and

$l = \{ i \mid \theta_i \text{ is a logarithmic monomial in } F \}$

be indexing sets for the monomials in $F$. Let $\theta_i = \theta_i a_i$ for all $i$ in $e \cup \{ n \}$, and let $\theta' = a_i a_i$ for all $i$ in $l$. We shall use the notation $[m]$ for the integer part of $m$.

Notice that the conditions on the rank of $\theta_n$ and the rank of $\theta_i$, $1 \leq i < n$, imply the existence of a monomial, say $\theta$, so that $a_n$ involves $\theta$ and the other arguments $a_j$, $j \in e \cup l$, are free of $\theta$. We shall denote the subfield of $E$ containing expressions which are free of both $\theta_n$ and $\theta$ by $D$. Thus $F = D(\theta)$. Finally, let $\ell = \{ i \mid \theta_i \text{ is a logarithmic monomial in } D \}$.

The proof of Theorem 2.2 will be broken down into a sequence of lemmas. Lemma 5.1 will provide a thorough analysis of the structure of the integral (provided, of course, the integral exists). The next three lemmas form a set of three mutually exclusive cases depending on whether $\theta$ is exponential (Lemma 5.2) or logarithmic (Lemmas 5.3 and 5.4). Finally, the proof of Theorem 2.2 is presented.

**Lemma 5.1.** Let $A\theta_n$ be as in Theorem 2.2. If $A\theta_n$ has an antiderivative in some $\text{erf}$-elementary extension of $E$, then there exists $B$ in $F = C(x, \theta_1, \ldots, \theta_{n-1})$, $d_i$ in $C$, and elements $u_i$ and $v_i$ algebraic over $E$ such that

$$A\theta_n = (B\theta_n) + \sum d_i u_i v_i,$$

where $v_i = (-u_i^2) v_i$. Furthermore, for each $i$, there exist rational numbers $r_{ij}$ where $2r_{ij}$ is an integer, and a constant $\tau_i$ so that the following hold:

(i) 

$$-u_i^2 = a_n + \sum r_{ij} a_j + \sum r_{ij} \theta_j + \tau_i,$$

(ii) 

$$v_i = \prod a_j^{r_{ij}} \prod \theta_j^{r_{ij} \theta_n},$$

(iii) 

$$\left( \prod a_j^{2r_{ij} - 2r_{ij} \theta_n} \prod \theta_j^{2r_{ij} - 2r_{ij} \theta_j} \right) \left( a_n + \sum r_{ij} a_j + \sum r_{ij} \theta_j + \tau_i \right).$$

is a perfect square in $F$.

**Proof.** If $A\theta_n$ has an antiderivative in some $\text{erf}$-elementary extension of $E$, then from
Theorem 2.1 there exist constants $c_i$ and $d_i$, elements $w_i$ in $E$, and elements $u_i$ and $v_i$ algebraic over $E$ such that

$$A\theta_n = w_0 + \sum c_i \frac{w_i}{u_i} + \sum d_i u_i v_i,$$

(5.5)


where $v_i = (-u_i^2)v_i$ and $u_i^2$, $v_i^2$ and $u_i v_i$ are in $E$. Since $v_i$ is an algebraic exponential over $E$, we have by Theorem 3.1 of Rothstein & Caviness (1979), that there exists rational numbers $r_i$ and $r_{ij}$ and a constant $\tau_i$ so that

$$-u_i^2 = r_i a_n + \sum r_{ij} a_j + \sum r_i \theta_j + \tau_i$$

and so

$$v_i = \eta_i \prod \theta_j^{a_j} \prod \theta_i^{\theta_i \theta_i},$$

(5.6)

where $\eta_i$ is in $C$.† We may assume, since $\eta_i$ is a multiplicative constant in $u_i v_i$, that $\eta_i = 1$ (i.e. absorb $\eta_i$ into $d_i$). Next notice that $u_i v_i$ is of the form $f_i \theta_i^{m}$ for some $f_i$ in $F$ and compare coefficients with respect to $\theta_i$ in equation (5.5). This yields $r_{in} = 1$, $w_0 = B \theta_n$, for some $B$ in $F$, and $w_j = 0$ for $i \neq j$. Also, since $E$ is factored, all of the factors in equation (5.6) are pairwise relatively prime elements of $C[x, \theta_1, \ldots, \theta_n]$. This coupled with the fact that $v_i^2$ is in $E$ implies that $2r_{ij}$ is an integer for all $j$. Thus equations (5.1), (5.2) and (5.3) have been proven.

