On Inversion of Some Meromorphic Matrices

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ABSTRACT

A problem of inversion of a real symmetric matrix meromorphically depending on a parameter is discussed. Such a matrix arises in a problem of perturbation of a positive semidefinite operator by an operator of finite rank. A number of modal synthesis methods deal with such matrices, referred to as characteristic matrices, in problems of investigating complex conservative systems. It is shown that instead of numerical inversion of the matrix at each given value of the parameter one can calculate the inverse matrix by using an explicit formula similar to that for the matrix itself. To diminish the truncation error in practical calculations, the representation of the inverse matrix via a series with accelerated convergence is obtained.

1. INTRODUCTION

A problem of inversion of a real symmetric meromorphic matrix of the form

$$\chi(\lambda) = I - \sum_{k=1}^{r} \frac{f_{0k} f_{0k}^T}{\lambda} + \sum_{k=1}^{\infty} \frac{f_k f_k^T}{\lambda_k - \lambda}, \qquad \lambda, \lambda_k > 0, \qquad f_{0k},\ f_k \in R^n, \tag{1}$$

is discussed. Such matrices occur in many applied problems. For example, a problem of perturbation of an operator having nonnegative discrete spectrum by a positive semidefinite operator of a finite rank can be reduced to the problem of investigating a matrix of the form (1) [1] (intermediate problems of the second kind). In structural dynamics some modal synthesis methods give rise to such matrices in problems of investigating complex conservative structures consisting of substructures whose dynamic characteristics are known [2–6]. Some of methods for

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calculating dynamic characteristics of modified structures, namely, the reanalysis methods based on local modifications, deal with such matrices [2, 7].

Note that sometimes (1) can be transformed to an equivalent form which contains a series with accelerated convergence ([3, 5, 8] and Section 5 of the paper). As in practice the series in (1) is approximated by a finite sum, such formulas result in less truncation error.

In some problems (e.g., problems of forced vibrations of complex or modified structures, problems of inverting perturbed operators) one needs the inverse of a matrix of the form (1). If the order of the matrix is large and the inverse matrix has to be calculated for many values of the parameter λ, it is desirable to have formulas for quick inversion of the matrix. Besides, the explicit formulas may be very useful in analytical calculations.

The subject of the investigation in this paper is matrices of the form (1) such that the inverse matrices can also be represented in a form analogous to (1), i.e., the inverse matrix is also a meromorphic function of λ with poles of the first order. We will not establish necessary conditions that guarantee that the inverse matrix may be presented in such a form. Instead of this, we will consider in the next section a class of matrices, so-called characteristic matrices, important from the standpoint of their applications. We will show that under some conditions both the characteristic matrix and its inverse can be represented in the form (1). So the aim of Section 2 is to demonstrate that the class of matrices under investigation is not empty.

The main results of the paper are established in Sections 3 and 4. In Section 3 an explicit formula for the inverse matrix, similar to that of (1), is obtained, the right side of the formula being represented in terms of some numbers and vectors related to the matrix $\chi(\lambda)$. An equivalent representation for the inverse matrix via a series with accelerated convergence is given in Section 4.

2. CHARACTERISTIC MATRIX AND ITS INVERSE FOR A PROBLEM OF PERTURBATION OF A POSITIVE SEMIDEFINITE OPERATOR BY AN OPERATOR OF FINITE RANK

Let us define a real symmetric operator-valued function, associated with the operator function $A(\lambda)$, by the formula

$$\chi(\lambda) = I + \Phi A^{-1}(\lambda)\Phi^* , \tag{2}$$

where Φ is an operator of finite rank n, Φ^* is its adjoint, I is an identity matrix of order n, and

$$A(\lambda) = H - \lambda M$$

with H positive semidefinite and M positive definite. It will be shown in this section that under some conditions the matrix (2) and its inverse can be written in the form (1). To substantiate the necessity of investigating matrices (2) and to obtain the representation (1) for them, consider the following problem. (As the subject of the paper is inversion of matrices of the form (1) and to save room, the following discussion in this section will be given without rigorous treatment. Other examples of the occurrence of matrices (2) associated with mechanical problems may be found in References [3–5].)

Let H be a positive semidefinite (in the general case unbounded) operator with a domain dense in a Hilbert space. Let r be a dimension of the null space of H, and φ_{01} ...,φ_{0r} be an orthonormal basis in Ker H: $H\varphi_{0i} = 0$, $\varphi_{0i}^*\varphi_{0k} \equiv (\varphi_{0i}, \varphi_{0k}) = \delta_{ik}$, $i, k = 1, \ldots, r$. Here δ_{ik} is the Kronecker delta, and the parentheses denote an inner product. Let λ_k and φ_k be nonzero eigenvalues and corresponding eigenfunctions of H: $H\varphi_k = \lambda_k\varphi_k$, $k = 1, \ldots, \infty$. Denote by H_1 the restriction of H to the subspace of its domain orthogonal to the null space of H. It is evident that eigenvalues and eigenfunctions of H_1 are nonzero eigenvalues λ_k and corresponding eigenfunctions φ_k of H, and that in contrast to H, the operator H_1 is invertible.

