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# Convex hull realizations of the multiplihedra

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ABSTRACT

We present a simple algorithm for determining the extremal points in Euclidean space whose convex hull is the $n$th polytope in the sequence known as the multiplihedra. This answers the open question of whether the multiplihedra could be realized as convex polytopes. We use this realization to unite the approach to $A_n$-maps of Iwase and Mimura to that of Boardman and Vogt. We include a review of the appearance of the $n$th multiplihedron for various $n$ in the studies of higher homotopy commutativity, (weak) $n$-categories, $A_\infty$-categories, deformation theory, and moduli spaces. We also include suggestions for the use of our realizations in some of these areas as well as in related studies, including enriched category theory and the graph-associahedra.



**Contents**



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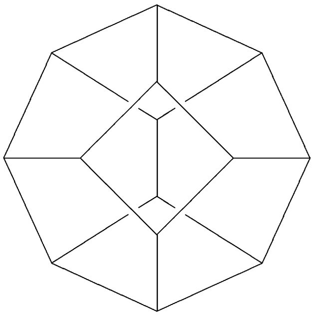
## 1. Introduction

The associahedra are the famous sequence of polytopes denoted $\mathcal{K}(n)$ from [27] which characterize the structure of weakly associative products. $\mathcal{K}(1) = \mathcal{K}(2) =$ a single point, $\mathcal{K}(3)$ is the line segment, $\mathcal{K}(4)$ is the pentagon, and $\mathcal{K}(5)$ is the following 3-d shape:

$\mathcal{K}(5) =$



The original examples of weakly associative product structure are the $A_n$ spaces, topological $H$-spaces with weakly associative multiplication of points. Here “weak” should be understood as “up to homotopy”. That is, there is a path in the space from $(ab)c$ to $a(bc)$. An $A_\infty$-space $X$ is characterized by its admission of an action

$$\mathcal{K}(n) \times X^n \to X$$

for all $n$. The axioms of this action as given in [27] are an example of the definition of an operad algebra action as given in [23].

The complexes now known as the multiplihedra, usually denoted $\mathcal{J}(n)$, were first discussed by Stasheff in [28]. Stasheff described how to construct the 1-skeleton of these complexes, but stopped short of a full combinatorial description. The multiplihedra were introduced in order to approach a full description of the category of $A_\infty$ spaces by providing the underlying structure for morphisms which preserved the structure of the domain space “up to homotopy” in the range. Thus the multiplihedra are used to recognize the $A_\infty$ (as well as $A_n$) maps. An $A_\infty$-map $f : X \to Y$ is characterized by its admission of an action

$$\mathcal{J}(n) \times X^n \to Y$$

for all $n$. This action obeys axioms as given in [28], including the crucial requirement that the action for $n = 1$ has the effect of taking a point $x \in X$ to $f(x) \in Y$.

In [3] Boardman and Vogt take up the challenge of a complete description of the category of $A_\infty$ spaces and maps (and their $A_n$ versions.) Their approach is to use sequences of spaces of binary trees with interior edges given a length in $[0, 1]$. They show that the space of such trees with $n$ leaves (under certain equivalence relations regarding length zero edges) is precisely the $n$th associahedron. They then develop several homotopy equivalent versions of a space of *painted* binary trees with interior edges of length in $[0, 1]$. These they use to define maps between $A_\infty$ spaces which preserve the multiplicative structure up to homotopy. A later definition of the same sort of map was published by Iwase and Mimura in [12]. They give the first detailed definition of the sequence of complexes $\mathcal{J}(n)$ now known as the multiplihedra, and describe their combinatorial properties. A good review of the combinatorics of their definition is in [13]. This latter reference also shows how the permuto-associahedra can be decomposed by a combinatorial use of the multiplihedra.

The overall structure of the associahedra is that of a topological operad, with the composition given by inclusion. The multiplihedra together form a bimodule (left and right module) over this operad, with the action again given by inclusion. That is, there exist inclusions:

$$\mathcal{K}(k) \times \big(\mathcal{J}(j_1) \times \cdots \times \mathcal{J}(j_k)\big) \hookrightarrow \mathcal{J}(n)$$

where $n$ is the sum of the $j_i$. This is the left module structure. The right module structure is from existence of inclusions:

$$\mathcal{J}(k) \times \big(\mathcal{K}(j_1) \times \cdots \times \mathcal{K}(j_k)\big) \hookrightarrow \mathcal{J}(n)$$

where $n$ is the sum of the $j_i$. This structure mirrors the fact that the spaces of painted trees form a bimodule over the operad of spaces of trees, where the compositions and actions are given by the grafting of trees, root to leaf. The definition of operad bimodule we use here is that stated in some detail in [18] and briefly in [11]. A related definition of operad bimodules can be found in [20, p. 138].

The multiplihedra appear frequently in higher category theory. The definitions of bicategory and tricategory homomorphisms each include commuting pasting diagrams as seen in [14] and [9] respectively. The two halves of the axiom for a bicategory homomorphism together form the boundary of the multiplihedra $\mathcal{J}(3)$, and the two halves of the axiom for a tricategory homomorphism together form the boundary of $\mathcal{J}(4)$. Since weak $n$-categories can be understood as being the algebras of higher operads, these facts can be seen as the motivation for defining morphisms of operad (and $n$-operad) algebras in terms of their bimodules. This definition is mentioned in [2] and developed in detail in [11]. In the latter paper it is pointed out that the bimodules in question must be co-rings, which have a co-multiplication with respect to the bimodule product over the operad.

The study of the $A_\infty$ spaces and their maps is still in progress. There is an open question about the correct way of defining composition of these maps in order to form a category. On p. 100 in [3] an obvious sort of composition is described as not being associative in the obvious way. A diagonal map is constructed for the multiplihedra in [25], extrapolated from an analogous diagonal on the associahedra. These maps allow a functorial monoidal structure for certain categories of $A_\infty$-algebras and $A_\infty$-categories. Different, conjecturally equivalent, versions of diagonals on the associahedra are presented in [19] and [16]. The painted trees and realizations in the current paper may help in defining analogous diagonals of the multiplihedra based upon these latter sources. Eventually it needs to be understood whether any of the possible diagonals make the multiplihedra into a co-ring as defined in [11], as well as how such a structure relates to the canonical compositions defined in [3].

The multiplihedra have appeared in many areas related to deformation theory and $A_\infty$ category theory. The 3-dimensional version of the multiplihedron is called by the name Chinese lantern diagram in [30], and used to describe deformation of functors. There is a forthcoming paper by Woodward and Mau in which a new realization of the multiplihedra as moduli spaces of disks with additional structure is presented [22]. This realization promises to help allow the authors and their collaborators to define $A_n$-functors as in [21], as well as morphisms of cohomological field theories. There are also interesting questions about the extension of $A_n$-maps, as in [10], and about the transfer of $A_\infty$ structure through these maps, as in [17]. In the latter there is an open question about canonical decompositions of the multiplihedra. The realizations we describe here lend themselves well to experimentation upon such decompositions.

The purpose of this paper is to describe how to represent Boardman and Vogt's spaces of painted trees with $n$ leaves as convex polytopes which are combinatorially equivalent to the CW-complexes described by Iwase and Mimura. Our algorithm for the vertices of the polytopes is flexible in that it allows an initial choice of a constant $q \in (0,1)$. The boundary of the open unit interval corresponds to certain quotient spaces of the multiplihedron. In the limit as $q \to 1$ the convex hull approaches that of Loday's convex hull representation of the associahedra as described in [15]. The limit as $q \to 1$ corresponds to the case for which the mapping strictly respects the multiplication.

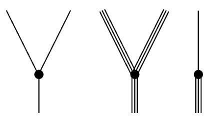
The limit of our algorithm as $q \to 0$ represents the case for which multiplication in the domain of the morphism in question is strictly associative. The case for which multiplication in the range is strictly associative was found by Stasheff in [28] to yield the associahedra. It was long assumed that the case for which the domain was associative would likewise yield the associahedra, but we demonstrate in [7] that this is not so. In the limit as $q \to 0$ the convex hulls instead approach a newly discovered sequence of polytopes. The low-dimensional terms of this new sequence may be found in [24] within the axioms for pseudomonoids in a monoidal bicategory, or in [4] within the axioms of enriched bicategories. Recall that when both the range and domain are strictly associative the multiplihedra become the cubes, as seen in [3].

The results in this paper support several related efforts of further research. The first is to describe important quotients of the multiplihedra, an effort brought to fruition in [7]. Another effort underway is the generalization of the multiplihedron and its quotients by analogy to the graph-associahedra introduced by Carr and Devadoss, in [5]. The graph-multiplihedra will be presented in [6].

An overview of the rest of this paper is as follows: In Section 2 we review the definition and properties of the multiplihedra, introducing a recursive combinatorial definition (using the painted trees of [3]) of the complex $\mathcal{J}(n)$ with the properties described in [12]. In Section 3 we briefly give some new and provocative combinatorial results related to the counting of the vertices of $\mathcal{J}(n)$. In Section 4 we describe the method for finding geometric realizations of the multiplihedra as convex hulls. The main result is that these convex hulls are indeed combinatorially equivalent to Stasheff's multiplihedra. In Section 5 we relate our geometric realization to the spaces of trees defined by Boardman and Vogt. This is done by defining a space of level trees that obeys the requirements in [3] and which in proof (2) of Lemma 1 is shown directly to be homeomorphic to our convex hull. Section 6 contains the proof of the main result by means of explicit bounding hyperplanes for the convex hulls.

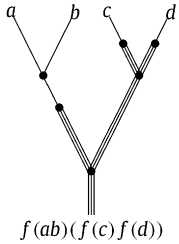
## 2. Facets of the multiplihedra

Pictures in the form of *painted binary trees* can be drawn to represent the multiplication of several objects in a monoid, before or after their passage to the image of that monoid under a homomorphism. We use the term "painted" rather than "colored" to distinguish our trees with two edge colorings, "painted" and "unpainted", from the other meaning of colored, as in colored operad or multicategory. We will refer to the exterior vertices of the tree as the root and the leaves, and to the interior vertices as nodes. This will be handy since then we can reserve the term "vertices" for reference to polytopes. A painted binary tree is painted beginning at the root edge (the leaf edges are unpainted), and always painted in such a way that there are only three types of nodes. They are:



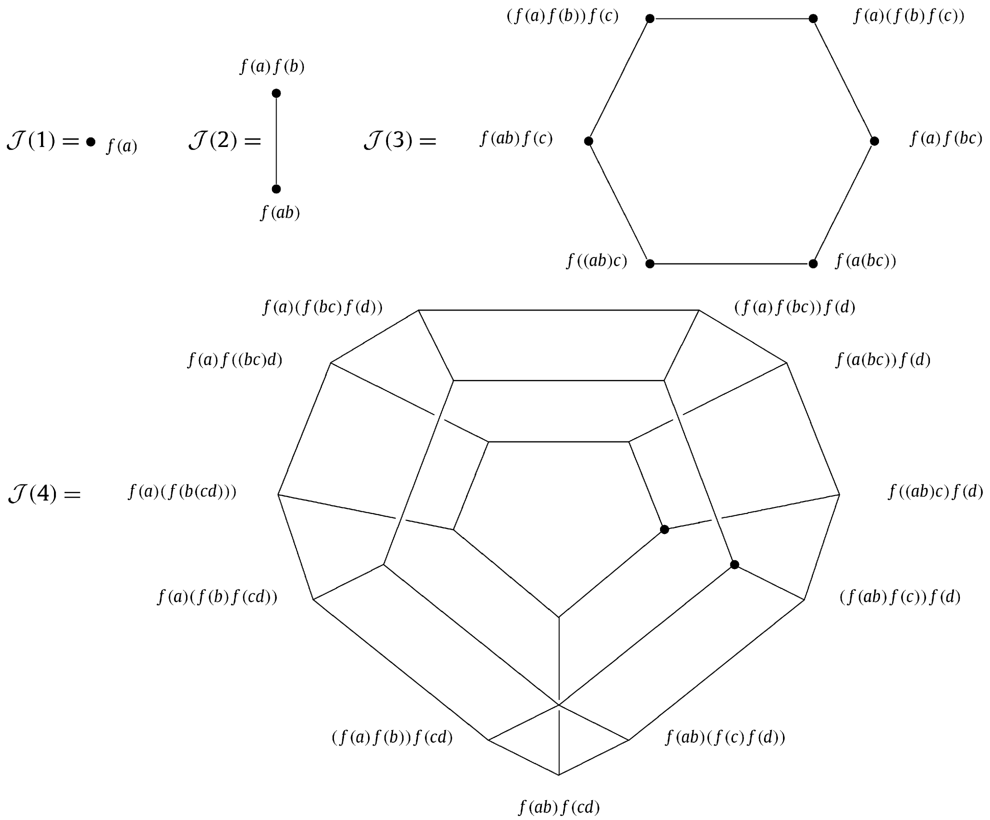
(1) (2) (3)

This limitation on nodes implies that painted regions must be connected, that painting must never end precisely at a trivalent node, and that painting must proceed up both branches of a trivalent node. To see the promised representation we let the left-hand, type (1) trivalent node above stand for multiplication in the domain; the middle, painted, type (2) trivalent node above stand for multiplication in the range; and the right-hand type (3) bivalent node stand for the action of the mapping. For instance, given $a, b, c, d$ elements of a monoid, and $f$ a monoid morphism, the following diagram represents the operation resulting in the product $f(ab)(f(c)f(d))$.