Consider now the expression $u_i v_i$ and write

$$u_i v_i = \frac{2u_i^2(u_i^2)^3}{2u_i^2(u_i^2)^3} = 2u_i^2 u_i v_i$$

Since the expression on the right is in $E$ we have $u_i v_i$ in $E$. Hence $-u_i^2 v_i^2$ is a perfect square in $E$. $(-u_i^2 v_i^2 = (\sqrt{-1}u_i v_i)^2)$. Combining this with equations (5.2) and (5.3) yields

$$(a_n + \sum r_{ij} a_j + \sum r_i \theta_j + \tau_i)(\prod j \theta_j^{a_j} \prod \theta_i^{\theta_i \theta_i}) = S_i^2 \theta_i^2,$$

where $S_i$ is some element of $F$. We now divide both sides by $\theta_i^2$ and by $\prod j \theta_j^{a_j} \prod \theta_i^{\theta_i \theta_i}$ still leaving a perfect square on the right. That is

$$(a_n + \sum r_{ij} a_j + \sum r_i \theta_j + \tau_i)(\prod j \theta_j^{2a_j} \prod \theta_i^{2\theta_i}) = R_i^2,$$

where $R_i$ is some element in $F$. This completes the proof.

At this point we introduce the abbreviated notation $\Pi_i$ for the power product $\prod j a_j^{2r_{ij} - 2\theta_i \theta_i} \prod \theta_j^{2r_{ij} - 2\theta_i \theta_i}$ and observe that, for each $i$ and $j$, the exponent $2r_{ij} - 2\theta_i \theta_i$ is either 0 or 1 (since $2r_{ij}$ is always an integer). This implies, since $E$ is factored, that $\Pi_i$ is always a squarefree product of exponential monomials and logarithmic arguments. Therefore, by examining only the field $E$ and its generators (but not the integrand), one can construct a finite set of possible candidates for $\Pi_i$.

The perfect square in statement (5.4) is an important key to the proof, for it allows us (in Lemmas 5.2 and 5.3, below), to reconstruct the elements $u_i$ and $v_i$ which appear in

† Although not explicitly stated in Rothstein & Caviness (1979), the uniqueness of $r_i$, $r_{ij}$, and $\tau_i$ follows easily from their Theorem 3.1 (see also Cherry, 1984).

‡ We are assuming here, as we may, that no subset of the terms in $\sum d_j u_j v_i$ sum to a function with an elementary integral, for such a subset could be absorbed into the elementary part of the integral. Thus $r_{im} = 1$ for all $i$. 
equation (5.1). This will be accomplished with the corollaries of Theorem 3.1. Once these elements are known the Main Theorem part (b) in Risch (1969) can be applied. This theorem reduces the calculation of the element $B$ and the constants $d_i$ to the solution of a system of linear equations with coefficients in $C$. There is one case (Lemma 5.4, below), where this approach will not work. This case will be reduced to the decision procedure for $\Sigma$-decompositions (Theorem 4.2).

**Lemma 5.2.** Let $A_0, E, F, D,$ and $\theta$ be as above and suppose $\theta = x$ or that $\theta$ is an exponential monomial over $D$. Then one can determine in a finite number of steps if $A_0$ has an antiderivative in some erf-elementary extension of $E$ and if so can find an element $B$ in $F$, constants $d_i$ and elements $u_i$ and $v_i$ algebraic over $E$ satisfying equation (5.1).

**Proof.** Writing $a_\theta$ as $p(\theta)/q(\theta)$ where $p$ and $q$ are relatively prime and $q$ is monic, statement (5.4) of Lemma 5.1 yields

$$\Pi_i \left( \frac{p(\theta)}{q(\theta)} + \sum_\ell r_{\ell i} a_\ell + \sum_\ell r_{\ell i} \theta_\ell + r_i \right) = R_i^2,$$

where $R_i$ is in $F$. Recall from the proceeding discussion that there is only a finite set of possible values for $\Pi_i$ and that they can be enumerated. Also notice that, in this case, the terms $\sum_\ell r_{\ell i} a_\ell + \sum_\ell r_{\ell i} \theta_\ell + r_i$ are in $D$. We may therefore take each candidate $\Pi_i$ and by Corollary 3.3 find all values of $\beta_i$ for which

$$\Pi_i \left( \frac{p(\theta)}{q(\theta)} + \beta_i \right)$$

is a perfect square. In each case we determine if $\beta_i$ can be written (uniquely) as $\sum_\ell r_{\ell i} a_\ell + \sum_\ell r_{\ell i} \theta_\ell + r_i$ for some $r_{\ell i}$ and $r_i$. Now for each set of $r_{\ell i}$ and $r_i$ which are determined in this manner we check whether

$$\Pi_i = \prod_\ell a_{\ell i}^2 \theta_{\ell i}^{-2(r_{\ell i})} \prod_\ell \theta_{\ell i}^{2(r_{\ell i})}.$$

If this equality holds, then $u_i$ and $v_i$ can be determined from equations (5.2) and (5.3). Thus we have reconstructed all the possible elements $u_i$ and $v_i$ which could occur in (5.1). Risch’s Main Theorem part (b) from Risch (1969) can now be applied to calculate $B$ and the constants $d_i$ or to determine that they do not exist.

**Lemma 5.3.** Let $A_0, E, F, D,$ and $\theta$ be as above and suppose that $\theta$ is a logarithmic monomial over $D$. Let $a_\theta = p(\theta)/q(\theta)$ where $p$ and $q$ are relatively prime and $q$ is monic, and assume that either $q(\theta) \neq 1$ or that $\deg(p(\theta)) > 2$. Then one can determine in a finite number of steps if $A_0$ has an antiderivative in some erf-elementary extension of $E$ and if so can find an element $B$ in $F$, constants $d_i$ and elements $u_i$ and $v_i$ satisfying equation (5.1).