Denote by K_1 the inverse operator: $K_1 = H_1^{-1}$. It is well known (e.g., [1]) that the eigenfunctions of K_1 are those of H_1 and its eigenvalues are the reciprocals of λ_k:

$$K_1\varphi_k = \frac{1}{\lambda_k}\varphi_k, \quad k = 1, \ldots, \infty.$$

Let us suppose that K_1 is a positive definite completely continuous operator. Then [1] all eigenvalues λ_k can be ordered in a nondecreasing fashion: $0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$; the eigenfunctions φ_k form a complete orthonormal basis in the domain of K_1: $\varphi_i^*\varphi_k = \delta_{ik}$; and the operator K_1 and resolvent operator $(H_1 - \lambda I)^{-1}$ can be represented in the form

$$K_1 = \sum_{i=1}^{\infty} \frac{\varphi_i\varphi_i^*}{\lambda_i},$$

$$(H_1 - \lambda I)^{-1} = \sum_{i=1}^{\infty} \frac{\varphi_i\varphi_i^*}{\lambda_i - \lambda}. \tag{3}$$

To obtain a formula for the resolvent operator $(H - \lambda I)^{-1}$ one has to extend the operator $(H_1 - \lambda I)^{-1}$ to all of the Hilbert space. Consider a function f belonging to Ker H: $f = \sum_{i=1}^{r} \varphi_{0i}(\varphi_{0i}, f)$. Taking into account that $H\varphi_{0i} = 0$, one obtains

$$(H - \lambda I)f = (H - \lambda I)\sum_{i=1}^{r} \varphi_{0i}(\varphi_{0i}, f) = -\lambda \sum_{i=1}^{r} \varphi_{0i}(\varphi_{0i}, f) = -\lambda f.$$

It follows from the last formula that

$$(H-\lambda I)^{-1}f=-\frac{1}{\lambda}f=-\sum_{i=1}^{r}\frac{\varphi_{0i}(\varphi_{0i},f)}{\lambda}=\left(-\sum_{i=1}^{r}\frac{\varphi_{0i}\varphi_{0i}^{*}}{\lambda}\right)f\,.$$

Summarizing, one arrives at the formula

$$(H-\lambda I)^{-1}=-\sum_{i=1}^{r}\frac{\varphi_{0i}\varphi_{0i}^{*}}{\lambda}+\sum_{i=1}^{\infty}\frac{\varphi_{i}\varphi_{i}^{*}}{\lambda_i-\lambda}\,. \tag{4}$$

It is not difficult to show that the bilinear representation (4) is also valid for the weighted resolvent operator $(H-\lambda M)^{-1}$, with M positive definite. In this case λ_i and φ_i must satisfy the equation $H\varphi_i=\lambda_i M\varphi_i$ and the orthogonality and normalization conditions $(\varphi_i, M\varphi_k)=\delta_{ik}$, $(\varphi_{0i}, M\varphi_{0k})=\delta_{ik}$. Note that the results established do not depend on whether the weighted eigenvalue problem is considered or not, as that does not influence the general properties of the characteristic matrix (2). So in the following discussions one can consider the operator M as an identity operator.

Consider now a perturbation of the operator H by a positive semidefinite operator of finite rank,

$$\widetilde{H}=H+\Phi^{*}\Phi, \tag{5}$$

where Φ is defined on the domain of H and its range is n-dimensional space (i.e., for any function φ belonging to the domain of H one has $\Phi\varphi\in R^{n}$), and where Φ^{*} is the adjoint of Φ. It is known that if the resolvent operator $(H-\lambda M)^{-1}$ can be efficiently calculated, the problem of finding the spectrum of $\widetilde{H}$ can be reduced to the problem of investigating some n-dimensional symmetric characteristic matrix. Examples of such reduction may be found, e.g., in [1]. To define the characteristic matrix here we will take advantage of the results of [3], which are presented below without proof.

Let Γ, α, and β be linear operators (to save room we omit here discussions concerning ranges and domains of the operators) and Γ be invertible. Then the perturbed operator $\widetilde{\Gamma}=\Gamma+\alpha\beta$ is invertible if and only if the characteristic operator

$$\chi=I+\beta\Gamma^{-1}\alpha \tag{6}$$

is invertible. If $\widetilde{\Gamma}$ is invertible then

$$\widetilde{\Gamma}^{-1}=\Gamma^{-1}-\Gamma^{-1}\alpha\chi^{-1}\beta\Gamma^{-1}\,. \tag{7}$$

Otherwise the dimensions of the null spaces of $\widetilde{\Gamma}$ and χ are equal and there exists a one-to-one relationship between vectors from the null spaces:

$$\xi=\Gamma^{-1}\alpha\eta,\quad \eta=-\beta\xi,\quad \xi\in\operatorname{Ker}\widetilde{\Gamma},\quad \eta\in\operatorname{Ker}\chi\,.$$

The formulas (6) and (7) are well known in matrix analysis — they are used to calculate the inverse of an invertible matrix perturbed by a matrix of small rank [9, Section 0.7]. If the operator Γ is perturbed by a finite-rank operator, as in the above problem discussed, the characteristic operator is a matrix.

Denote $A(\lambda) = H - \lambda M$ and $\widetilde{A}(\lambda) = \widetilde{H} - \lambda M$. It follows from (5) that

$$\widetilde{A}(\lambda) = H + \Phi^*\Phi - \lambda M = A(\lambda) + \Phi^*\Phi.$$

Applying (6) to the operators $A(\lambda)$ and $\widetilde{A}(\lambda)$, one obtains the formula (2) for the characteristic matrix $\chi(\lambda)$, where $A^{-1}(\lambda) \equiv (H - \lambda M)^{-1}$ is given by (4). Substituting (4) into (2), we see that the characteristic matrix has the form (1), where $f_{0k} = \Phi\varphi_{0k}$, $f_k = \Phi\varphi_k$. Properties of the characteristic-matrix eigenvalues, including those that describe their behavior in a neighborhood of the matrix poles, are discussed in [4].

It can be seen from (7) that to calculate the operator $\widetilde{A}^{-1}(\lambda)$ one needs the matrix $\chi^{-1}(\lambda)$. We obtain now a general representation for the inverse of the characteristic matrix (2). We will see that the inverse matrix can also be written in a form similar to that of (1).