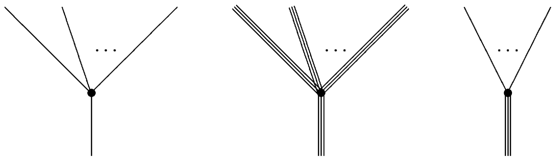



Of course in the category of associative monoids and monoid homomorphisms there is no need to distinguish the product $f(ab)(f(c)f(d))$ from $f(abcd)$. These diagrams were first introduced by Boardman and Vogt in [3] to help describe multiplication in (and morphisms of) topological monoids that are not strictly associative (and whose morphisms do not strictly respect that multiplication). The $n$th multiplihedron is a CW-complex whose vertices correspond to the unambiguous ways of multiplying and applying an $A_\infty$-map to $n$ ordered elements of an $A_\infty$-space. Thus the vertices correspond to the binary painted trees with $n$ leaves. The edges of the multiplihedra correspond to either an association $(ab)c \to a(bc)$ or to a preservation $f(a)f(b) \to f(ab)$. The associations can either be in the range: $(f(a)f(b))f(c) \to f(a)(f(b)f(c))$; or the image of a domain association: $f((ab)c) \to f(a(bc))$.

Here are the first few low-dimensional multiplihedra. The vertices are labeled, all but some of those in the last picture. There the bold vertex in the large pentagonal facet has label $((f(a)f(b))f(c))f(d)$ and the bold vertex in the small pentagonal facet has label $f(((ab)c)d)$. The others can be easily determined based on the fact that those two pentagons are copies of the associahedron $\mathcal{K}(4)$, that is to say all their edges are associations.

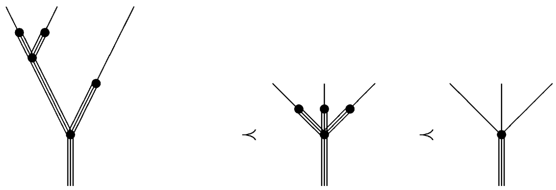



Faces of the multiplihedra of dimension greater than zero correspond to painted trees that are no longer binary. Here are the three new types of node allowed in a general painted tree. They correspond to the node types (1), (2) and (3) in that they are painted in similar fashion. They generalize types (1), (2), and (3) in that each has greater or equal valence than the corresponding earlier node type.



(4) (5) (6)

**Definition 1.** By *refinement* of painted trees we refer to the relationship: $t$ refines $t'$ means that $t'$ results from the collapse of some of the internal edges of $t$. This is a partial order on $n$-leaved painted trees, and we write $t \prec t'$. Thus the binary painted trees are refinements of the trees having nodes of type (4)–(6). *Minimal refinement* refers to the covering relation in this poset: $t$ minimally refines $t''$ means that $t$ refines $t''$ and also that there is no $t'$ such that both $t$ refines $t'$ and $t'$ refines $t''$. Here is an example of a chain in the poset of 3-leaved painted trees:

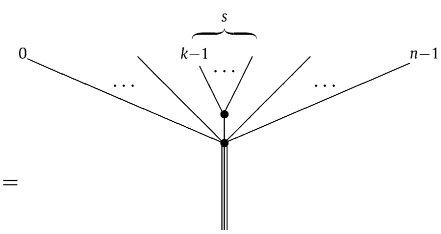


Iwase and Mimura, rather than stating a recursive definition of $\mathcal{J}(n)$, give a geometric definition of the CW-complex and then prove combinatorial facts about its faces. Here we reverse that order and use their theorems about facets as our definition.

Our recursive Definition 4 of the $n$th multiplihedron is stated by describing the type and number of the facets, or $(n-2)$-dimensional cells. Then the boundary of $\mathcal{J}(n)$ is given as the gluing together of these facets along $(n-3)$-dimensional cells with matching associated painted trees. Finally $\mathcal{J}(n)$ is defined as the cone on this boundary. It turns out that the faces can be indexed by, or labeled by, the painted trees in such a way that the face poset of the $n$th multiplihedron is equivalent to the face poset of the $n$-leaved painted trees. This recasting of the definition allows the two main goals of the current paper: to unite the viewpoints of [12] and [3], and to do so via a convex polytope realization.

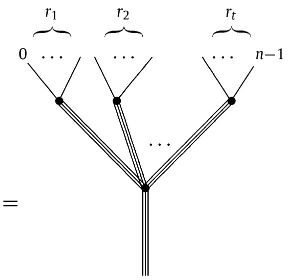
Recall that we refer to an unpainted tree with only one node as a corolla. A *painted corolla* is a painted tree with only one node, of type (6). A facet of the multiplihedron corresponds to a minimal refinement of the painted corolla: either a painted tree with only one, unpainted, interior edge, or to a tree with all its interior edges attached to a single painted node (type (2) or (5)).

**Definition 2.** A *lower tree* $l(k,s)$ is determined by a selection of $s$ consecutive leaves of the painted corolla, $1 < s \leqslant n$, which will be the leaves of the subtree which has the sole interior edge as its root edge.

$l(k,s) =$ 


To each lower tree corresponds a *lower facet* of the multiplihedron, which in [12] are denoted $\mathcal{J}_k(r,s)$ where $r = n+1-s$. Here $k-1$ numbers the first leaf of the $s$ consecutive leaves. In the complex $\mathcal{J}(n)$ defined in [12] the lower facet $\mathcal{J}_k(r,s)$ is a combinatorial copy of the complex $\mathcal{J}(r) \times \mathcal{K}(s)$.

**Definition 3.** The *upper trees* $u(t; r_1, \ldots, r_t)$ with all interior (necessarily painted) edges attached to a single painted node will appear thus:

$u(t; r_1, \ldots, r_t) =$ 


In [12] the corresponding *upper facets* are labeled $\mathcal{J}(t; r_1, \ldots, r_t)$. Here $t$ is the number of painted interior edges and $r_i$ is the number of leaves in the subtree supported by the $i$th interior edge. In the complex $\mathcal{J}(n)$ defined in [12] the upper facet $\mathcal{J}(t; r_1, \ldots, r_t)$ is a combinatorial copy of the complex $\mathcal{K}(t) \times \mathcal{J}(r_1) \times \cdots \times \mathcal{J}(r_t)$.

Here is a quick count of upper and lower facets, agreeing precisely with that given in [12].

**Theorem 1.** *(See [12].) The number of facets of the nth multiplihedron is*:

$$\frac{n(n-1)}{2} + 2^{(n-1)} - 1.$$

**Proof.** The number of lower trees is $\frac{n(n-1)}{2}$. This follows easily from summing the ways of choosing $s$ consecutive leaves. Note that this count includes one more than the count of the facets of the associahedron, since it includes the possibility of selecting all $n$ leaves.

The upper trees are determined by choosing any size $k$ proper subset of the "spaces between branches" of the painted corolla, $1 \leqslant k < n-1$. Each set of consecutive "spaces between branches" in that list of $k$ chosen spaces determines a set of consecutive leaves which will be the leaves of a subtree (that is itself a painted corolla) with its root edge one of the painted interior edges. If neither of the adjacent spaces to a given branch are chosen, its leaf will be the sole leaf of a subtree that is a painted corolla with only one leaf. Thus we count upper trees by $\sum_{k=0}^{n-2} \binom{n-1}{k} = 2^{(n-1)} - 1$. $\square$

The construction of the $n$th multiplihedron may be inductively accomplished by collecting its facets, and then labeling their faces. The following definition is identical to the properties demonstrated in [12].

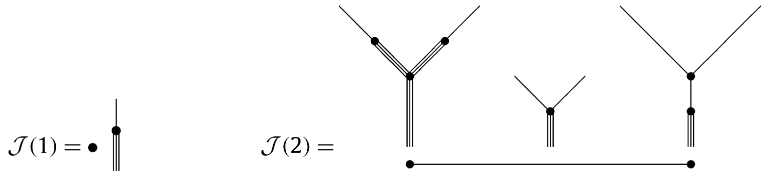
**Definition 4.** The first multiplihedron denoted $\mathcal{J}(1)$ is defined to be the single point $\{*\}$. It is associated to the painted tree with one leaf, and thus one type (3) internal node. Assume that the $\mathcal{J}(k)$ have been defined for $k = 1, \ldots, n-1$. To $\mathcal{J}(k)$ we associate the $k$-leaved painted corolla. We define an $(n-2)$-dimensional CW-complex $\partial\mathcal{J}(n)$ as follows, and then define $\mathcal{J}(n)$ to be the cone on $\partial\mathcal{J}(n)$. Now the top-dimensional cells of $\partial\mathcal{J}(n)$ (upper and lower facets of $\mathcal{J}(n)$) are in bijection with the set of upper and lower trees as described in Definitions 2 and 3, and have the respective product structures $\mathcal{K}(t) \times \mathcal{J}(r_1) \times \cdots \times \mathcal{J}(r_t)$ and $\mathcal{J}(r) \times \mathcal{K}(s)$.

Each sub-facet of an upper or lower facet is labeled with a tree that is a refinement of the upper or lower tree as follows: Since the facets are products, their sub-facets in turn are products of faces (of smaller associahedra and multiplihedra) whose dimensions sum to $n-3$. Each of these sub-facets thus comes (inductively) with a list of associated trees. There will always be a unique way of grafting the trees on this list to construct a painted tree that is a minimal refinement of the upper or lower tree associated to the facet in question. For the sub-facets of an upper facet the recipe is to paint entirely the $t$-leaved tree associated to a face of $\mathcal{K}(t)$ and to graft to each of its branches in turn the trees associated to the appropriate faces of $\mathcal{J}(r_1)$ through $\mathcal{J}(r_t)$ respectively. A sub-facet of the lower facet $\mathcal{J}_k(r, s)$ inductively comes with pair of trees. The recipe for assigning our sub-facet an $n$-leaved minimal refinement of the $n$-leaved minimal lower tree $l(k, s)$ is to graft the unpainted $s$-leaved tree to the $k$th leaf of the painted $r$-leaved tree.