**Proof.** Applying Lemma 5.1, rewrite statement (5.4) as

$$\Pi_i \left( \frac{p(\theta)}{q(\theta)} + r_i \theta + \sum_\ell r_{\ell i} a_\ell + \sum_\ell r_{\ell i} \theta_\ell + r_i \right) = R_i^2,$$

where $r_i$ is a rational number and $R_i$ is in $F$. Note that in this case $\Pi_i$ is in $D$.

We can again complete squares and calculate all values $r_i$, $r_{\ell i}$, $\tau_i$ and $R_i$ satisfying
equation (5.7). This is done as follows: For each candidate $\Pi_i$ apply Corollary 3.2 to find $\alpha$ and $\beta$ such that $\Pi_i\rho(\theta)/q(\theta) + \alpha \theta + \beta$ is a perfect square. Let $r_i = \alpha/\Pi_i$ and check if $2r_i$ is an integer. Next, find $r_i$ and $\tau_i$ such that

$$
\frac{\beta}{\Pi_i} = \sum_{e} r_{ij}a_j + \sum_{e} r_{ij}b_j + \tau_i.
$$

If such $r_i$ and $\tau_i$ exist and

$$
\Pi_i = \prod_{i}^{(k-1)} a^{2r_{ij} - 2r_{ij}!} \prod_{i}^{(k-1)} \theta_{ij}^{2r_{ij} - 2r_{ij}!},
$$

then $u_i$ and $v_i$ are determined from equations (5.2) and (5.3). Once all terms, $u'_i v_i$, which can occur in (5.1) have been generated, we again apply the Main Theorem part (b) from Risch (1969).

**Lemma 5.4.** Let $A\theta\alpha$, $E$, $F$, $D$, and $\theta$ be as above and suppose that $\theta$ is a logarithmic monomial over $D$. Further, assume that $a_{n} = r\theta^2 + s\theta + t$, where $r$, $s$, and $t$ are in $D$ and $r \neq 0$. Then one can determine in a finite number of steps if $A\theta\alpha$ has an antiderivative in some elementary extension of $E$ and if so can find an element $B$ in $F$, constants $d_i$, and elements $u_i$ and $v_i$ satisfying equation (5.1).

**Proof.** From statement (5.4) we have

$$
\Pi_i(r\theta^2 + s\theta + t + r_i\theta + \beta_i) = R_i^2,
$$

where

$$
\beta_i = \sum_{e} r_{ij}a_j + \sum_{e} r_{ij}b_j + \tau_i
$$

and $R_i$ is in $F$. Setting $R_i = \sqrt{-1}(g_i\theta + h_i)$ for some $g_i$ and $h_i$ in $D$ yields

$$
\Pi_i(r\theta^2 + s\theta + t + r_i\theta + \beta_i) = -(g_i\theta + h_i)^2
$$

and so

$$
u_i = -(r\theta^2 + s\theta + t + r_i\theta + \beta_i)^{1/2} = \frac{g_i\theta + h_i}{\Pi_i^{1/2}}.
$$

Equating the coefficients of $\theta^2$ and $\theta$ in equation (5.8) we have

$$
\Pi_i r = -g_i^2
$$

and

$$
\Pi_i(s + r_i) = -2g_i h_i.
$$

From equation (5.10) we see that $\Pi_i$ may be uniquely determined by examining the multiplicities of the factors of $r$ (and so $\Pi_i$ does not depend on $i$). This now allows us to determine $g_i = (-\Pi_i r)^{1/2}$. Note that the positive square root can be taken for by equation (5.9), a sign change in $g_i$ can be replaced by a sign change in $u_i$. This will only change the sign of the constant $d_i$ in equation (5.1). From now on we will drop the subscripts on $\Pi_i$ and $g_i$ since these values are known and do not depend on $i$.

Writing $u_i$ as in equation (5.9) and observing that $v_i$ can be written as

$$
v_i = \Pi^{1/2} \prod_{i} a_{ij}^{(i)} \prod_{i} \theta_{ij}^{(i)},
$$

rewrite equation (5.1) as

$$
A\theta\alpha = (B\theta\alpha)' + \sum_{i} d_i \left(\frac{g\theta + h_i}{\Pi^{1/2}}\right)^{\Pi^{1/2} \prod_{i} a_{ij}^{(i)}} \prod_{i} \theta_{ij}^{(i)}.
$$

The remainder of the proof is broken down into the following four stages:
Stage (i): $B$ is calculated by considering the partial fraction decomposition of $A$. This reduces equation (5.12) to

$$ \sum_{i} d_i \left( \frac{g \theta + h_i}{1 + \theta^2} \right)^t \prod_{j=1}^{N} \theta_{r_{ij}}^{(r_{ij})}, $$

(5.13)

where $A_1$ and $A_0$ are in $D$. If equation (5.12) is not reduced to this form, then $A \theta_n$ cannot be integrated in terms of elementary functions and error functions.