We have arrived at the characteristic matrix (2) by considering a perturbation of an operator, whose spectrum and eigenfunctions are known (or can be easily found), by a positive semidefinite operator of finite rank. Let us consider a problem inverse, in some sense, to the above problem. Namely, starting with the perturbed operator $\widetilde{A}(\lambda)$ (or $\widetilde{H}$), consider an operator defined by the formula

$$A(\lambda) = \widetilde{A}(\lambda) - \Phi^*\Phi\ .$$

That is, we consider now a problem of perturbation of the operator $\widetilde{A}(\lambda)$ by the negative semidefinite operator $-\Phi^*\Phi$. Applying (6), one obtains the characteristic matrix $\widetilde{\chi}(\lambda)$ of this problem:

$$\widetilde{\chi}(\lambda) = I - \Phi\widetilde{A}^{-1}(\lambda)\Phi^*\ . \tag{8}$$

LEMMA 1. *If the operators $A(\lambda)$ and $\widetilde{A}(\lambda)$ are invertible at given λ, then*

$$\chi^{-1}(\lambda) = \widetilde{\chi}(\lambda)\ .$$

Proof. Take advantage of (7) to calculate $\widetilde{A}^{-1}(\lambda)$:

$$\widetilde{A}^{-1}(\lambda) = A^{-1}(\lambda) - A^{-1}(\lambda)\Phi^*\chi^{-1}(\lambda)\Phi A^{-1}(\lambda)\ .$$

Substituting the last formula into (8) and using (2), one obtains

$$
\begin{aligned}
\widetilde{\chi}(\lambda) &= I - \Phi A^{-1}(\lambda)\Phi^* + \Phi A^{-1}(\lambda)\Phi^*\chi^{-1}(\lambda)\Phi A^{-1}(\lambda)\Phi^* \\
&= I - [\chi(\lambda) - I] + [\chi(\lambda) - I]\chi^{-1}(\lambda)[\chi(\lambda) - I] = \chi^{-1}(\lambda).
\end{aligned}
$$

∎

Note now that the operator $\widetilde{H}$ also has nonnegative spectrum, as it is obtained by adding the positive semidefinite operator $\Phi^*\Phi$ to the operator H, and its resolvent operator $(\widetilde{H} - \lambda M)^{-1}$ can also be represented in the form of a bilinear series (4). Taking into account that $\widetilde{A}^{-1}(\lambda) = (\widetilde{H} - \lambda M)^{-1}$, we obtain

$$
\widetilde{A}^{-1}(\lambda) = -\sum_{i=1}^{\tilde{r}} \frac{\widetilde{\varphi}_{0i}\widetilde{\varphi}_{0i}^*}{\lambda} + \sum_{i=1}^{\infty} \frac{\widetilde{\varphi}_i\widetilde{\varphi}_i^*}{\tilde{\lambda}_i - \lambda}\,, \tag{9}
$$

where the eigenvalues $\tilde{\lambda}_i$ and eigenfunctions $\widetilde{\varphi}_{0i}$, $\widetilde{\varphi}_i$ satisfy equations $\widetilde{H}\widetilde{\varphi}_{0i} = 0$, $\widetilde{H}\widetilde{\varphi}_i = \tilde{\lambda}_i M\widetilde{\varphi}_i$ and conditions $\widetilde{\varphi}_{0i}^* M\widetilde{\varphi}_{0k} = \delta_{ik}$, $\widetilde{\varphi}_i^* M\widetilde{\varphi}_k = \delta_{ik}$, and where $\tilde{r}$ is the dimension of the null space of $\widetilde{H}$. Substituting (9) into (8) and using Lemma 1, one arrives at the formula

$$
\chi^{-1}(\lambda) = I + \sum_{i=1}^{\tilde{r}} \frac{h_{0i}h_{0i}^T}{\lambda} - \sum_{i=1}^{\infty} \frac{h_i h_i^T}{\tilde{\lambda}_i - \lambda}\,. \tag{10}
$$

where $h_{0i} = \Phi\widetilde{\varphi}_{0i}$ and $h_i = \Phi\widetilde{\varphi}_i$ are n-dimensional vectors.

Thus we have obtained the general form of the inverse matrix analogous to that of the characteristic matrix itself. The right side of (10) is given in terms of the eigenvalues and eigenfunctions of the perturbed operator $\widetilde{H}$. Our aim now is to represent it in terms of some numbers and vectors related to $\chi(\lambda)$.

3. FORMULA FOR THE INVERSE MATRIX

Up to this point we have addressed ourselves to the problem of investigation of the perturbed operator to obtain the form (1) for the characteristic matrix (2) and the general form (10) of its inverse. We have established that there exists a class of matrices of the form (1) for which it is known that their inverses can be represented in the form (10) where $\tilde{\lambda}_i > 0$ and h_i, h_{0i} are unknown numbers and vectors to be found. Our aim is to specify (10) and to give the way to find the numbers $\tilde{\lambda}_i$ and vectors h_i. From this point on, to establish and to prove the main results we will need only the formula (1) and the representation (10).

Clearly the set of poles in (10) must contain all those values of λ for which

$\chi(\lambda)$ exists and is singular. In the general case some pole λ_i of $\chi(\lambda)$ may also be a pole of $\chi^{-1}(\lambda)$. The latter case needs special treatment, as $\chi(\lambda_i)$ does not exist.

Let r_k be the number of terms in (1) having pole λ_k. Rewrite (1) in a neighborhood of λ_k, separating out those terms that have pole λ_k:

$$\chi(\lambda) = \chi_{\text{reg}}(\lambda) + \frac{[\, f_k\,]\,[\, f_k\,]^T}{\lambda_k - \lambda} . \qquad (11)$$

Here and in what follows, $[\, f_k\,]$ denotes the $n \times r_k$ matrix formed by the vectors f_k, and the matrix $\chi_{\text{reg}}(\lambda)$ exists at $\lambda = \lambda_k$. Note that we can always consider only the case where the rank of $[\, f_k\,]$ is equal to r_k and $r_k \leq n$. Indeed, let the rank q of $[\, f_k\,]$ be less than r_k. As $C = [\, f_k\,]\,[\, f_k\,]^T$ is a positive semidefinite matrix of rank q, it can be factorized in the form [9, Section 7.2] $C = [\, \tilde{f}_k\,]\,[\, \tilde{f}_k\,]^T$ where $[\, \tilde{f}_k\,]$ is an $n \times q$ matrix and $q \leq n$. Thus, if $q < r_k$ we can replace the matrix $[\, f_k\,]\,[\, f_k\,]^T$ in (11) by $[\, \tilde{f}_k\,]\,[\, \tilde{f}_k\,]^T$, the rank of $[\, \tilde{f}_k\,]$ being equal to the number of its columns.