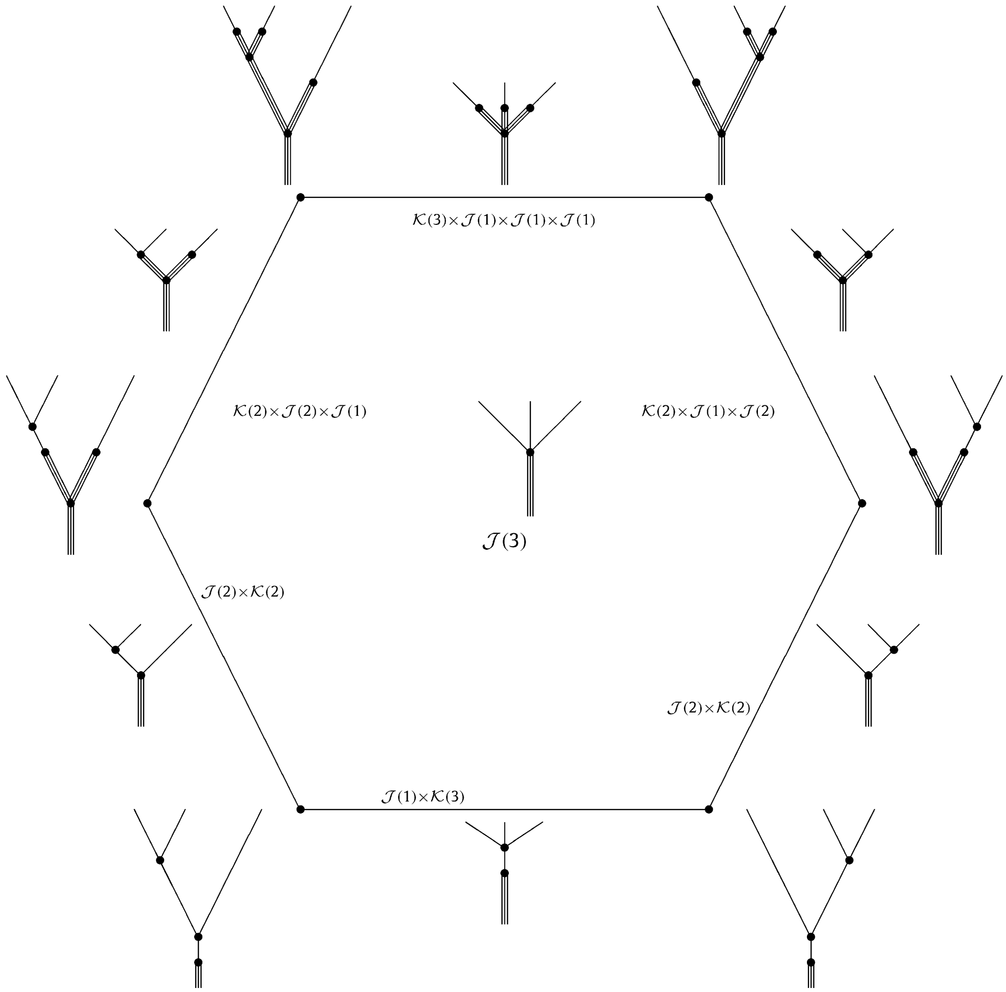
The intersection of two facets in the boundary of $\mathcal{J}(n)$ occurs along sub-facets of each which have associated painted trees that are identical. Then $\mathcal{J}(n)$ is defined to be the cone on $\partial\mathcal{J}(n)$. To $\mathcal{J}(n)$ we assign the painted corolla of $n$ leaves.

**Remark 1.** The listing of types and enumeration of facets above corresponds to properties (2-a) through (2-c) of [12]. The intersection of facets described in the definition corresponds to properties (c-1) through (c-4) in [12].

**Example 1.** We label the faces of the complexes with corresponding trees. Here is the point $\mathcal{J}(1)$ and the segment $\mathcal{J}(2)$, with the upper facet $\mathcal{K}(2) \times \mathcal{J}(1) \times \mathcal{J}(1)$ on the left and the lower facet $\mathcal{J}(1) \times \mathcal{K}(2)$ on the right:




And here is the complex $\mathcal{J}(3)$. The product structure of facets is listed. Notice how the sub-facets (vertices) are labeled. For instance, the upper right vertex is labeled by a tree that could be constructed by grafting three copies of the single leaf painted corolla onto a completely painted binary tree with three leaves, or by grafting a single leaf painted corolla and a 2-leaf painted binary tree onto the leaves of a 2-leaf (completely) painted binary tree.




## 3. Vertex combinatorics

Now for a new result about the counting of the binary painted trees with $n$ leaves.

**Theorem 2.** *The number of vertices $a_n$ of the nth multiplihedron is given recursively by*:

$$a_n = C(n-1) + \sum_{i=1}^{n-1} a_i a_{n-i}$$

*where $a_0 = 0$ and $C(n-1)$ are the Catalan numbers, which count binary (unpainted) trees as well as the vertices of the associahedron.*

**Proof.** The Catalan numbers $C(n-1)$ count those vertices which correspond to the painted binary trees with $n$ leaves which have only the root painted, that is only nodes of type (1) and (3). Now we count the trees for which the initial (lowest) trivalent node is painted (type (2)). Each of these consists of a choice of two painted binary subtrees whose root is the initial painted node, and whose leaves must sum to $n$. Thus we sum over the ways that $n$ can be split into two natural numbers. □

**Remark 2.** This formula gives the sequence which begins:

$$0, 1, 2, 6, 21, 80, 322, 1348, 5814, \ldots.$$

It is sequence A121988 of the On-line Encyclopedia of integer sequences. The recursive formula above yields the equation

$$A(x) = xc(x) + \big(A(x)\big)^2$$

where $A(x)$ is the ordinary generating function of the sequence $a_n$ above and $c(x)$ is the generating function for the Catalan numbers $C(n)$. (So $xc(x)$ is the generating function for the sequence $\{C(n-1)\}_{n=0}^{\infty}$.) Recall that $c(x) = \frac{1-\sqrt{1-4x}}{2x}$. Thus by use of the quadratic formula we have

$$A(x) = \frac{1 - \sqrt{2\sqrt{1-4x} - 1}}{2}.$$

It is not hard to check that therefore $A(x) = xc(x)c(xc(x))$. The Catalan transform of a sequence $b_n$ with generating function $B(x)$ is defined in [1] as the sequence with generating function $B(xc(x))$. Since $xc(x)$ is the generating function of $C(n-1)$ then the number of vertices of the $n$th multiplihedron is given by the Catalan transform of the Catalan numbers $C(n-1)$. Thus the theorems of [1] apply, for instance: a formula for the number of vertices is given by

$$a_n = \frac{1}{n}\sum_{k=1}^{n}\binom{2n-k-1}{n-1}\binom{2k-2}{k-1}; \qquad a_0 = 0.$$

We note that $A(x) = B(x)c(B(x))$ for $B(x) = xc(x)$. It may be that taking a generating function $B(x)$ to the new one given by $B(x)c(B(x))$ is the definition of a new kind of Catalan transform that would be interesting to study in its own right.

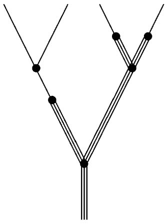
## 4. An algorithm for the extremal points

In [15] Loday gives an algorithm for taking the binary trees with $n$ leaves and finding for each an extremal point in $\mathbf{R}^{n-1}$; together whose convex hull is $\mathcal{K}(n)$, the $(n-2)$-dimensional associahedron. Note that Loday writes formulas with the convention that the number of leaves is $n+1$, where we instead always use $n$ to refer to the number of leaves. Given a (non-painted) binary $n$-leaved tree $t$, Loday arrives at a point $M(t)$ in $\mathbf{R}^{n-1}$ by calculating a coordinate from each trivalent node. These are ordered left to right based upon the ordering of the leaves from left to right. Following Loday we number the leaves $0, 1, \ldots, n-1$ and the nodes $1, 2, \ldots, n-1$. The $i$th node is "between" leaf $i-1$ and leaf $i$ where "between" might be described to mean that a rain drop falling between those leaves would be caught at that node. Each trivalent node has a left and right branch, each of which supports a subtree. To find the Loday coordinate for the $i$th node we take the product of the number of leaves of the left subtree ($l_i$) and the number of leaves of the right subtree ($r_i$) for that node. Thus $M(t) = (x_1, \ldots, x_{n-1})$ where $x_i = l_i r_i$. Loday proves that the convex hull of the points thus calculated for all $n$-leaved binary trees is the $n$th associahedron. He also shows that the points thus calculated all lie in the $(n-2)$-dimensional affine hyperplane $H$ given by the equation $x_1 + \cdots + x_{n-1} = S(n-1) = \frac{1}{2}n(n-1)$.

We adjust Loday's algorithm to apply to painted binary trees as described above, with only nodes of type (1), (2), and (3), by choosing a number $q \in (0,1)$. Then given a painted binary tree $t$ with $n$ leaves we calculate a point $M_q(t)$ in $\mathbf{R}^{n-1}$ as follows: we begin by finding the coordinate for each trivalent node from left to right given by Loday's algorithm, but if the node is of type (1) (unpainted, or colored by the domain) then its new coordinate is found by further multiplying its Loday coordinate by $q$. Thus

$$M_q(t) = (x_1, \ldots, x_{n-1}) \quad \text{where } x_i = \begin{cases} q l_i r_i, & \text{if node } i \text{ is type (1)}, \\ l_i r_i, & \text{if node } i \text{ is type (2)}. \end{cases}$$

Note that whenever we speak of the numbered nodes ($1, \ldots, n-1$ from left to right) of a binary tree, we are referring only to the trivalent nodes, of type (1) or (2). For an example, let us calculate the point in $\mathbf{R}^3$ which corresponds to the 4-leaved tree:

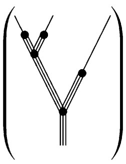
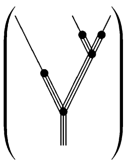
$t =$ 

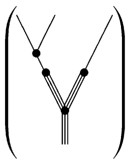
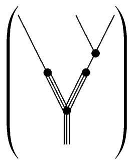
Now $M_q(t) = (q, 4, 1)$.

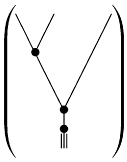
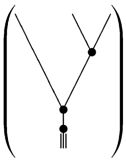
**Theorem 3.** *The convex hull of all the resulting points $M_q(t)$ for $t$ in the set of $n$-leaved binary painted trees is the $n$th multiplihedron. That is, our convex hull is combinatorially equivalent to the CW-complex $\mathcal{J}(n)$ defined by Iwase and Mimura, and is homeomorphic to the space of level (painted) trees defined by Boardman and Vogt.*

The proof will follow in Section 6.

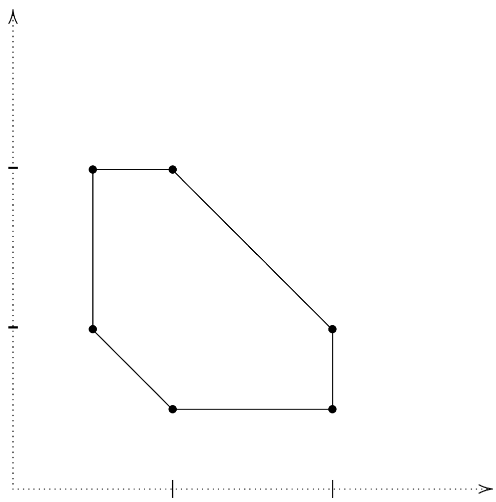
**Example 2.** Here are all the painted binary trees with 3 leaves, together with their points $M_q(t) \in \mathbf{R}^2$.