Stage (ii): An expression, $\Phi$, depending only on the known quantities $A_1$, $A_0$, $g$, $\Pi$ and $s$ is calculated such that

$$ \Phi = \sum_{i} d_i \prod_{j=1}^{N} \theta_{r_{ij}}^{(r_{ij})}. $$

(5.14)

Stage (iii): The unknown quantities $d_i$ and $[r_{ij}]$ are determined by calculating the $\Sigma$-decomposition in equation (5.14). We show that the decomposition is restricted by a computable function and that the degree of the decomposition is 2. If the $\Sigma$-decomposition does not exist, then $A \theta_n$ cannot be integrated in terms of elementary functions and error functions.

Stage (iv): Conclude the algorithm by calculating, for each term in equation (5.14), a term $d_i \mu_i \nu_i$ in equation (5.1).

Stage (i): To show that the element $B$ in (5.1) can be calculated using an undetermined coefficients scheme, first decompose $A$ and $B$ into partial fractions to obtain:

$$ \left[ A_{m+2} \theta^{m+2} + \cdots + A_0 + \sum_{i=1}^{K} \sum_{j=1}^{K_i} A_{ij} \right] \theta_n $$

$$ = \left[ B_m \theta^m + \cdots + B_0 + \sum_{i=1}^{K} \sum_{j=1}^{K_i} B_{ij} \right] \theta_n + \sum_{i=1}^{K} \sum_{j=1}^{K_i} d_i \left( \frac{g \theta + h_i}{1 + \theta^2} \right)^t \theta_n. $$

(5.15)

We solve for $B_{1k_1}, \ldots, B_{kk_k}, \ldots, B_1$ as follows: Differentiating and comparing terms yields $A_{1k_1} + 1 = -k_1 B_{1k_1} p_1 \text{ (modulo } p_1)$. Now proceed as in Risch (1969), page 180: calculate polynomials $R$ and $S$ so that $R p_1 + S p_1 = A_{1k_1} + 1$ and $\text{deg}(S) < \text{deg}(p_1)$. Then, $B_{1k_1} = -S/k_1$. In this way we also determine $B_{2k_2}, \ldots, B_{kk_k}$. Now replace $A \theta_n$ with

$$ A \theta_n - \left[ \left( \frac{B_{1k_1}}{p_1} + \cdots + \frac{B_{kk_k}}{p_k} \right) \theta_n \right] $$

and repeat this process. Eventually, $B$ is reduced to a polynomial and equation (5.15) becomes

$$ \left( A_{m+2} \theta^{m+2} + \cdots + A_0 \right) \theta_n $$

$$ = \left[ (B_m \theta^m + \cdots + B_0) + (r \theta^2 + s \theta + t) (B_m \theta^m + \cdots + B_0) \right] \theta_n + \sum_{i=1}^{K} \sum_{j=1}^{K_i} d_i \left( \frac{g \theta + h_i}{1 + \theta^2} \right)^t \theta_n. $$

In a similar manner we may solve for $B_m, \ldots, B_1$ until we have

$$ \left( A_2 \theta^2 + A_1 \theta + A_0 \right) \theta_n = \left[ B_0 + (r \theta^2 + s \theta + t) B_0 \right] \theta_n + \sum_{i=1}^{K} \sum_{j=1}^{K_i} d_i \left( \frac{g \theta + h_i}{1 + \theta^2} \right)^t \theta_n. $$
At this point there are two possibilities. If \( r' \neq 0 \), then
\[
B_0 = A_2 / r'.
\]
Otherwise, \( r' = 0 \) and so by equation (5.10) we have \( g' = 0 \) and \( \Pi = 1 \). Therefore,
\[
\sum d_i \left( \frac{g\theta + h}{\Pi^{1/2}} \right) v_i
\]
is in \( D \) and
\[
B_0 = \frac{A_1}{2r\theta + \bar{v}}.
\]
Note that \( 2r\theta + \bar{v} \neq 0 \), since \( \theta \) is a monomial over \( D \). Thus we have determined \( B \) and reduced equation (5.15) to equation (5.13). This completes stage (i).

Stage (ii): Consider now equation (5.13). Performing the differentiation and collecting terms yields
\[
A_1 \theta + A_0 = \left[ \sum d_i \left( g' - \frac{g\Pi'}{2\Pi} \right) \prod \frac{d[v]}{e} \prod \frac{d[y]}{e} \right] \theta
\]
\[
+ \left[ \sum d_i \left( g\theta' + h'_i - \frac{h_i\Pi'}{2\Pi} \right) \prod \frac{d[v]}{e} \prod \frac{d[y]}{e} \right].
\]
(5.16)

Again, we examine two cases: \( r' \neq 0 \) and \( r' = 0 \).