The following lemma allows one to determine if any pole of $\chi(\lambda)$ is a pole of $\chi^{-1}(\lambda)$.

LEMMA 2. *A pole λ_k of $\chi(\lambda)$ is a pole of $\chi^{-1}(\lambda)$ if and only if matrix*

$$\chi_k = \begin{bmatrix} \chi_{\text{reg}}(\lambda_k) & [\, f_k\,] \\ [\, f_k\,]^T & 0 \end{bmatrix} \qquad (12)$$

of order $n + r_k$ is singular. The number $\tilde{r}_k$ of terms in the right side of (10) having pole λ_k is equal to dim Ker χ_k.

Proof. Denote by $\nu_i(\lambda)$ and $y_i(\lambda)$ the eigenvalues and orthonormal eigenvectors of the matrix $\chi(\lambda)$:

$$\chi(\lambda) y_i(\lambda) = \nu_i(\lambda) y_i(\lambda), \quad i = 1, \ldots, n, \quad \lambda \neq \lambda_j, \quad j = 1, \ldots, \infty . \qquad (13)$$

It is proved in [4] that when λ tends to λ_k, there exist limits $y_p(\lambda_k)$ of all eigenvectors of $\chi(\lambda)$ and that r_k eigenvalues have infinite discontinuities while the others have removable singularities. It is not difficult to prove that in addition to this, all eigenvectors and the eigenvalues that have removable singularities are differentiable functions of λ at λ_k.

It is well known that $\chi(\lambda)$ and $\chi^{-1}(\lambda)$ can be presented in the form

$$\chi(\lambda) = \sum_{i=1}^{n} \nu_i(\lambda) y_i(\lambda) y_i^T(\lambda), \quad \chi^{-1}(\lambda) = \sum_{i=1}^{n} \frac{y_i(\lambda) y_i^T(\lambda)}{\nu_i(\lambda)} . \qquad (14)$$

Clearly $\chi^{-1}(\lambda)$ has pole λ_k if and only if some of eigenvalues $\nu_i(\lambda)$ have zero limits when λ tends to λ_k, and the number of terms in (10) having pole λ_k is equal to the number of eigenvalues $\nu_i(\lambda)$ having zero limits.

Let $\lim_{\lambda\to\lambda_k} \nu_p(\lambda) = 0$ for some number p. Then

$$\lim_{\lambda\to\lambda_k} \chi(\lambda) y_p(\lambda) = 0 .$$

Substituting (11) in the last formula, one obtains

$$\lim_{\lambda\to\lambda_k} \chi(\lambda) y_p(\lambda) = \chi_{\text{reg}}(\lambda_k) y_p(\lambda_k) + \lim_{\lambda\to\lambda_k} \frac{[\, f_k\,][\, f_k\,]^T y_p(\lambda)}{\lambda_k - \lambda} = 0 .$$

In order for the last limit to exist we must have

$$[\, f_k\,]^T y_p(\lambda_k) = 0 .$$

Applying L'Hospital's rule, one obtains

$$\lim_{\lambda\to\lambda_k} \chi(\lambda) y_p(\lambda) = \chi_{\text{reg}}(\lambda_k) y_p(\lambda_k) - [\, f_k\,][\, f_k\,]^T y_p'(\lambda_k) = 0 ,$$

where $y_p'(\lambda_k) = \lim_{\lambda\to\lambda_k} y_p'(\lambda)$. Summarizing and denoting $h_p = y_p(\lambda_k)$, $z_p = -[\, f_k\,]^T y_p'(\lambda_k)$, one arrives at the conclusion that the system of equations with respect to vectors $h_p \in R^n$ and $z_p \in R^{r_k}$ given by

$$\chi_{\text{reg}}(\lambda_k) h_p + [\, f_k\,]\, z_p = 0, \tag{15}$$

$$[\, f_k\,]^T h_p = 0 \tag{16}$$

has nontrivial solutions, and hence the matrix (12) is singular.

If $\tilde{r}_k$ eigenvalues have zero limits, the corresponding limit eigenvectors h_p are linearly independent and hence the corresponding solutions of the system (15)–(16) are linearly independent vectors in $(n + r_k)$-dimensional space. Hence we obtain the inequality

$$\tilde{r}_k \leq \dim \operatorname{Ker} \chi_k . \tag{17}$$

Now we prove the sufficiency of the statement of the lemma. Let the matrix (12) be singular. Then there exist vectors $h_p \in R^n$ and $z_p \in R^{r_k}$ satisfying Equations (15)–(16). Define the vector $v_p(\lambda)$ by the formula

$$v_p(\lambda) = h_p + (\lambda_k - \lambda)[\, f_k\,]([\, f_k\,]^T [\, f_k\,])^{-1} z_p . \tag{18}$$

As the rank of the matrix $[\, f_k\,]$ is assumed to be equal to the number of its columns, r_k, the matrix in parentheses is of full rank and hence is invertible. We prove that the limit of the vector $\chi(\lambda) v_p(\lambda)$ as $\lambda \to \lambda_k$ is equal to the zero vector. Using the formulas (11), (18) and Equations (15)–(16), one obtains

$$\lim_{\lambda\to\lambda_k} \chi(\lambda) v_p(\lambda) = \lim_{\lambda\to\lambda_k} \left(\chi_{\text{reg}}(\lambda) + \tfrac{[\, f_k\,][\, f_k\,]^T}{\lambda_k - \lambda} \right)$$
$$\{h_p + (\lambda_k - \lambda)[\, f_k\,]([\, f_k\,]^T [\, f_k\,])^{-1} z_p\} = \chi_{\text{reg}}(\lambda_k) h_p + [\, f_k\,]\, z_p = 0 .$$

It is evident that the equation

$$\lim_{\lambda \to \lambda_k} \chi(\lambda) v_p(\lambda) = 0$$

means that there exists an eigenvalue of the characteristic matrix that has zero limit when $\lambda \to \lambda_k$, the limit of the corresponding eigenvector being equal to h_p.