$M_q$  $= (1, 2)$, $M_q$  $= (2, 1)$

$M_q$  $= (q, 2)$ $M_q$  $= (2, q)$

$M_q$  $= (q, 2q)$, $M_q$  $= (2q, q)$

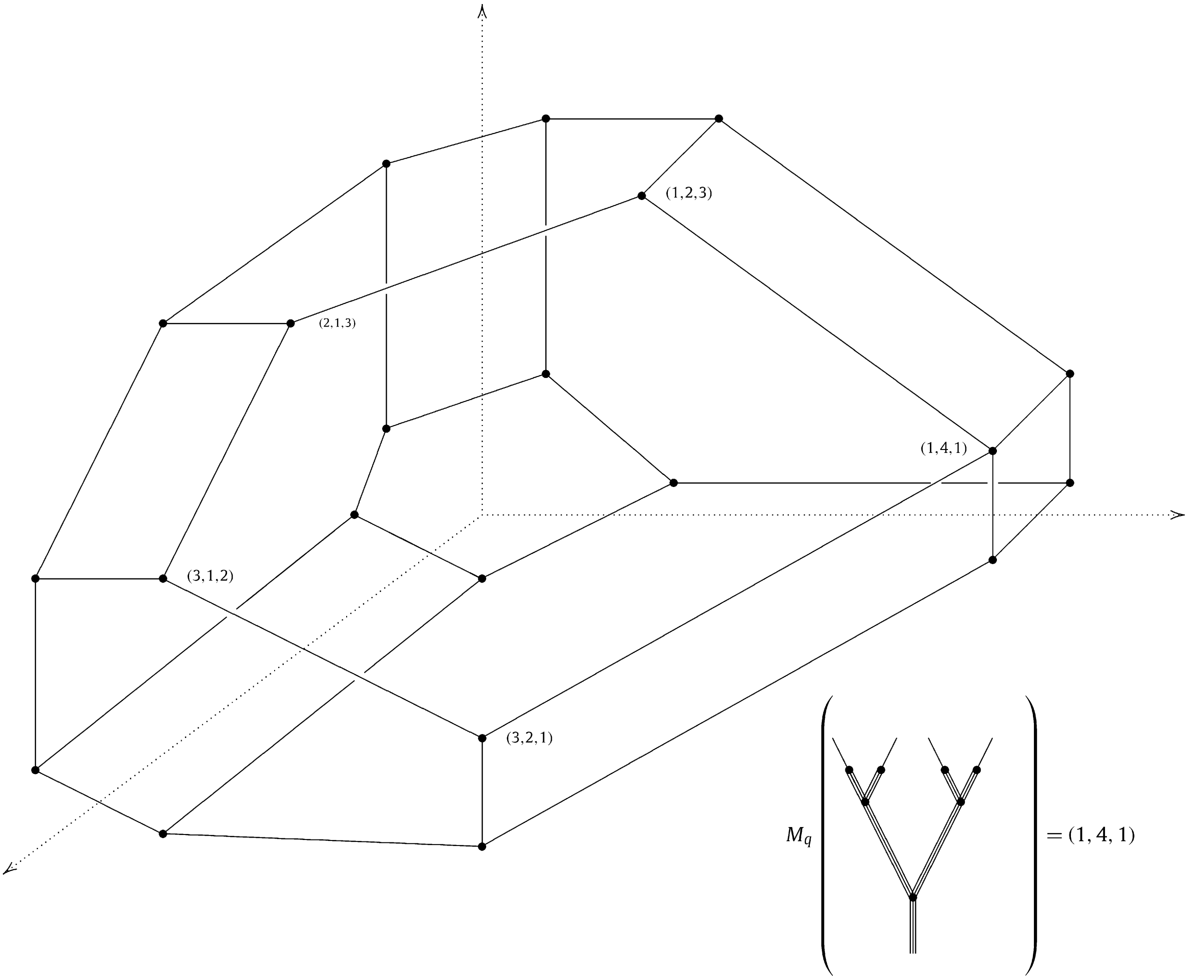
Thus for $q = \frac{1}{2}$ we have the six points $\{(1, 2), (2, 1), (\frac{1}{2}, 2), (2, \frac{1}{2}), (\frac{1}{2}, 1), (1, \frac{1}{2})\}$. Their convex hull appears as follows:



**Example 3.** The list of vertices for $\mathcal{J}(4)$ based on painted binary trees with 4 leaves, for $q = \frac{1}{2}$, is:

| | | | | |
|---|---|---|---|---|
| $(1,2,3)$ | $(1/2,2,3)$ | $(1/2,2/2,3)$ | $(1/2,2/2,3/2)$ | |
| $(2,1,3)$ | $(2,1/2,3)$ | $(2/2,1/2,3)$ | $(2/2,1/2,3/2)$ | |
| $(3,1,2)$ | $(3,1/2,2)$ | $(3,1/2,2/2)$ | $(3/2,1/2,2/2)$ | |
| $(3,2,1)$ | $(3,2,1/2)$ | $(3,2/2,1/2)$ | $(3/2,2/2,1/2)$ | |
| $(1,4,1)$ | $(1/2,4,1)$ | $(1,4,1/2)$ | $(1/2,4,1/2)$ | $(1/2,4/2,1/2)$ |

These are suggestively listed as a table where the first column is made up of the coordinates calculated by Loday for $\mathcal{K}(4)$, which here correspond to trees with every trivalent node entirely painted. The rows may be found by applying the factor $q$ to each coordinate in turn, in order of increasing size of those coordinates. Here is the convex hull of these points, where we see that each row of the table corresponds to shortest paths from the big pentagon to the small one. Of course sometimes there are multiple such paths.




**Remark 3.** The largest pentagonal facet of this picture corresponds to the bottom pentagonal facet in the drawing of $\mathcal{J}(4)$ on p. 53 of [28], and to the pentagonal facet labeled $d_{(0,1)}$ in the diagram of $\mathcal{J}(4)$ in Section 5 of [25]. Just turn the page 90 degrees clockwise to see the picture of $\mathcal{J}(4)$ that is Fig. 1(a) of this paper. The $n$th multiplihedron as a complex can be seen as a subdivision of the complex $\mathcal{K}(n) \times I$. Indeed the drawing of $\mathcal{J}(4)$ in [28] appears as a pentagonal cylinder. The drawing in Fig. 1(a) of this paper can be seen as a view of that cylinder from below. In [26] the authors give an alternative definition of $\mathcal{J}(n)$ based on the subdivision of the cylinder with $\mathcal{K}(n)$ base.

**Remark 4.** Recall that $\mathcal{P}(n)$ is defined as the convex hull of all the permutations of the point $(1,2,\ldots,n)$ in $\mathbf{R}^n$. In [25], the authors review the cellular projection from the permutohedron $\mathcal{P}(n)$ to the associahedron $\mathcal{K}(n-1)$ first described in [29]. They factor this projection through a new cellular projection $\pi : \mathcal{P}(n) \to \mathcal{J}(n)$, generated by identifying certain level trees

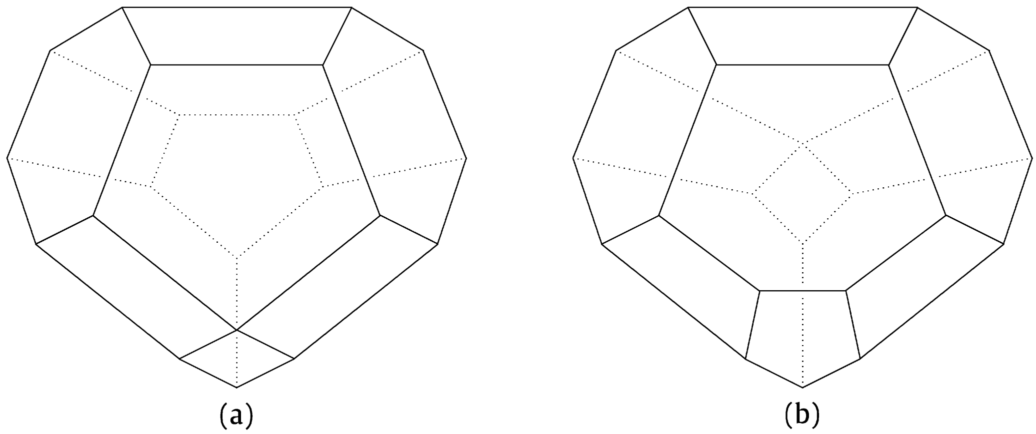



**Fig. 1.** The main character: Two views of the 3-d multiplihedron $\mathcal{J}(4)$.

labeling the faces of $\mathcal{P}(n)$. It may be possible to find a new convex hull realization of the multiplihedron $\mathcal{J}(n)$, this time included within a hyperplane of $\mathbf{R}^n$, by use of this projection $\pi$. Conjecturally a set of identified vertices of $\mathcal{P}(n)$ (that is, a preimage of $\pi$ restricted to 0-cells) might be replaced by a single appropriate point in $\mathbf{R}^n$ in order to produce the vertices of $\mathcal{J}(n)$.

To see a rotatable version of the convex hull which is the fourth multiplihedron, enter the following homogeneous coordinates into the Web Demo of polymake (with option visual), at http://www.math.tu-berlin.de/polymake/index.html#apps/polytope. Indeed polymake was instrumental in the experimental phase of this research [8].

*POINTS*

1 1 2 3

1 1/2 2 3

1 1/2 2/2 3

1 1/2 2/2 3/2

1 2 1 3

1 2 1/2 3

1 2/2 1/2 3

1 2/2 1/2 3/2

1 3 1 2

1 3 1/2 2

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1 3 2 1

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1 3 2/2 1/2

1 3/2 2/2 1/2

1 1 4 1

1 1/2 4 1

1 1 4 1/2

1 1/2 4 1/2

1 1/2 4/2 1/2

## 5. Spaces of painted trees

Boardman and Vogt develop several homotopy equivalent versions of the space of $n$-leaved painted trees. We choose to focus on one version which has the advantage of reflecting the intuitive dimension of the multiplihedra. The points of this space are based on the binary painted trees with the three types of nodes pictured in the introduction. The leaves are always colored by the domain $X$ (here we say unpainted), and the root is always colored by the range, $Y$ (here we say painted).

To get a point of the space each interior edge of a given binary painted tree with $n$ leaves is assigned a value in $[0, 1]$. The result is called a *painted metric tree*. When none of the trivalent nodes are painted (that is, disallowing the second node type), and with the equivalence relations we will review shortly, this will become the space $SM\mathcal{U}(n, 1)$ as defined in [3]. Allowing all three types of nodes gives the space

$$HW(\mathcal{U} \otimes \mathcal{L}_1)\big(n^0, 1^1\big).$$

(In [3] the superscripts denote the colors, so this denotes that there are $n$ inputs colored "0" and one output colored "1". This is potentially confusing since these numbers are also used for edge lengths, and so in this paper we will denote coloring with the shaded edges and reserve the values to denote edge lengths.)

We want to consider the retract of this space to the *level trees*, denoted in [3]

$$LW(\mathcal{U} \otimes \mathcal{L}_1)\big(n^0, 1^1\big).$$

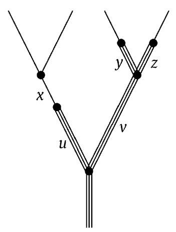
The definition in [3] simply declares that a level tree is either a tree that has one or zero nodes, or a tree that decomposes into level trees. The authors then unpack the definition a bit to demonstrate that the effect of their recursive requirement is to ensure that the space of 2-leaved level trees has dimension 1. They declare in general that their space of $n$-leaved level trees will have the expected form, that is, will be homeomorphic to a closed $(n-1)$-dimensional ball. We give here a specific way to realize a space of trees satisfying the recursive requirement and having the expected form. Again the requirement will ensure that a decomposition of level trees will always be into level trees.

We will denote our version of the space of level trees with $n$ leaves by $LW\mathcal{U}(n)$. It is defined in Definition 6 as the space of painted metric trees, after introducing relations on the lengths of edges.

**Definition 5.** We first describe a space corresponding to each painted binary tree $t$. We denote it $W(t)$. A point in $W(t)$ is an assignment of edge lengths to the interior edges of $t$. Edge lengths can be chosen freely from the interval $[0, 1]$ subject to the following conditions. At each trivalent node of a tree $t$ there are two subtrees with their root that node. The left subtree is defined by the tree with its rooted edge the left-hand branch of that node and the right subtree is likewise supported by the right-hand branch. The conditions are that for each node of type (2) we have an equation relating the painted interior edge lengths of the left subtree and the right subtree (interior with respect to the original $t$). Let $u_1, \dots, u_k$ be the lengths of the painted interior edges of the left subtree and let $v_1, \dots, v_j$ be the painted lengths of the right subtree. Let $p_u$ be the number of leaves of the left subtree and let $p_v$ be the number of leaves of the right subtree. The equation to be obeyed is

$$\frac{1}{p_u}\sum_{i=1}^{k} u_i = \frac{1}{p_v}\sum_{i=1}^{j} v_i.$$

For example consider the edge lengths $u, v, x, y, z \in [0, 1]$ assigned to the following tree:




The relations on the lengths then are the equations:

$$y = z \quad \text{and} \quad \frac{1}{2}u = \frac{1}{2}(v + y + z).$$

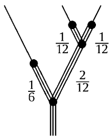
Note that this will sometimes imply that lengths of certain edges are forced to take values only from $[0, p]$, $p < 1$. In [3] the definition of the level trees is given by an inductive property, which guarantees that decompositions of the trees will always be into level trees. This seems equivalent to our requirement that the nodes be of types (1)–(6). The relations on edge length serve to ensure that this requirement is preserved even as some edges go to zero.