First assume that \( r' \neq 0 \) and suppose that
\[
g' - \frac{g\Pi'}{2\Pi} = 0.
\]
Then
\[
2g' = \Pi'
\]
and so
\[
g^2 = \bar{c} \Pi
\]
where \( \bar{c} \) is some constant. This contradicts the fact that \( \Pi \) is squarefree. Thus,
\[
g' - \frac{g\Pi'}{2\Pi} \neq 0
\]
and we have
\[
\frac{A_1}{g' - \frac{g\Pi'}{2\Pi}} = \sum d_i \prod \frac{d[v]}{e} \prod \frac{d[y]}{e}.
\]

Now assume that \( r' = 0 \). By equation (5.10) we have \( g' = 0 \) and \( \Pi = 1 \). This along with equation (5.11) implies that
\[
h'_i = - \frac{s'}{2g}.
\]
With these simplifications, equation (5.16) implies that
\[
\frac{A_0}{g\theta' - \frac{s'}{2g}} = \sum d_i \prod \frac{d[v]}{e} \prod \frac{d[y]}{e},
\]
where
\[
g\theta' - \frac{s'}{2g} \neq 0,
\]
since \( \theta \) is a monomial over \( D \).
Thus in both cases we have calculated an element $\Phi$ of $D$ which satisfies equation (5.14) concluding stage (ii). The analysis of this $\Sigma$-decomposition is carried out in stage (iii).

Stage (iii): First rewrite equation (5.14) as

$$\Phi = \sum d_i a_i a_i^{(\alpha)} \prod_{j} a_j^{(r_j)} \prod_{\theta_j} \theta_j^{(r_j)}$$

and recall that $\{a\} \cup \{a_j\} \cup \{\theta_j\}_{j\in e}$ is a set of pairwise relatively prime, irreducible elements of $C[x, \theta_1, \ldots, \theta_{n-1}]$. Thus, equation (5.17) is a $\Sigma$-decomposition of $\Phi$ over $C$, where

$$\Sigma = (a, a_1, \ldots, a_i, \ldots, \theta_1, \ldots, \theta_j, \ldots), \ i \in l, \ j \in e.$$

We claim that this decomposition is restricted by a computable function and that the degree of the decomposition is less than or equal to two.

From equations (5.8), (5.10) and (5.11) we have

$$\frac{(s + r_i)^2}{4r} - t = \beta_i$$

and so

$$\frac{(s + r_j)^2}{4r} - t = \sum \omega_j \omega_i + \sum \theta_i \theta_j + \tau,$$

(5.18)

Therefore for each $i$ the values $r_j$ are uniquely determined by $r_i$. This observation essentially shows that equation (5.17) is a restricted $\Sigma$-decomposition which we now formally show. First recall that each $[r_j]$ is either $r_j$ or $r_j-1/2$ depending on the factors of $\Pi$ and let

$$J = \{j \in \Pi | a_j \text{ divides } \Pi \} \cup \{j \in e | \theta_j \text{ divides } \Pi \}.$$

Notice that given $[r_i]$ we can compute

$$r_i = \begin{cases} [r_i] + 1/2 & \text{if } a \text{ divides } \Pi \\ [r_i] & \text{otherwise.} \end{cases}$$

Then compute $r_j$ from (5.18) and finally

$$[r_j] = \begin{cases} r_j - 1/2 & \text{if } j \text{ is in } J \\ r_j & \text{otherwise.} \end{cases}$$

We therefore define a function $\phi: T \rightarrow Q^{[l+[e]}$ as follows: For $a$ in $Z$ let

$$\bar{a} = \begin{cases} a + 1/2 & \text{if } a \text{ divides } \Pi \\ a & \text{otherwise,} \end{cases}$$

and let $T$ be the set of all integers, $a$, for which

$$\frac{(s + \bar{a})^2}{4r} - \tau = \sum \omega_j \omega_i + \sum \theta_i \theta_j + \tau$$

(5.19)

for some rational $\omega_j$ and $\tau$ in $C$. Now for each $j$ in equation (5.19) form

$$\bar{a}_j = \begin{cases} a_j - 1/2 & \text{if } j \text{ is in } J \\ a_j & \text{otherwise,} \end{cases}$$

and let $\phi(a) = (a, \bar{a}_1, \bar{a}_2, \ldots)$. Finally, denote the preimage of $Z^{[l+[e] by T}$ and let $\bar{\Phi} = \Phi|_T$. Then equation (5.17) is a $\Sigma$-decomposition of $\Phi$ restricted by $\bar{\Phi}$. 
In order to show that the degree of equation (5.17) is less than or equal to two, we first prove the following claim: Let $\kappa$ be the number of terms in equation (5.17). Then, for each $j$, there exists a quadratic polynomial, $p_j(\alpha)$, such that $r_j = p_j(\alpha)$, $i = 1, \ldots, \kappa$. This is certainly true if $\kappa < 3$. Therefore let $\kappa \geq 3$ and suppose equation (5.18) holds for distinct $r_1$, $r_2$ and $r_3$. Now define the set $S$ as the collection of all elements in $D$ which can be written as $\sum_ r r_i a_i + \sum_ r r_i \theta_j + \tau$ for some constant $\tau$ and rational numbers $r_j$. Notice that $S$ is closed under addition, subtraction and multiplication by a rational number. From equation (5.18) we have