Let now $\dim \operatorname{Ker} \chi_k = q > 1$. Then there exist q linearly independent vectors

$$\begin{Bmatrix} h_1 \\ z_1 \end{Bmatrix}, \dots, \begin{Bmatrix} h_q \\ z_q \end{Bmatrix} \in R^{n+r_k}, \quad h_i \in R^n, \quad z_i \in R^{r_k} \tag{19}$$

satisfying Equations (15)–(16). To prove that in this case there exist q eigenvalues having zero limits it is sufficient to prove that the vectors $h_1, \dots, h_q$ are also linearly independent. Let us suppose that $h_1, \dots, h_q$ are linearly dependent. Then there exist numbers α_i, $i = 1, \dots, q$, which are not all zero, such that

$$\sum_{i=1}^{q} \alpha_i h_i = 0 . \tag{20}$$

As the vectors (19) are linearly independent, we must have $\sum_{i=1}^{q} \alpha_i z_i \neq 0$. Taking the linear combination of the vectors (19) with coefficients α_i, substituting the vector obtained into Equations (15)–(16), and taking into account (20), one arrives at the equation $[f_k] \sum_{i=1}^{q} \alpha_i z_i = 0$. This equation cannot be satisfied, as the rank of the matrix $[f_k]$ is equal to the number of its columns. The contradiction obtained means that the vectors $h_1, \dots, h_q$ are linearly independent and hence the inequality

$$q = \dim \operatorname{Ker} \chi_k \leq \tilde{r}_k \tag{21}$$

holds. Comparison of (17) and (21) gives $\tilde{r}_k = \dim \operatorname{Ker} \chi_k$. ■

So to determine if a given pole of $\chi(\lambda)$ is inherited by the inverse matrix, one has to investigate the matrix (12). But there is one pole — the zero pole — that does not need investigation, as, for any particular matrix (1), it does not belong to the set of poles of the inverse matrix. Namely, the following lemma is valid.

LEMMA 3. *The inverse matrix has no pole at* $\lambda = 0$.

Proof. We prove that for the case of a zero pole of $\chi(\lambda)$ the matrix (12) is always nonsingular. Then according to Lemma 2, $\lambda = 0$ is not a pole of the inverse matrix.

Suppose that it is not true and the matrix (12) is singular. That means that the equations

$$\chi_{\text{reg}}(0) h_0 + [f_0]\, z_0 = 0 , \tag{22}$$

$$[f_0]^T h_0 = 0 \tag{23}$$

have a nontrivial solution. Here $[\ f_0\ ]$ is the $n \times r$ matrix formed by vectors f_{01}, $\dots, f_{0r}$, and

$$\chi_{\text{reg}}(0) = I + \sum_{i=1}^{\infty} \frac{f_i f_i^T}{\lambda_i}, \quad \lambda_i > 0 . \tag{24}$$

Note that $h_0 \neq 0$, as the rank of the matrix $[\ f_0\ ]$ is equal to the number of its columns (see remark before Lemma 2). Taking the inner product of both sides of Equation (22) with the vector h_0 and using Equation (23), one obtains

$$\left(h_0, \chi_{\text{reg}}(0) h_0\right) = 0 .$$

On the other hand, substituting (24) into the left side of the last equation gives

$$\left(h_0, \chi_{\text{reg}}(0) h_0\right) = (h_0, h_0) + \sum_{i=1}^{\infty} \frac{(f_i, h_0)^2}{\lambda_i} \geq (h_0, h_0) > 0 .$$

Thus the assumption that the matrix (12) is singular leads to a contradiction. This proves the lemma. ∎

We proved that the first sum in the right side of (10) must be omitted. We now establish properties of the vectors h_i in the second sum of the formula. Let $\tilde{\lambda}_k$ be a pole of the inverse matrix. Let, for simplicity, $\tilde{r}_k = 1$, i.e., only one term in the right side of (10) has pole $\tilde{\lambda}_k$. Rewrite the formula in a neighborhood of $\tilde{\lambda}_k$ in the form

$$\chi^{-1}(\lambda) = -\frac{h_k h_k^T}{\tilde{\lambda}_k - \lambda} + \left(\chi^{-1}(\lambda)\right)_{\text{reg}} ,$$

where $\left(\chi^{-1}(\lambda)\right)_{\text{reg}}$ exists at $\tilde{\lambda}_k$.

For any λ belonging to a neighborhood of $\tilde{\lambda}_k$, $\lambda \neq \tilde{\lambda}_k$, we have

$$I = \chi(\lambda)\chi^{-1}(\lambda) = -\frac{\chi(\lambda) h_k h_k^T}{\tilde{\lambda}_k - \lambda} + \chi(\lambda)\left(\chi^{-1}(\lambda)\right)_{\text{reg}} . \tag{25}$$

Taking limits of the left and right sides of (25) when λ tends to $\tilde{\lambda}_k$, one obtains

$$I = \lim_{\lambda \to \tilde{\lambda}_k} \left\{ -\frac{\chi(\lambda) h_k h_k^T}{\tilde{\lambda}_k - \lambda} + \chi(\lambda)\left(\chi^{-1}(\lambda)\right)_{\text{reg}} \right\} . \tag{26}$$

Consider the two possible cases.