Before describing how to glue together all these subspaces for different trees to create the entire $LW\mathcal{U}(n)$ we show the following:

**Theorem 4.** *The dimension of the subspace $W(t)$ of $LW\mathcal{U}(n)$ corresponding to a given binary painted tree is $n-1$.*

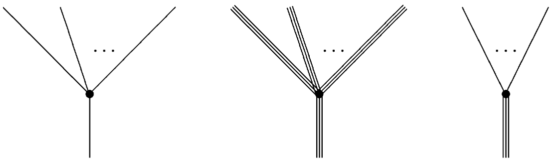
**Proof.** After assigning variables to the internal edges and applying the relations, the total number of free variables is at least the number of interior edges less the number of painted, type (2), nodes. This difference is always one less than the number of leaves. To see that the constraining equations really do reduce the number of free variables to $n-1$, notice what the equations imply about the painted interior edge lengths (the unpainted edge lengths are all free variables.) Beginning at the painted nodes which are closest to the leaves and setting equal to zero one of the two branches (a free variable) at each node it is seen that all the painted interior edge lengths are forced to be zero. Thus each painted node can only contribute one free variable—the other branch length must be dependent. Therefore, given a painted binary tree with $n$ leaves and $k$ internal edges, the space of points corresponding to the allowed choices for the edge values of that tree is the intersection of an $(n-1)$-dimensional subspace of $\mathbf{R}^k$ with $[0,1]^k$. We see this simply by solving the system of homogeneous equations indicated by the type (2) nodes and restricting our solution to the lengths in $[0,1]$.

In fact, the intersection just described is an $(n-1)$-dimensional polytope in $\mathbf{R}^k$. We see that this is true since there is a point in the intersection for which each of the coordinates is in the range $(0,\frac{1}{2}]$. To see an example of such a point we consider edge lengths of our binary tree such that the unpainted edges each have length $\frac{1}{2}$ and such that the painted edges have lengths in $(0,\frac{1}{2}]$. To achieve the latter we begin at the first painted type (2) node above the root, and consider the left and right subtrees. If the left subtree has only one painted edge we assign that edge the length $\frac{p}{2n}$ where $p$ is the number of leaves of the left subtree; but if not then we assign the root edge of the left subtree the length $\frac{p}{4n}$. We do the same for the right subtree, replacing $p$ with the number of leaves of the right subtree. This proceeds inductively up the tree. At a given type (2) node if its left/right $p'$-leaved subtree has only one painted edge we assign that edge the length $\frac{p'}{d}$ where $d$ is the denominator of the length assigned to the third edge (closest to the root) of the that node on the previous step; but if not then we assign the root edge of the left/right subtree the length $\frac{p'}{2d}$. This produces a set of non-zero lengths which obey the relations and are all $\leqslant \frac{1}{2}$. For example:




□

To describe the equivalence relations on our space we recall the trees with three additional allowed node types. They correspond to the node types (1), (2) and (3) in that they are painted in similar fashion.



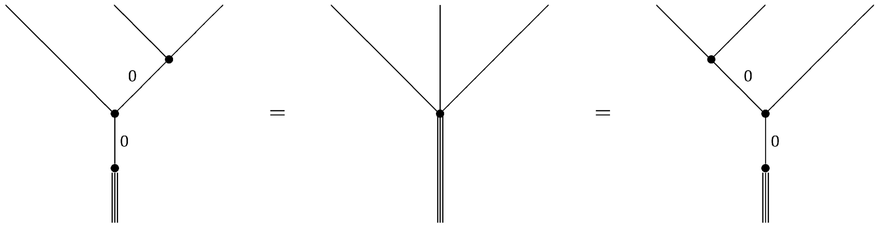
(4) (5) (6)

These nodes each have subtrees supported by each of their branches in order from left to right. The interior edges of each tree are again assigned lengths in $[0,1]$. The requirements on edge lengths which we get from each node of type (5) of valence $j+1$ are the equalities:

$$\frac{1}{p_1}\sum_{i=1}^{k_1} u_{1i} = \frac{1}{p_2}\sum_{i=1}^{k_2} u_{2i} = \cdots = \frac{1}{p_j}\sum_{i=1}^{k_j} u_{ji}$$

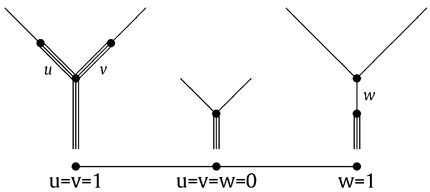
where $k_1,\ldots,k_j$ are the numbers of painted internal edges of each of the $j$ subtrees, and $p_1 \ldots p_j$ are the numbers of leaves of each of the subtrees. Now we review the equivalence relation on trees introduced in [3].

**Definition 6.** The space of painted metric trees with $n$ leaves $LW\mathcal{U}(n)$ is formed by first taking the disjoint union of the $(n-1)$-dimensional polytopes $W(t)$, one polytope for each binary painted tree. Then it is given the quotient topology (of the standard topology of the disjoint union of the polytopes in $\mathbf{R}^k$) under the following equivalence relation: Two trees are equivalent if they reduce to the same tree after completely collapsing their respective edges of length zero. This is why we call the variable assigned to interior edges "length" in the first place. By "same tree" we mean possessing the same painted tree structure and having the same lengths assigned to corresponding edges. For example one pair of equivalence relations appears as follows:

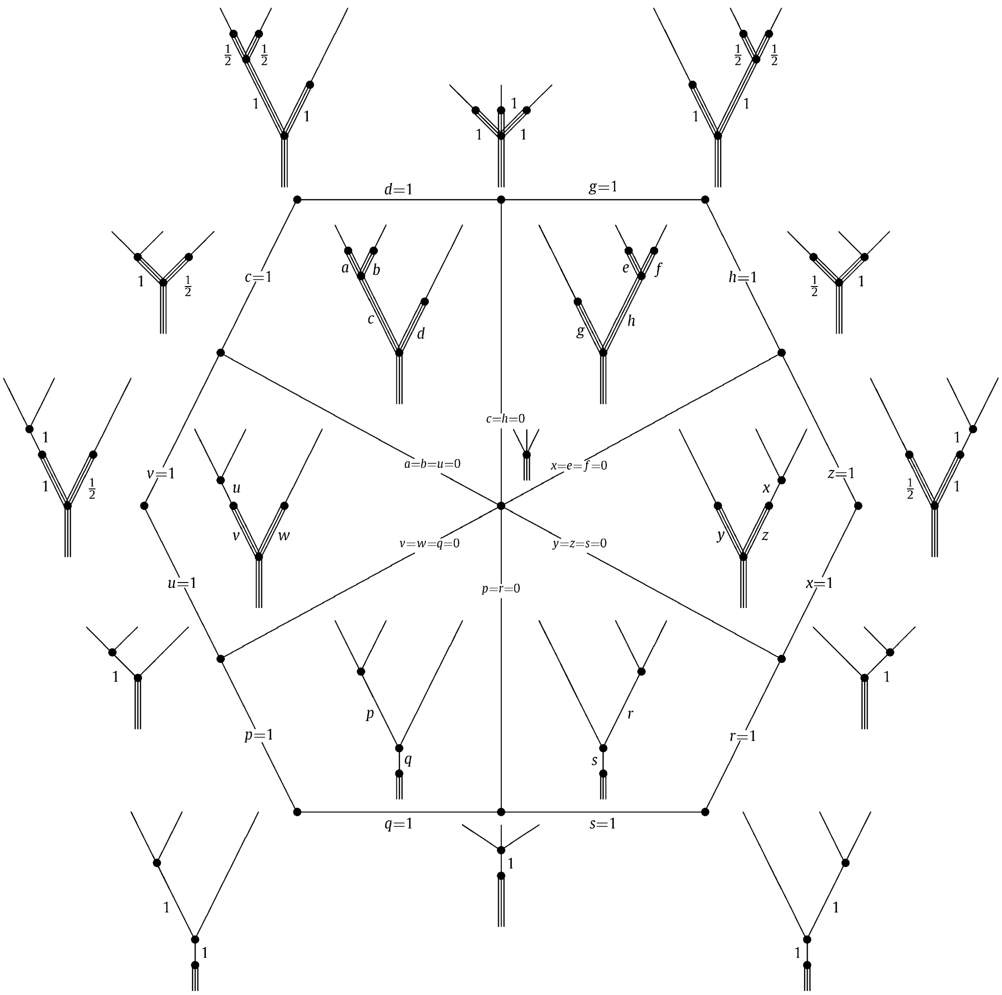



Note that an equivalence class of trees may always be represented by any one of several binary trees, with only nodes of type (1), (2), and (3), since we can reduce the valence of nodes within an equivalence class by introducing extra interior edges of length zero. However we often represent the equivalence class with the unique tree that shows no zero edges. We refer to this as the collapsed tree. Also note that the relations on the variable lengths of a tree which has some of those lengths set to zero are precisely the relations on the variables of the collapsed tree equivalent to it.

**Example 4.** $LW\mathcal{U}(1)$ is just a single point. Here is the space $LW\mathcal{U}(2)$, where we require $u = v$:



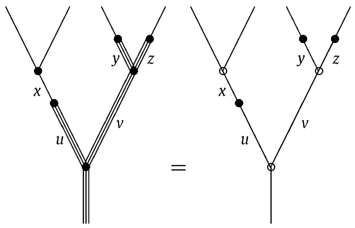

And here is the space $LW\mathcal{U}(3)$:



Note that the equations which the variables in $LW\mathcal{U}(3)$ must obey are:

$$a = b \quad \text{and} \quad d = \frac{1}{2}(a + b + c),$$
$$e = f \quad \text{and} \quad g = \frac{1}{2}(e + f + h),$$
$$w = \frac{1}{2}v \quad \text{and} \quad y = \frac{1}{2}z.$$

In [22] the space of painted metric trees (bicolored metric ribbon trees) is described in a slightly different way. First, the trees are not drawn with painted edges, but instead the nodes of type (3) are indicated by color, and the edges between the root and those nodes can be assumed to be painted. The correspondence is clear: for example,




Secondly, the relations required of the painted lengths are different. In [22] it is required that the sum of the painted lengths along a path from the root to a leaf must always be the same. For example, for the above tree, the new relations obeyed in [22] are $u = v + y = v + z$. This provides the same dimension of $n - 1$ for the space associated to a single binary tree with $n$ leaves as found in Theorem 4 in this paper.

Thirdly, the topology on the space of painted metric trees with $n$ leaves is described by first assigning lengths in $(0, \infty)$ and then defining the limit as some lengths in a given tree approach 0 as being the tree with those edges collapsed. This topology clearly is equivalent to the definition as a quotient space given here and in [3]. Thus we can use the results of [22] to show the following:

**Lemma 1.** *The space $LW\mathcal{U}(n)$ is homeomorphic to the closed ball in $\mathbf{R}^{n-1}$.*

**Proof.** (1) In [22] it is shown that the entire space of painted trees with $n$ leaves with lengths in $[0, \infty)$ is homeomorphic to $\mathbf{R}_+^{n-1} \cup \mathbf{0}$. (This is done via a homeomorphism to the space of quilted disks.) Thus if the lengths are restricted to lie in $[0, 1]$ then the resulting space is homeomorphic to the closed ball in $\mathbf{R}^{n-1}$. $\square$

However, we think it is valuable to see how the homeomorphism from the entire space of trees to the convex polytope might actually be constructed piecewise from smaller homeomorphisms based on specific $n$-leaved trees.