$$s^2 + 2r_1 s + r_1^2 \quad 4r$$

and

$$s^2 + 2r_2 s + r_2^2 \quad 4r$$

in $S$. Subtracting and multiplying by $4/(r_1 - r_2)$ we find that

$$\frac{2s}{r} + \frac{r_1 + r_2}{r}$$

is in $S$. Similarly we have

$$\frac{2s}{r} + \frac{r_1 + r_3}{r}$$

in $S$. If we now subtract these two expressions and multiply by $1/(r_2 - r_3)$, we have that $1/r$ is in $S$ and, therefore, $1/(4r)$ is in $S$. Now work backwards: Since $1/(4r)$ is in $S$ we have from equation (5.21) that $s/2r$ is in $S$. Then, from equation (5.20) we have $(s^2/4r) - t$ in $S$. Therefore there exist constants $w_i$ and rational numbers $w_j$ such that

$$\frac{1}{4r} = \sum_ e w_{2j} a_j + \sum_ e w_{2j} \theta_j + w_2$$

$$\frac{s}{2r} = \sum_ e w_{1j} a_j + \sum_ e w_{1j} \theta_j + w_1$$

$$\frac{s^2}{4r} - t = \sum_ e w_{0j} a_j + \sum_ e w_{0j} \theta_j + w_0.$$

Substituting these into equation (5.18) and collecting terms yields

$$\sum_ e (w_2 r_i^2 + w_1 r_i + w_0) a_j + \sum_ e (w_2 r_i^2 + w_1 r_i + w_0) \theta_j + (w_2 r_i^2 + w_1 + w_0)$$

Therefore for each $j$,

$$r_j = w_2 r_i^2 + w_1 r_i + w_0,$$

and hence

$$p_j(\alpha) = w_2 \alpha^2 + w_1 \alpha + w_0,$$

proving the claim.

Now consider again two cases. If $a$ divides $\Pi$, define

$$\tilde{p}_j(\alpha) = \begin{cases} 
  p_j(\alpha + 1/2) - 1/2 & \text{if } j \text{ is in } J \\
  p_j(\alpha + 1/2) & \text{otherwise},
\end{cases}$$

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and if \( a \) does not divide \( \Pi \), define

\[
\tilde{p}_j(x) = \begin{cases} 
-p_j(x) - 1/2 & \text{if } j \text{ is in } J \\
p_j(x) & \text{otherwise}.
\end{cases}
\]

We then have for each \( j \),

\[
[r_{ij}] = \tilde{p}_j([r_j]), \quad i = 1, \ldots, \kappa,
\]

proving that the degree of the \( \Sigma \)-decomposition in equation (5.17) is less than or equal to two.

We can now apply Theorem 4.2 (actually the multivariate generalisation), to determine in a finite number of steps if there exist constants \( d_i \) and integers \( [r_{ij}] \) satisfying equation (5.17). This completes stage (iii).

Stage (iv): In this final stage we generate, for each \( i \), a term \( d_i u_j v_j \) in equation (5.1) from the corresponding term \( d_i \prod_j d_j^{r_{ij}} \prod_j \beta_j^{r_{ij}} \) in equation (5.13). Fix \( i \). The notation \( d_i \), of course, refers to the same constant in equations (5.1) and (5.13). We let \( r_i = [r_i] + 1/2 \) if \( a \) divides \( \Pi \) and let \( r_i = [r_i] \) if \( a \) does not divide \( \Pi \). The \( r_{ij} \) are obtained in the same manner. That is,

\[
r_{ij} = \begin{cases} 
[r_{ij}] + 1/2 & \text{if } j \text{ is in } J \\
r_{ij} & \text{otherwise},
\end{cases}
\]

where \( J \) is the indexing set for the factors of \( \Pi \) defined in stage (iii). Next, calculate \( \tau_i \) from equation (5.18). Finally, equations (5.2) and (5.3) yield \( u_i \) and \( v_i \). This completes the case where \( r' = 0 \). If \( r' \neq 0 \) we need only to check that the expression \( A_0 \) in equation (5.16) also agrees with these values for \( d_i, r_i, r_{ij} \) and \( \tau_i \).

**Proof of Theorem 2.2.** Follows directly from Lemmas 5.2, 5.3, 5.4 and the observation that the linear case, where \( a_0 \) is linear in \( \theta \), cannot involve error functions. For assume otherwise and let \( a_n = r \theta + s \) for \( r \) and \( s \) in \( D \). Then statement (5.4) yields

\[
\Pi_i(r \theta + s + r_i \theta + \beta_i) = R_i^2,
\]

where \( R_i \) is in \( F \) and

\[
\beta_i = \sum_j r_{ij} a_j + \sum_j r_{ij} + \tau_i.
\]

Therefore, \( r + r_i = 0 \) which implies that \( r \) is rational. But this contradicts the assumption that \( \theta_i \) is reduced. We conclude that no error functions can occur in this case and proceed as in Risch (1969).