(1) $\chi(\tilde{\lambda}_k)$ *exists*. For the limit of the right side to exist we must have

$$\chi(\tilde{\lambda}_k) h_k = 0 , \tag{27}$$

i.e., $h_k \in \operatorname{Ker}\chi(\tilde{\lambda}_k)$. Premultiplying both sides of Equation (26) by h_k^T and applying L'Hospital's rule, one obtains

$$h_k^T = (h_k, \chi'(\tilde{\lambda}_k)h_k)h_k^T + h_k^T\chi(\tilde{\lambda}_k)\left(\chi^{-1}(\tilde{\lambda}_k)\right)_{\text{reg}} .$$

It follows from Equation (27) that the second term in the right side of the last equation is equal to zero, and we conclude that the equation holds if and only if

$$(h_k, \chi'(\tilde{\lambda}_k)h_k) = 1 . \tag{28}$$

Equation (27) and the normalization condition (28) uniquely define the vector h_k.

(2) $\chi(\tilde{\lambda}_k)$ *does not exist, i.e.,* $\tilde{\lambda}_k$ *is a pole of* $\chi(\lambda)$. Then according to Lemma 2 there exists a vector z_k such that h_k and z_k satisfy Equations (15)–(16). Premultiply the left and right sides of Equation (25) by $v_k^T(\lambda)$, where

$$v_k(\lambda) = h_k + (\tilde{\lambda}_k - \lambda)[\, f_k\,]([\, f_k\,]^T[\, f_k\,])^{-1}z_k , \tag{29}$$

and take limits of both sides when $\lambda \to \tilde{\lambda}_k$. We have

$$\lim_{\lambda\to\tilde{\lambda}_k} v_k^T(\lambda) = \lim_{\lambda\to\tilde{\lambda}_k}\left\{-\frac{\{\chi(\lambda)v_k(\lambda)\}^T h_k h_k^T}{\tilde{\lambda}_k - \lambda} + \{\chi(\lambda)v_k(\lambda)\}^T\left(\chi^{-1}(\lambda)\right)_{\text{reg}}\right\} . \tag{30}$$

As $\lim_{\lambda\to\tilde{\lambda}_k}\chi(\lambda)v_k(\lambda) = 0$ (see the proof of Lemma 2), the limit of the right side of the equation exists, with the second term being equal to zero. Apply L'Hospital's rule and (11) to calculate the first term in the right side of Equation (30):

$$\lim_{\lambda\to\tilde{\lambda}_k}\left\{-\frac{\{\chi(\lambda)v_k(\lambda)\}^T h_k h_k^T}{\tilde{\lambda}_k - \lambda}\right\} = \lim_{\lambda\to\tilde{\lambda}_k}\left\{v_k^T(\lambda)\chi'(\lambda)h_k h_k^T + \right.$$

$$\left.\{v_k'(\lambda)\}^T\chi(\lambda)h_k h_k^T\right\} = \lim_{\lambda\to\tilde{\lambda}_k}\left\{v_k^T(\lambda)\chi_{\text{reg}}'(\lambda)h_k h_k^T + \frac{v_k^T(\lambda)[\, f_k\,]\,[\, f_k\,]^T h_k h_k^T}{(\tilde{\lambda}_k - \lambda)^2}\right.$$

$$\left. + \{v_k'(\lambda)\}^T\chi_{\text{reg}}(\lambda)h_k h_k^T + \frac{\{v_k'(\lambda)\}^T[\, f_k\,]\,[\, f_k\,]^T h_k h_k^T}{\tilde{\lambda}_k - \lambda}\right\} .$$

In view of Equation (16), the second and the fourth terms in the right side of the formula are equal to zero. Using the formula (29) and Equation (15), one obtains

$$\lim_{\lambda\to\tilde{\lambda}_k}\left\{-\frac{\{\chi(\lambda)v_k(\lambda)\}^T h_k h_k^T}{\tilde{\lambda}_k - \lambda}\right\} = \left\{h_k^T\chi_{\text{reg}}'(\tilde{\lambda}_k)h_k - \right.$$

$$\left. z_k^T([\, f_k\,]^T[\, f_k\,])^{-1}[\, f_k\,]^T\chi_{\text{reg}}(\tilde{\lambda}_k)h_k\right\} h_k^T = \left\{h_k^T\chi_{\text{reg}}'(\tilde{\lambda}_k)h_k + \right.$$

$$z_k^T([\, f_k \,]^T[\, f_k \,])^{-1}[\, f_k \,]^T[\, f_k \,]z_k\Big\}\, h_k^T = \Big\{(h_k, \chi'_{\mathrm{reg}}(\tilde{\lambda}_k)h_k) + (z_k, z_k)\Big\}\, h_k^T \,.$$

Substituting this and (29) into Equation (30), one obtains

$$h_k^T = \Big\{(h_k, \chi'_{\mathrm{reg}}(\tilde{\lambda}_k)h_k) + (z_k, z_k)\Big\}\, h_k^T \,.$$

This equation holds if and only if

$$(h_k, \chi'_{\mathrm{reg}}(\tilde{\lambda}_k)h_k) + (z_k, z_k) = 1 \,. \tag{31}$$

Equations (15)–(16) and the normalization condition (31) uniquely define the vector h_k.

The case of multiple terms having pole $\tilde{\lambda}_k$ ($\tilde{r}_k > 1$) involves no additional difficulties. Following exactly the same procedure, one obtains that vectors h_k corresponding to the same pole $\tilde{\lambda}_k$ must be orthogonal to each other and satisfy a normalization condition (28) or (31).