**Proof.** (2) We will use the Alexander trick, which is the theorem that states that any homeomorphism from the bounding sphere of one disk to another bounding sphere of a second disk may be extended to the entire disks. We are using this to construct a homeomorphism $\varphi$ from the convex hull realization of $\mathcal{J}(n)$ to $LWU(n)$. First we consider the barycentric subdivision of the former $(n-1)$-dimensional polytope. Recalling that each face of $\mathcal{J}(n)$ is associated with a specific painted $n$-leaved tree $t$, we associate that same tree to the respective barycenter denoted $v(t)$.

We will be creating $\varphi$ inductively. We begin by defining it on the crucial barycenters. The barycenter of the entire polytope $\mathcal{J}(n)$ is associated to the painted corolla, and should be mapped to the equivalence class represented by the corolla—that is, the class of trees with all zero length interior edges.

The barycenters of facets of $\mathcal{J}(n)$ are each associated to a lower or upper tree. Since the relations on variable edge lengths are preserved by collapsing zero edges, we can see that each of these facet trees correspond to a one-dimensional subset of the space of metric trees. Upper trees have one fewer relation than the number of painted interior edges (and no other interior edges) while lower trees have a single interior edge. The barycenters of lower facets are mapped to the class represented by their respective tree with edge length 1. The barycenters of upper facets are mapped to the class represented by their respective trees with maximal edge lengths. The maximal lengths are found by choosing an edge with maximal valence type (6) node, and assigning length 1 to that edge. The other lengths are then determined. Examples of this are shown by the facets of the hexagon that is $LWU(3)$ above.

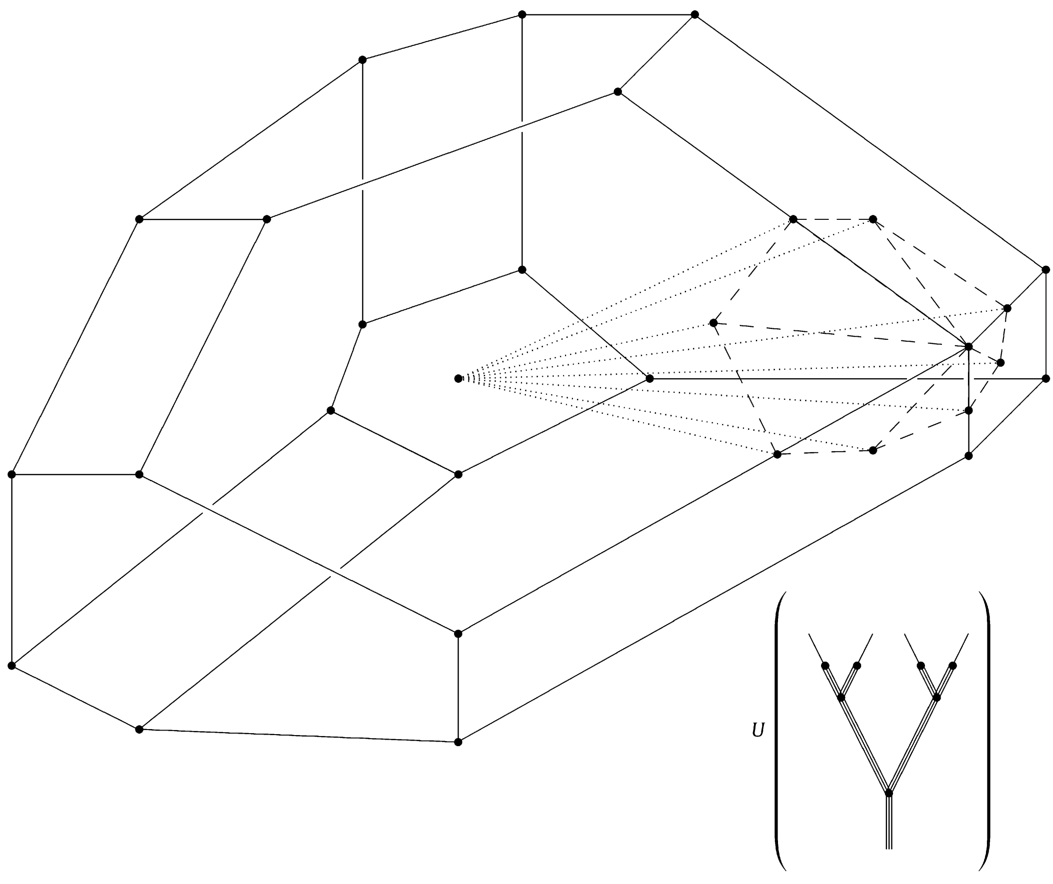
Now consider a particular binary painted tree $t$, associated to a vertex $v(t) = M_q(t)$ of $\mathcal{J}(n)$. The simplicial complex made up of all the simplices in the barycentric subdivision which contain $v(t)$ we denote $U(t)$. $U(t)$ is spanned by the vertices $v(t')$ for all $t' \prec t$. Recall that $t' \prec t$ denotes that $t'$ refines $t$, which means that $t$ results from the collapse of some of the internal edges of $t'$. $U(t)$ is homeomorphic to the $(n-1)$-disk. Next we will extend our choice of images of the facet barycenters for facets adjacent to $v(t)$ to a homeomorphism $\varphi_t : U(t) \to W(t)$. This extension will be done incrementally where the increments correspond to the refinement of trees, so that the piecewise defined mapping $\varphi(x) = \varphi_t(x)$; $x \in U(t)$ (with one piece defined on $U(t)$ for each binary $n$-leaved $t$) will be well defined, 1-1, and onto $LWU(n)$. $U(t)$ for a particular 4-leaved tree is pictured as a subset of the convex hull realization of $\mathcal{J}(4)$ just following this proof.

The incremental construction of our homeomorphism $\varphi_t$ is by way of subdividing the respective boundaries of $U(t)$ and $W(t)$ based upon tree refinement. For each tree $t' \prec t$, let $p$ be the number of free variables in the metric version of $t'$ (so $n-(p+1)$ is the dimension of the face associated to $t'$), and define $U(t')$ to be the sub-complex of $p$-simplices of $U(t)$ spanned by $v(t')$ and all the $v(t'')$ for $t'' \prec t'$. $U(t')$ is a $p$-disk by construction. Also define $W(t')$ to be the subspace of the boundary of $W(t)$ given by all those equivalence classes which can be represented by a metric version of $t'$, with interior edge lengths in $[0, 1]$. By a parallel argument to Theorem 4 $W(t')$ is also a $p$-disk.

To establish the base case we consider a facet barycenter (with associated tree $t' \prec t$). The barycenter $v(t')$ and the barycenter of $\mathcal{J}(n)$ form a copy of $S^0$ bounding the 1-simplex $U(t')$. Now the 1-dimensional subset $W(t')$ of the boundary of $W(t)$ is made up of equivalence classes of trees represented by metric versions of $t'$. The boundary of this 1-disk is the copy of $S^0$ given by the tree with all zero lengths and the tree with maximal length. Thus we can extend that choice of images made above to a homeomorphism $\varphi_{t'}$ of the 1-disks for each $t'$.

For an arbitrary tree $t'$ the boundary of $U(t')$ is a $(p-1)$-spherical simplicial complex that is made up of two $(p-1)$-disks. The first *interior* disk is the union of $U(t'')$ for $t'' \prec t'$. Each $(p-1)$-simplex in this first disk contains the barycenter of $\mathcal{J}(n)$. Each $(p-1)$-simplex in the second *exterior* disk contains $v(t)$. The shared boundary of the two disks is a $(p-2)$-sphere. The boundary of $W(t')$ is also made up of two $(p-1)$-disks. The first disk is the union of $W(t'')$ for $t'' \prec t'$. The second disk is the collection of equivalence classes of metric trees represented by $t'$ with at least one edge set equal to 1. Now we can build $\varphi_t$ inductively by assuming it to be defined on the disks: $U(t'') \to W(t'')$ for all trees $t'' \prec t'$. This assumed mapping may then be restricted to a homeomorphism of the $(p-2)$-spheres that are the respective boundaries of the interior disks, which in turn can then be extended to the exterior disks and thus the entire $(p-1)$-spherical boundaries of $U(t')$ and $W(t')$. From there the homeomorphism can be extended to the entire $p$-disks: $U(t') \to W(t')$. This continues inductively until, after the last extension, the resulting homeomorphism is called $\varphi_t : U(t) \to W(t)$.

Now by construction the map $\varphi : \mathcal{J}(n) \to LWU(n)$ given by $\varphi(x) = \varphi_t(x)$; $x \in U(t)$ is well defined, continuous, bijective and open.




□

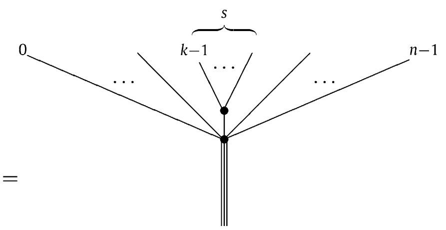
## 6. Proof of Theorem 3

To demonstrate that our convex hulls are each combinatorially equivalent to the corresponding convex CW-complexes defined by Iwase and Mimura, we need only check that they both have the same vertex–facet incidence. We will show that for each $n$ there is an isomorphism $f$ between the vertex sets (0-cells) of our convex hull and $\mathcal{J}(n)$ which preserves the sets of vertices corresponding to facets; i.e. if $S$ is the set of vertices of a facet of our convex hull then $f(S)$ is a vertex set of a facet of $\mathcal{J}(n)$.

To demonstrate the existence of the isomorphism, noting that the vertices of $\mathcal{J}(n)$ correspond to the binary painted trees, we only need to check that the points we calculate from those binary painted trees are indeed the vertices of their convex hull. The isomorphism implied is the one that takes a vertex associated to a certain tree to the 0-cell associated to the same tree. Now a given facet of $\mathcal{J}(n)$ corresponds to a tree $T$ which is one of the two sorts of trees pictured in Definitions 3 and 2. To show that our implied isomorphism of vertices preserves vertex sets of facets we need to show that for each $T$ there is one facet that is the convex hull of the points corresponding to the binary trees which are *refinements* of $T$. By refinement of painted trees we refer to the relationship: $t$ refines $t'$ if $t'$ results from the collapse of some of the internal edges of $t$. Note that the two sorts of trees pictured in Definitions 3 and 2 are each a single collapse away from being the painted corolla.

The proofs of both key points will proceed in tandem, and will be inductive. The main strategy will be to define a dimension $n-2$ affine hyperplane $H_q(T)$ in $\mathbf{R}^{n-1}$ for each of the upper and lower facet trees $T$ (as drawn in the Definitions 3 and 2), and then to show that these are the proper bounding hyperplanes of the convex hull (i.e. that each actually contains a facet). The definition of hyperplane will actually generalize our algorithm for finding a point $M_q(t)$ in $\mathbf{R}^{n-1}$ from a binary tree $t$ with $n$ leaves. The proof of Theorem 3 will however not use these hyperplanes directly, but recast them in a weighted version. Then they will be recovered when the weights are all set equal to 1.