### 6. Examples

**Example 6.1.** Consider \( \int e^{x} - e^{2x} \, dx \). Here the field \( E = C(\theta_1 = e^{2x}, \theta_2 = e^{x} - e^{2x}) \) satisfies the hypotheses of Theorem 2.2 and, in the proof of this theorem, we apply Lemma 5.2 with \( \theta = \theta_1 \), \( p(\theta)/q(\theta) = -\theta + x \) and \( \beta_i = r_{ii}(2x) + \tau_i \). Given the possible values for \( \Pi_i \): \( \Pi_1 = 1 \) and \( \Pi_2 = \theta_1 \), we determine all values \( \beta_i \) such that \( \Pi_i(p(\theta)/q(\theta) + \beta_i) \) is a perfect square. This yields \( \Pi_1 = \theta_1 \) and \( \beta_i = -x \). Thus, \( r_{11} = -1/2 \), \( \tau_1 = 0 \) and, from equations (5.2) and (5.3), \( u_1 = e^x \) and \( v_1 = e^{-e^{2x}} \). An application of the Main Theorem, part (b), from Risch (1969) yields \( \int e^{x} - e^{2x} \, dx = \text{erf} (e^3) \).

**Example 6.2.** Consider

\[
\int \left( \frac{1}{2 \log (x)} - \frac{1}{x \log^2 (x)} \right) e^{\frac{-x}{2 \log (x)} - \frac{1}{4 \log^2 (x)}} \, dx.
\]
Here

\[ E = C(x, \log(x), e^{\log^2(x)} - \frac{1}{2} \log(x)), \quad \theta = \log(x), \quad p(\theta) = \frac{-x}{\log^2(x)} - \frac{1}{2} \log(x) \]

and the possibilities for \( \Pi \) are \( \Pi_1 = x \) and \( \Pi_2 = 1 \). We apply Lemma 5.3. Equation (5.7) becomes

\[ \Pi_1 \left( \frac{-x}{\log^2(x)} - \frac{1}{2} \log(x) + r_1 \log(x) + \tau_1 \right) = \mathcal{R}_1^2. \]

The only solution is \( \Pi_1 = x, \ r_1 = \frac{1}{2} \) and \( \tau_1 = 0 \), which yields \( u_1 = \sqrt{x/\log(x)} \) and \( v_1 = e^{\log^2(x)} \). We again apply (Risch, 1969), part (b), yielding

\[ \int \left( \frac{1}{2 \log(x)} - \frac{1}{x \log^2(x)} \right) e^{-x} e^{\frac{1}{2} \log(x)} \, dx = \text{erf} \left( \frac{\sqrt{x}}{\log(x)} \right). \]

**Example 6.3.** Consider

\[ \int \left( \frac{\log(x)}{2} + 1 \right) e^{-x} \log^2(x) - \frac{1}{2} \log(x) \, dx. \]

We apply Lemma 5.4 with \( \theta = \log(x), r = -x \) and \( s = -\frac{1}{2} \). From equation (5.10) we have \( \Pi = x \) and \( g = x \). Since

\[ A = A_1 + A_0 = \frac{\log(x)}{2} + 1 \]

is linear in \( \theta \), there is no elementary part (i.e. \( B = 0 \)). Hence, we move to stage (ii), apply the subcase where \( r \neq 0 \) and consider the corresponding \( \Sigma \)-decomposition

\[ \frac{A_1}{g'} = 1 = \sum d_i x^{r_i}. \]

This yields \( [r_1] = 0, \ d_1 = 1 \) and, since \( x \) divides \( \Pi \), \( r_1 = \frac{1}{2} \). Now, by equations (5.2) and (5.3), \( u_1 = \sqrt{x} \log(x) \) and \( v_1 = e^{-x} \log^2(x) \). Thus,

\[ \int \left( \frac{\log(x)}{2} + 1 \right) e^{-x} \log^2(x) - \frac{1}{2} \log(x) \, dx = \text{erf} \left( \sqrt{x \log(x)} \right). \]

**Example 6.4.** Consider

\[ \int \frac{3x^6 + 1}{x^5} e^{-\log^2(x)} \, dx. \]

This is also an example of Lemma 5.4. This time \( r = -1, \ s = 0, \ t = 0 \) and so \( \Pi = x \) and \( g = 1 \). The corresponding \( \Sigma \)-decomposition is

\[ \frac{A_0}{g'} = \frac{3x^6 + 1}{x^4} = \Sigma d_i x^{r_i}, \]

which yields \( r_1 = -4, \ d_1 = 1, \ r_2 = 2 \) and \( d_2 = 3 \). From equation (5.18) calculate \( \tau_1 = -4 \) and \( \tau_2 = -1 \). Then,

\[ u_1 = \log(x) + 2, \ u_2 = \log(x) - 1, \ v_1 = \frac{e^{-\log^2(x)}}{x^4}, \ v_2 = x e^{-\log^2(x)} \]
and
\[ \int \frac{3x^6 + 1}{x^5} e^{-\log^2(x)} \, dx = u_1' v_1 + 3u_2' v_2. \]

Finally, following the notation of section 2, we let
\[ \lambda_1 = e^{-4}, \quad \bar{v}_1 = \frac{e^{-\log^2(x) - 4}}{x^4}, \quad \lambda_2 = e^{-1}, \quad \bar{v}_2 = x e^{-\log^2(x) - 1}, \]
and we have
\[ \int \frac{3x^6 + 1}{x^5} e^{-\log^2(x)} \, dx = e^{4} \text{erf}(\log(x) + 2) + 3 \text{erf}(\log(x) - 1). \]

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References
To state the main theorem on error functions from Cherry (1983), we need the following definitions.