The results of this section are summed up in the following theorem.

THEOREM 1. *The inverse of the matrix (1) can be represented in the form*

$$\chi^{-1}(\lambda) = I - \sum_{i=1}^{\infty} \frac{h_i h_i^T}{\tilde{\lambda}_i - \lambda} \,, \tag{32}$$

with positive poles $\tilde{\lambda}_i$ ordered in a nondecreasing fashion: $0 < \tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots \to \infty$. The set of the poles consists of those values $\tilde{\lambda}_i$ for which $\chi(\tilde{\lambda}_i)$ exists and is singular and of those poles of $\chi(\lambda)$ for which matrix (12) is singular.

If $\chi(\tilde{\lambda}_k)$ exists, then $h_k \in Ker\, \chi(\tilde{\lambda}_k)$ and satisfies the condition

$$(h_k, \chi'(\tilde{\lambda}_k)h_k) = 1 \,.$$

If $\tilde{\lambda}_k$ is a pole of $\chi(\lambda)$ and r_k is the number of terms in (1) that have pole $\tilde{\lambda}_k$, then there exists a vector $z_k \in R^{r_k}$ such that the vectors h_k and z_k satisfy Equations (15)–(16) and the condition

$$(h_k, \chi'_{\mathrm{reg}}(\tilde{\lambda}_k)h_k) + (z_k, z_k) = 1 \,.$$

If $\tilde{r}_k$ members of the series in the right side of (32) have the same pole $\tilde{\lambda}_k$, then the $\tilde{r}_k$ corresponding vectors h_k are orthogonal to each other.

Thus we have obtained the desired bilinear formula for the matrix $\chi^{-1}(\lambda)$. The formula (32) may be very useful in analytical calculations, but in practice one usually has a limited number of eigenvalues $\tilde{\lambda}_k$ and corresponding vectors

h_k. Therefore the series in (32) is approximated by a finite sum. To diminish the truncation error it is desirable to have another representation for $\chi^{-1}(\lambda)$ with more quickly converging series.

4. CALCULATION OF THE INVERSE MATRIX BY MEANS OF SERIES WITH ACCELERATED CONVERGENCE

It follows from (32) that $\chi^{-1}(0)$ exists and

$$\chi^{-1}(0) = I - \sum_{i=1}^{\infty} \frac{h_i h_i^T}{\tilde{\lambda}_i} . \tag{33}$$

Let us denote for brevity $Q \equiv \chi^{-1}(0)$. Suppose we have calculated Q exactly in some way or another without using (33). Adding Q to the right side of (32) and subtracting the right side of (33) from it, one obtains

$$\begin{aligned} \chi^{-1}(\lambda) &= I - \sum_{i=1}^{\infty} \frac{h_i h_i^T}{\tilde{\lambda}_i - \lambda} - \left(I - \sum_{i=1}^{\infty} \frac{h_i h_i^T}{\tilde{\lambda}_i} \right) + Q \\ &= Q - \lambda \sum_{i=1}^{\infty} \frac{h_i h_i^T}{\tilde{\lambda}_i (\tilde{\lambda}_i - \lambda)} . \end{aligned} \tag{34}$$

It is evident that the series in the right side of (34) converges more quickly than that in (32) and hence the truncation error of the last formula is less than that of (32) under the condition that the same number of terms of the series is used in both formulas. To take advantage of (34) we need a formula for Q different from (33).

We take advantage of the spectral representation (14) for the inverse matrix. It follows from the results of [4] that when λ tends to zero, r eigenvalues of the matrix (1) tend to $-\infty$, while the limits of the others are finite. Let the eigenvalues $\nu_k(\lambda)$ be ordered as follows: $\nu_1(\lambda) \geq \nu_2(\lambda) \geq \cdots \geq \nu_n(\lambda)$. Then when $\lambda \to 0$ the first $m \equiv n - r$ eigenvalues have finite limits. Denote these limits by ν_k^0, and the limits of the corresponding eigenvectors by y_k^0, $k = 1, \ldots, m$. It follows from (14) that

$$Q \equiv \chi^{-1}(0) = \lim_{\lambda \to 0} \chi^{-1}(\lambda) = \lim_{\lambda \to 0} \sum_{i=1}^{n} \frac{y_i(\lambda) y_i^T(\lambda)}{\nu_i(\lambda)} = \sum_{i=1}^{m} \frac{y_i^0 (y_i^0)^T}{\nu_i^0} . \tag{35}$$

This formula gives us a way of calculating the matrix Q, different from that of (33), by finding limit eigenvalues and eigenvectors. Moreover, it turns out that to obtain Q there is no need to calculate the limits of $\nu_i(\lambda)$ and $y_i(\lambda)$.

Let us consider the m-dimensional subspace of R^n orthogonal to the vectors $f_{01},\dots,f_{0r}$, and let $v_1,\dots,v_m$ be an orthonormal basis in this subspace. Denote by V the $n\times m$ matrix $V = [v_1, \dots, v_m]$. Evidently, $V^T V = I$, where I is an identity matrix of order m.

THEOREM 2. *The inverse of the matrix (1) can be calculated by the formula*

$$\chi^{-1}(\lambda) = \sum_{i=1}^{m} \frac{Vu_i(Vu_i)^T}{\mu_i} - \lambda \sum_{i=1}^{\infty} \frac{h_i h_i^T}{\tilde{\lambda}_i(\tilde{\lambda}_i - \lambda)}, \tag{36}$$

where μ_i and u_i are the eigenvalues and eigenvectors of the positive definite matrix

$$D = V^T \chi_{\text{reg}}(0) V, \tag{37}$$

and $\chi_{\text{reg}}(\lambda)$ is given by (24).

Proof. Taking into account (34) and (35), we must prove that the limit eigenvalues ν_i^0 are equal to the eigenvalues μ_i of the matrix D and that the corresponding limit eigenvectors y_i^0 and eigenvectors u_i of D are related by the formula $y_i^0 = Vu_i$.