**Definition 7.** The lower facets $\mathcal{J}_k(r,s)$ correspond to lower trees such as:

$l(k,s) =$ 


These are assigned a hyperplane $H_q(l(k,s))$ determined by the equation

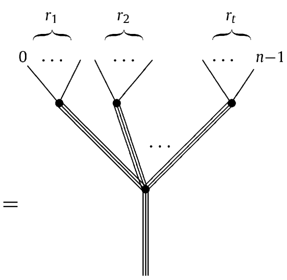
$$x_k + \cdots + x_{k+s-2} = \frac{q}{2}s(s-1).$$

Recall that $r$ is the number of branches extending from the lowest node, and $r+s=n+1$. Thus $1 \leqslant k \leqslant r$. Notice that if $s=n$ (so $r=k=1$) then this becomes the hyperplane given by

$$x_1 + \cdots + x_{n-1} = \frac{q}{2}n(n-1) = qS(n-1).$$

Therefore the points $M_q(t)$ for $t$ a binary tree with only nodes type (1) and (3) will lie in the hyperplane $H_q(l(1,n))$ by Lemma 2.5 of [15]. (Simply multiply both sides of the relation proven there by $q$.) Also note that for $q=1$ (thus disregarding the painting) that these hyperplanes are an alternate to the bounding hyperplanes of the associahedron defined by Loday using admissible shuffles. Our hyperplanes (for $q=1$) each have the same intersection with the hyperplane $H$ as does the corresponding hyperplane $H_\omega$ defined by Loday (for $\omega$ corresponding to the unpainted version of our tree $l(k,s)$.)

**Definition 8.** The upper facets $\mathcal{J}(t;r_1,\ldots,r_t)$ correspond to upper trees such as:

$u(t;r_1,\ldots,r_t) =$ 

These are assigned a hyperplane $H_q(u(t; r_1, \ldots, r_t))$ determined by the equation

$$x_{r_1} + x_{(r_1+r_2)} + \cdots + x_{(r_1+r_2+\cdots+r_{t-1})} = \frac{1}{2}\left(n(n-1) - \sum_{i=1}^{t} r_i(r_i - 1)\right)$$

or equivalently:

$$x_{r_1} + x_{(r_1+r_2)} + \cdots + x_{(r_1+r_2+\cdots+r_{t-1})} = \sum_{1\leqslant i<j\leqslant t} r_i r_j.$$

Note that if $t = n$ (so $r_i = 1$ for all $i$), this becomes the hyperplane given by

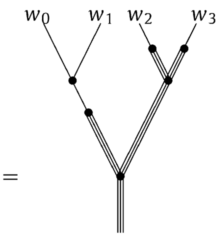
$$x_1 + \cdots + x_{n-1} = \frac{1}{2}n(n-1) = S(n-1).$$

Therefore the points $M_q(t)$ for $t$ a binary tree with only nodes type (2) and (3) will lie in the hyperplane $H$ by Lemma 2.5 of [15] (using notation $S(n)$ and $H$ as in that source).

In order to prove Theorem 3 it turns out to be expedient to prove a more general result. This consists of an even more flexible version of the algorithm for assigning points to binary trees in order to achieve a convex hull of those points which is the multiplihedron. To assign points in $\mathbf{R}^{n-1}$ to the binary painted trees with $n$ leaves, we not only choose a value $q \in (0, 1)$ but also an ordered $n$-tuple of positive integers $w_0, \ldots, w_{n-1}$. Now given a tree $t$ we calculate a point $M_q^w(t)$ in $\mathbf{R}^{n-1}$ as follows: we begin by assigning the weight $w_i$ to the $i$th leaf. We refer to the result as a weighted tree. Then we modify Loday's algorithm for finding the coordinate for each trivalent node by replacing the number of leaves of the left and right subtrees with the sums of the weights of the leaves of those subtrees. Thus we let $L_i = \sum w_k$ where the sum is over the leaves of the subtree supported by the left branch of the $i$th node. Similarly we let $R_i = \sum w_k$ where $k$ ranges over the leaves of the subtree supported by the right branch. Then

$$M_q^w(t) = (x_1, \ldots, x_{n-1}) \quad \text{where } x_i = \begin{cases} qL_iR_i, & \text{if node } i \text{ is type (1)}, \\ L_iR_i, & \text{if node } i \text{ is type (2)}. \end{cases}$$

Note that the original points $M_q(t)$ are recovered if $w_i = 1$ for $i = 0, \ldots, n-1$. Thus proving that the convex hull of the points $M_q^w(t)$ where $t$ ranges over the binary painted trees with $n$ leaves is the $n$th multiplihedron will imply the main theorem. For an example, let us calculate the point in $\mathbf{R}^3$ which corresponds to the 4-leaved tree:

$t =$ 


Now $M_q^w(t) = (qw_0w_1, (w_0 + w_1)(w_2 + w_3), w_2w_3)$. To motivate this new weighted version of our algorithm we mention that the weights $w_0, \ldots, w_{n-1}$ are to be thought of as the sizes of various trees to be grafted to the respective leaves. This weighting is therefore necessary to make the induction go through, since the induction is itself based upon the grafting of trees.

**Lemma 2.** *For $q = 1$ the convex hull of the points $M_q^w(t)$ for $t$ an $n$-leaved binary tree gives the $n$th associahedron.*

**Proof.** Recall that for $q = 1$ we can ignore the painting, and thus for $w_i = 1$ for $i = 0, \ldots, n-1$ the points we calculate are exactly those calculated by Loday's algorithm. Now for arbitrary weights $w_0, \ldots, w_{n-1}$ we can form from each weighted tree $t$ (with those weights assigned to the respective leaves) a non-weighted tree $t'$ formed by grafting a corolla with $w_i$ leaves onto the $i$th leaf of $t$. Note that for binary trees which are refinements of $t'$ the coordinates which correspond to the nodes of $t'$ below the grafting receive precisely the same value from Loday's algorithm which the corresponding nodes of the original weighted tree received from the weighted algorithm. Now since Loday's algorithm gives the vertices of the associahedra, then the binary trees which are refinements of $t'$ give the vertices of $\mathcal{K}(n) \times \mathcal{K}(w_0) \times \cdots \times \mathcal{K}(w_{n-1})$. If we restrict our attention in each entire binary refinement of $t'$ to the nodes of (the refinements of) the grafted corolla with $w_i$ leaves we find the vertices of $\mathcal{K}(w_i)$. The definition of a cartesian product of polytopes guarantees that the vertices of the product are points which are cartesian products of the vertices of the operands. Polytopes are also combinatorially invariant under change of basis, and so we can rearrange the coordinates of our vertices to put all the coordinates corresponding to the nodes of (the refinements of) the grafted corollas at the end of the point, leaving the coordinates corresponding to the nodes below the graft in order at the beginning of the point. Thus the nodes below the grafting correspond to the vertices of $\mathcal{K}(n)$, and so the weighted algorithm (with $q = 1$) does give the vertices of $\mathcal{K}(n)$. $\square$

**Lemma 3.** *For $q = 1$ the points $M_q^w(t)$ for $t$ an $n$-leaved binary tree all lie in the $(n-2)$-dimensional affine hyperplane of $\mathbf{R}^{n-1}$ given by the equation*:

$$x_1 + \cdots + x_{n-1} = \sum_{1 \leqslant i < j \leqslant (n-1)} w_i w_j.$$

**Proof.** In Lemma 2.5 of [15] it is shown inductively that when $w_i = 1$ for $i = 1, \ldots, n-1$ then the point $M_1^{1,\ldots,1}(t) = M(t) = (x_1, \ldots, x_{n-1})$ satisfies the equation $\sum_{i=1}^{n-1} x_i = \frac{1}{2}n(n-1)$. As in the proof of the previous lemma we replace the weighted tree $t$ with the non-weighted $t'$ formed by grafting an arbitrary binary tree with $w_i$ leaves to the $i$th leaf of $t$. Let $m = \sum_{i=1}^{n-1} w_i$. Thus the point $M_1^{1,\ldots,1}(t') = M(t') = (x_1, \ldots, x_m)$ satisfies the equation

$$\sum_{i=1}^{m-1} x_i = \frac{1}{2}m(m-1) = \frac{1}{2}\sum_{i=1}^{n-1} w_i \left(\sum_{i=1}^{n-1} w_i - 1\right).$$

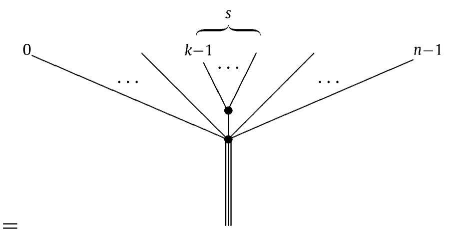
Also the coordinates corresponding to the nodes of the grafted tree with $w_i$ leaves sum up to the value $\frac{1}{2}w_i(w_i - 1)$. Thus the coordinates corresponding to the nodes below the graft, that is, the coordinates of the original weighted tree $t$, sum up to the difference:

$$\frac{1}{2}\left(\sum_{i=1}^{n-1} w_i \left(\sum_{i=1}^{n-1} w_i - 1\right) - \sum_{i=1}^{n-1} w_i(w_i - 1)\right) = \sum_{1 \leqslant i < j \leqslant (n-1)} w_i w_j. \qquad \square$$

Since we are proving that the points $M_q^w(t)$ are the vertices of the multiplihedron, we need to define hyperplanes $H_q^w(t)$ for this weighted version which we will show to be the bounding hyperplanes when $t$ is a facet tree.

**Definition 9.** Recall that the lower facets $\mathcal{J}_k(r, s)$ correspond to lower trees such as:

$l(k, s) =$




These are assigned a hyperplane $H_q^w(l(k, s))$ determined by the equation

$$x_k + \cdots + x_{k+s-2} = q\left(\sum_{(k-1) \leqslant i < j \leqslant (k+s-2)} w_i w_j\right).$$

Recall that $r$ is the number of branches from the lowest node, and $r + s = n + 1$.

**Lemma 4.** *For any painted binary tree $t$ the point $M_q^w(t)$ lies in the hyperplane $H_q^w(l(k, s))$ iff $t$ is a refinement of $l(k, s)$. Also the hyperplane $H_q^w(l(k, s))$ bounds below the points $M_q^w(t)$ for $t$ any binary painted tree.*

**Proof.** By Lemma 3 we have that any binary tree $t$ which is a refinement of the lower tree $l(k, s)$ will yield a point $M_q^w(t)$ which lies in $H_q^w(l(k, s))$. To see this we simply note that the nodes in $t$ associated to the coordinates $x_k, \ldots, x_{k+s-2}$ in $M_q^w(t)$ will each be of type (1), and so we multiply by $q$ on both sides of the equation proven in the lemma.

We now demonstrate that if a binary tree $t$ is not a refinement of a lower tree $l(k, s)$ then the point $M_q^w(t)$ will have the property that

$$x_k + \cdots + x_{k+s-2} > q\left(\sum_{(k-1) \leqslant i < j \leqslant (k+s-2)} w_i w_j\right).$$

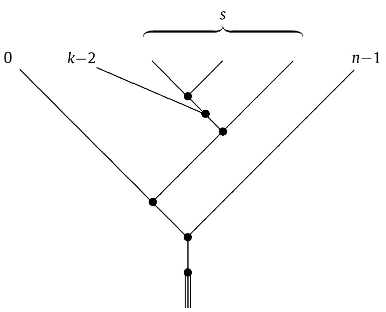
Recall that the trees which are refinements of $l(k, s)$ have all their nodes inclusively between $k$ and $k + s - 2$ of type (1). Now if $t$ has these same $s - 1$ nodes $k, \ldots, k + s - 2$ all type (1) and is not a refinement of $l(k, s)$ then there is no node in $t$ whose deletion results in the separation of only the leaves $k - 1, \ldots, k + s - 2$ from the rest of the leaves of $t$. Let $t'$ be the subtree of $t$ determined by taking as its root the node furthest from the root of $t$ whose deletion results in the separation of all the leaves $k - 1, \ldots, k + s - 2$ from the rest of the leaves of $t$. Thus $t'$ will have more than just those $s$ leaves, say those leaves of $t$ labeled $k - p, \ldots, k + p' - 2$ where $p \geqslant 1$, $p' \geqslant s$ and at least one of the inequalities strict. Since the situation is symmetric we just consider the case where $p' = s$ and $p > 1$. Then we have an expression for the sum of all the coordinates whose nodes are in $t'$ and can write:

$$x_k + \cdots + x_{k+s-2} = q\left(\sum_{(k-p)\leqslant i<j\leqslant(k+s-2)} w_i w_j\right) - q(x_{k-p+1} + \cdots + x_{k-1}). \tag{$*$}$$

Notice that the first sum on the right-hand side of $(*)$ contains

$$x_{k-p+1} + \cdots + x_{k-1} + \sum_{(k-1)\leqslant i<j\leqslant(k+s-2)} w_i w_j.$$

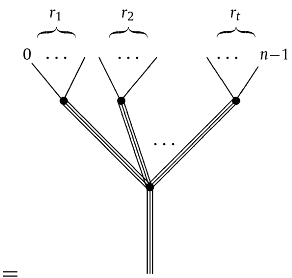
(There is no overlap between the coordinate values here and the sum since each of the terms in $x_{k-p+1} + \cdots + x_{k-1}$ contains a factor from $w_{k-p}, \ldots, w_{k-2}$.) The first sum on the right-hand side of $(*)$ also contains at least one term $w_m w_j$ where $(k-p) \leqslant m \leqslant (k-2)$ and where $w_m w_j$ does not occur as a term in $x_{k-p+1} + \cdots + x_{k-1}$, else the leaf labeled by $m$ would not lie in $t'$. Thus we have the desired inequality. Here is a picture of an example situation, where $p = 2$. Note that the key term $w_m w_j$ in the above discussion is actually $w_{k-2} w_{k+1}$ in this picture.