Let $F = C(x, \theta_1, \ldots, \theta_n)$ be an elementary extension of $C(x)$, where $C$ is the field of constants and $x$ is a solution to $x' = 1$. Rearrange the $\theta$'s into a tower

$$C(x) = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_r = F,$$

where

$$F_i = F_{i-1}(\theta_{i1}, \ldots, \theta_{i\beta})$$

for $i = 1, \ldots, r$, and where one of the following holds for each $\theta_{ij}$:

(i) $\theta_{ij}$ is algebraic over $F_{i-1}$ and transcendental over $F_{i-2}$.
(ii) $\theta_{ij} = a_{ij}/a_{ij}'$ for some non-zero $a_{ij}$ in $F_{i-1}$, where $a_{ij}$ is not in $F_{i-2}$.
(iii) $\theta_{ij} = \theta_{ij}'/a_{ij}$ for some $a_{ij}$ in $F_{i-1}$, where $a_{ij}$ is not in $F_{i-2}$.

The rank of a tower of elementary fields $F = C(x, \theta_1, \ldots, \theta_n)$, denoted $\text{rank}(F)$, is defined as the tuple $(m_r, \ldots, m_1, 1)$, where $m_i$ is the transcendence degree of $F_i$ over $F_{i-1}$, for $1 \leq i \leq r$.

An element $a$ in $F$ has rank $k$ if $a$ is an element of $F_k$ and $a$ is not an element of $F_{k-1}$. Of course, if $\theta_i$ is transcendental over $F = C(x, \theta_1, \ldots, \theta_{i-1})$ for $1 \leq i \leq n$, then this definition reduces to the definition given in section 2.

Given two sequences $(m_r, \ldots, m_1, 1)$ and $(\tilde{m}_r, \ldots, \tilde{m}_1, 1)$, we say that $(m_r, \ldots, m_1, 1) < (\tilde{m}_r, \ldots, \tilde{m}_1, 1)$ if $r < s$ or if $r = s$ and $(m_r, \ldots, m_1, 1) < (\tilde{m}_r, \ldots, \tilde{m}_1, 1)$ in the usual lexicographic ordering.

Finally, we say that $F$ is reduced if for all elementary extensions, $\tilde{F}$, of $C(x)$, which are algebraic over $F$, with the same set of constants, we have $\text{rank}(\tilde{F}) \geq \text{rank}(F)$.

**Example** A.1. The field $F = C(x, \log(x), \exp(\frac{1}{2}\log(x)+x))$ is not reduced since $\text{rank}(F) = (1, 1, 1)$ and $\tilde{F} = C(x, \sqrt{x}, \log(x), \exp(x))$, which is algebraic over $F$, has a rank of $(2,1)$.

The next theorem, which is a generalisation of Proposition 5.2 from Singer et al. (1983) gives an effective means for determining whether a given transcendental elementary field is reduced. The proof can be found in Cherry (1983), (Theorem 3.1). We shall denote the indexing set for the exponentials of rank $i$ in $F$, $\{j/\theta_{ij} = \exp(a_{ij})\}$, by $e_i$ and shall denote the union of these by $e$. The indexing for the logarithmic monomials will be handled similarly.

**Theorem A.1.** Let $F = C(x, \theta_1, \ldots, \theta_n)$ be a transcendental elementary extension of $C(x)$ and in addition assume that $F$ is factored. Then $F$ is reduced if and only if for each $k = 2, 3, \ldots, r$ the following holds:

$$\sum_{e_k} \alpha_{ki}a_{ij} + \sum_{i=1}^{k-2} \beta_{k-1,j} \theta_{k-1,j} \in F_{k-2} \text{ for rational numbers } \alpha_{ki}, \beta_{k-1,j} \text{ implies } \alpha_{ki} = \beta_{k-1,j} = 0.$$

The main theorem on error functions from Cherry (1983) follows. As we have already indicated, the proof of this theorem is essentially a series of reductions to Theorem 2.2.

**Theorem A.2.** Let $E = C(x, \theta_1, \ldots, \theta_n)$ be a differential field of characteristic zero with algebraically closed subfield of constants $C$. Assume $x$ is transcendental over $C$ and a
solution to $x' = 1$, $\theta_i$ is a monomial over $C(x, \theta_1, \ldots, \theta_{i-1})$ for each $i$, and $E = C(x, \theta_1, \ldots, \theta_2)$ is factored and reduced. Given $\gamma$ in $E$, one can decide in a finite number of steps if $\gamma$ has an antiderivative in some erf-elementary extension of $E$, and if so, find constants $c_i$ and $d_i$, elements $w_i$ in $E$ and $u_i, v_i$ algebraic over $E$ such that

$$\gamma = w'_0 + \sum c_i \frac{w'_i}{w_i} + \sum d_i u'_i v_i,$$

where $w'_i = (-u'_i) v_i$. 