$t =$ 


Now if in the situation for which there does not exist a node of $t$ which if deleted would separate exactly the leaves $k-1, \ldots, k+s-2$ from the other leaves and root of $t$, there are also some of the nodes in $k, \ldots, k+s-1$ of type (2), the inequality still holds, and now to a greater degree since some of the factors of $q$ are missing from the right-hand side.

If there does exist a node of $t$ which if deleted would separate exactly the leaves $k-1, \ldots, k+s-2$ from the other leaves and root of $t$, but $t$ is not a refinement of $l(k, s)$ due to the painting (some of the nodes in $k, \ldots, k+s-1$ are of type (2)), then the inequality holds precisely because the only difference left to right is that the right-hand side has fewer terms multiplied by the factor of $q$. $\square$

**Definition 10.** Recall that the upper facets $\mathcal{J}(t; r_1, \ldots, r_t)$ correspond to upper trees such as:

$u(t; r_1, \ldots, r_t) =$ 


These are assigned a hyperplane $H_q^w(u(t; r_1, \ldots, r_t))$ determined by the equation

$$x_{r_1} + x_{(r_1+r_2)} + \cdots + x_{(r_1+r_2+\cdots+r_{t-1})} = \sum_{1\leqslant i<j\leqslant t} R_i R_j$$

where $R_i = \sum w_j$ where the sum is over the leaves of the $i$th subtree (from left to right) with root the type (5) node; the index $j$ goes from $(r_1 + r_2 + \cdots + r_{i-1})$ to $(r_1 + r_2 + \cdots + r_i - 1)$ (where $r_0 = 0$). Note that if $t = n$ (so $r_i = 1$ for all $i$) that this becomes the hyperplane given by

$$x_1 + \cdots + x_{n-1} = \sum_{1\leqslant i<j\leqslant n-1} w_i w_j.$$

**Lemma 5.** *For any painted binary n-leaved tree t the point $M_q^w(t)$ lies in the hyperplane $H_q^w(u(t; r_1, \ldots, r_t))$ iff t is a refinement of $u(t; r_1, \ldots, r_t)$. Also the hyperplane $H_q^w(u(t; r_1, \ldots, r_t))$ bounds above the points $M_q^w(t)$ for t any binary painted tree.*

**Proof.** Now by Lemma 3 we have that any binary tree $t$ which is a refinement of the upper tree $u(t; r_1, \ldots, r_t)$ will yield a point $M_q^w(t)$ which lies in $H_q^w(u(t; r_1, \ldots, r_t))$. To see this we simply note that the coordinates $x_{r_1}, x_{(r_1+r_2)}, \ldots, x_{(r_1+r_2+\cdots+r_{t-1})}$ in $M_q^w(t)$ will each be assigned the same value as if the original upper tree had $r_i = 1$ for all $i$ but where the weights given were $R_0, \ldots, R_{n-1}$.

We now demonstrate that if a binary tree $T$ is not a refinement of an upper tree $u(t; r_1, \ldots, r_t)$ then the point $M_q^w(T)$ will have the property that

$$x_{r_1} + x_{(r_1+r_2)} + \cdots + x_{(r_1+r_2+\cdots+r_{t-1})} < \sum_{1 \leqslant i < j \leqslant t} R_i R_j.$$

Recall that $R_i = \sum_j w_j$ where the sum is over the leaves of the $i$th subtree (from left to right) with root the type (5) node; the index $j$ goes from $(r_1 + r_2 + \cdots + r_{i-1})$ to $(r_1 + r_2 + \cdots + r_i - 1)$ (where $r_0 = 0$.) If $T$ is not a refinement of $u(t; r_1, \ldots, r_t)$ then for some of the partitioned sets of $r_i$ leaves in the partition $r_1, \ldots, r_t$ it is true that there does not exist a node of $T$ which if deleted would separate exactly the leaves in that set from the other leaves and root of $T$. Thus the proof here will use the previous result for the lower trees. First we consider the case for which $T$ is entirely painted—it has only type (2) nodes. Now by Lemma 3 the total sum of the coordinates of $M_q^w(T)$ will be equal to $\sum_{1 \leqslant i < j \leqslant n-1} w_i w_j$. Consider a (partitioned) set of $r_m$ leaves (starting with leaf $k-1$) in the partition $r_1, \ldots, r_t$ for which there does not exist a node of $T$ which if deleted would separate exactly the leaves in that set from the other leaves and root of $T$. (Here $k - 1 = r_1 + r_2 + \cdots + r_{m-1}$.) Let $P_m$ be the sum of the $r_m - 1$ coordinates $x_k + \cdots + x_{k+r_m-2}$. We have by the same argument used for lower trees that

$$P_m > \sum_{(k-1) \leqslant i < j \leqslant (k+r_m-2)} w_i w_j.$$

Now for this $T$, for which some of the partitioned sets of $r_i$ leaves in the partition $r_1, \ldots, r_t$ there does not exist a node of $T$ which if deleted would separate exactly the leaves in that set from the other leaves and root of $T$, we have:

$$x_{r_1} + x_{(r_1+r_2)} + \cdots + x_{(r_1+r_2+\cdots+r_{t-1})} = \sum_{1 \leqslant i < j \leqslant n-1} w_i w_j - \sum_{m=1}^{t} P_m < \sum_{1 \leqslant i < j \leqslant t} R_i R_j.$$

If a tree $T'$ has the same branching structure as $T$ but with some nodes of type (1) then the argument still holds since the argument from the lower trees still applies. Now for a tree $T$ whose branching structure is a refinement of the branching structure of the upper tree $u(t; r_1, \ldots, r_t)$, but which has some of its nodes $r_1, (r_1 + r_2), \ldots, (r_1 + r_2 + \cdots + r_{t-1})$ of type (1), the inequality holds simply due to the application of some factors $q$ on the left-hand side. □

**Proof of Theorem 3.** Now we may proceed with our inductive argument. The base case of $n = 2$ leaves is trivial to check. The points in $\mathbf{R}^1$ are $w_0 w_1$ and $q w_0 w_1$. Their convex hull is a line segment, combinatorially equivalent to $\mathcal{J}(2)$. Now we assume that for all $i < n$ and for arbitrary $q \in (0, 1)$ and for positive integer weights $w_0, \ldots, w_{i-1}$, the convex hull of the points

$$\left\{ M_q^w(t) \;\middle|\; t \text{ is a painted binary tree with } i \text{ leaves} \right\}$$

in $\mathbf{R}^{i-1}$ is combinatorially equivalent to the complex $\mathcal{J}(i)$, and that the points $M_q^w(t)$ are the vertices of the convex hull. Now for $i = n$ we need to show that the equivalence still holds. Recall that the two items we plan to demonstrate are that the points $M_q^w(t)$ are the vertices of their convex hull and that the facet of the convex hull corresponding to a given lower or upper tree $T$ is the convex hull of just the points corresponding to the binary trees that are refinements of $T$. The first item will be seen in the process of checking the second.

Given an $n$-leaved lower tree $l(k, s)$ we have from Lemma 4 that the points corresponding to binary refinements of $l(k, s)$ lie in an $(n-2)$-dimensional hyperplane $H_q^w(l(k, s))$ which bounds the entire convex hull. To see that this hyperplane does indeed contain a facet of the entire convex hull we use the induction hypothesis to show that the dimension of the convex hull of just the points in $H_q^w(l(k, s))$ is $n - 2$. Recall that the tree $l(k, s)$ is the result of grafting an unpainted $s$-leaved corolla onto leaf $k - 1$ of an $r$-leaved painted corolla. Thus the points $M_q^w(t)$ for $t$ a refinement of $l(k, s)$ have coordinates $x_k, \ldots, x_{k+s-1}$ which are precisely those of the associahedron $\mathcal{K}(s)$, by Lemma 2 (after multiplying by $q$). Now considering the remaining coordinates, we see by induction that they are the coordinates of the multiplihedron $\mathcal{J}(r)$. This is by process of considering their calculation as if performed on an $r$-leaved weighted tree $t'$ formed by replacing the subtree of $t$ (with leaves $x_{k-1}, \ldots, x_{k+s-1}$) with a single leaf of weight $\sum_{j=k-1}^{k+s-1} w_j$. Now after a change of basis to reorder the coordinates, we see that the points corresponding to the binary refinements of $l(k, s)$ are the vertices of a polytope combinatorially equivalent to $\mathcal{J}(r) \times \mathcal{K}(s)$ as expected. Since $r + s = n + 1$ this polytope has dimension $r - 1 + s - 2 = n - 2$, and so is a facet of the entire convex hull.

Given an $n$-leaved upper tree $u(t, r_1, \ldots, r_t)$ we have from Lemma 5 that the points corresponding to binary refinements of $u(t, r_1, \ldots, r_t)$ lie in an $(n-2)$-dimensional hyperplane $H_q^w(u(t, r_1, \ldots, r_t))$ which bounds the entire convex hull. To see that this hyperplane does indeed contain a facet of the entire convex hull we use the induction hypothesis to show that the dimension of the convex hull of just the points in $H_q^w(u(t, r_1, \ldots, r_t))$ is $n - 2$. Recall that the tree $u(t, r_1, \ldots, r_t)$ is the result of grafting painted $r_i$-leaved corollas onto leaf $i$ of a $t$-leaved completely painted corolla. Thus the points $M_q^w(t)$ for $T$ a refinement of $u(t, r_1, \dots, r_t)$ have coordinates corresponding to the nodes in the $i$th subtree which are precisely those of the multiplihedron $\mathcal{J}(r_i)$, by the inductive hypothesis. Now considering the remaining coordinates, we see by Lemma 2 that they are the coordinates of the associahedron $\mathcal{K}(t)$. This is by process of considering their calculation as if performed on an $t$-leaved weighted tree $T'$ formed by replacing each (grafted) subtree of $T$ (with $r_i$ leaves) with a single leaf of weight $\sum_j w_j$, where the sum is over the $r_i$ leaves of the $i$th grafted subtree. Now after a change of basis to reorder the coordinates, we see that the points corresponding to the binary refinements of $u(t, r_1, \dots, r_t)$ are the vertices of a polytope combinatorially equivalent to $\mathcal{K}(t) \times \mathcal{J}(r_1) \times \cdots \times \mathcal{J}(r_t)$ as expected. Since $r_1 + \cdots + r_t = n$ this polytope has dimension $t - 2 + (r_1 - 1) + (r_2 - 1) + \cdots + (r_t - 1) = n - 2$, and so is a facet of the entire convex hull.

Since each $n$-leaved binary painted tree is a refinement of some upper and or lower trees, then the point associated to that tree is found as a vertex of some of the facets of the entire convex hull, and thus is a vertex of the convex hull. This completes the proof. Recall that in Lemma 6 we have already shown that our convex hull is homeomorphic to the space of painted trees $LW\mathcal{U}(n)$. □

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