Representing the matrix (1) in a neighborhood of $\lambda = 0$ in the form (the notation used was introduced in Lemma 3)

$$\chi(\lambda) = \chi_{\text{reg}}(\lambda) - \frac{[\,f_0\,]\,[\,f_0\,]^T}{\lambda},$$

and substituting the formula into the equations (13), one obtains

$$\chi_{\text{reg}}(\lambda) y_i(\lambda) - \frac{[\,f_0\,][\,f_0\,]^T y_i(\lambda)}{\lambda} = \nu_i(\lambda) y_i(\lambda)\,.$$

Taking limits of both sides of the equation when λ tends to zero for $i = 1, \dots, m$ and taking into account that the first m limit eigenvalues are finite, we have

$$\chi_{\text{reg}}(0) y_i^0 - \lim_{\lambda\to 0} \frac{[\,f_0\,][\,f_0\,]^T y_i(\lambda)}{\lambda} = \nu_i^0 y_i^0, \quad i = 1, \dots, m\,.$$

For the limit in the left side of the equation to exist the equations

$$[\,f_0\,]^T y_i^0 = 0, \quad i = 1, \dots, m \tag{38}$$

must hold. Applying L'Hospital's rule, one obtains

$$\chi_{\text{reg}}(0) y_i^0 - [\,f_0\,][\,f_0\,]^T y_i'(0) = \nu_i^0 y_i^0, \quad i = 1, \dots, m\,. \tag{39}$$

It follows from the equations (38) that $y_i^0 = Vu_i$ where $u_i \in R^m$. Premultiplying both sides of the equations (39) by V^T, one obtains

$$V^T \chi_{\text{reg}}(0) V u_i = \nu_i^0 V^T V u_i, \quad i = 1, \ldots, m .$$

Taking into account (37) and that $V^T V = I$, one obtains

$$D u_i = \nu_i^0 u_i, \quad i = 1, \ldots, m ,$$

i.e., ν_i^0 and u_i are eigenvalues and eigenvectors of the matrix D. ■

5. SOME REMARKS ON USING FORMULAS WITH ACCELERATED CONVERGENCE

To apply Theorem 2 in practice one needs a formula for calculating the matrix $\chi_{\text{reg}}(0)$ different from that of (24), as the latter contains infinite series and, thus, does not allow one to calculate $\chi_{\text{reg}}(0)$ with the necessary accuracy. It seems likely that one cannot obtain such a formula considering only the matrix (1), but must turn back to the original problem which led to (1). Consider again the problem of perturbation of an operator discussed in Section 2. We will show what is needed to obtain the desired formula and at the same time will obtain an accelerated representation for the characteristic matrix itself.

Let the unperturbed operator H be simple enough so that its restriction H_1 can be inverted immediately, either analytically or by means of some numerical procedure, i.e., one can calculate the operator K_1 without using the bilinear representation (3). Adding the operator K_1 to the right side of (4) and subtracting the right side of (3) from it, one obtains

$$\begin{aligned} A^{-1}(\lambda) \equiv (H - \lambda I)^{-1} &= K_1 - \sum_{i=1}^{\infty} \frac{\varphi_i \varphi_i^*}{\lambda_i} - \sum_{i=1}^{r} \frac{\varphi_{0i}\varphi_{0i}^*}{\lambda} + \sum_{i=1}^{\infty} \frac{\varphi_i \varphi_i^*}{\lambda_i - \lambda} \\ &= K_1 - \sum_{i=1}^{r} \frac{\varphi_{0i}\varphi_{0i}^*}{\lambda} + \lambda \sum_{i=1}^{\infty} \frac{\varphi_i \varphi_i^*}{\lambda_i(\lambda_i - \lambda)} \end{aligned}$$

(some other representations for $A^{-1}(\lambda)$ are given in [5]). Substituting this into (2), one arrives at the formula for the characteristic matrix with a more quickly converging series:

$$\chi(\lambda) = I + \Phi K_1 \Phi^* - \sum_{i=1}^{r} \frac{f_{0i} f_{0i}^T}{\lambda} + \lambda \sum_{i=1}^{\infty} \frac{f_i f_i^T}{\lambda_i(\lambda_i - \lambda)} . \quad (40)$$

The right sides of (1) and (40) are equal, but the latter provides less truncation error if the same number of terms of the series is used to approximate $\chi(\lambda)$. It follows from (40) that

$$\chi_{\text{reg}}(0) = I + \Phi K_1 \Phi^* \ .$$

We see that if one has an opportunity to calculate operator K_1 one can use the series with accelerated convergence to calculate both the characteristic matrix and its inverse.

In mechanical problems the operator K_1 is referred to as the generalized static Green operator. For some not very complicated systems it can be calculated by means of explicit formulas. For example, suppose one is investigating vibrations of a system consisting of a finite number of beams and/or rods interacting at a finite number of points. This problem can be formulated in terms of the problem of perturbation of a positive semidefinite operator by an operator of finite rank [5], the unperturbed operator being defined by differential operators governing vibrations of the isolated subsystems. To take advantage of the formula (40) for calculating the characteristic matrix and the formula (36) for its inverse, one needs the generalized static Green operators of the isolated subsystems. For the case of free-free beams (rods) they are integral operators the kernels of which are the generalized static Green functions. Explicit formulas for generalized static Green functions of nonuniform free-free rods and Euler-Bernoulli beams are given in [5]. The more complicated case is considered in [10], where explicit formulas for the generalized static Green operator of a free-free 3D nonuniform beam with oscillators are obtained.

6. CONCLUSION

The problem of inversion of a real symmetric meromorphic matrix of the form (1) has been investigated. It is assumed that the inverse matrix is also meromorphic and its poles are of the first order. Such a matrix arises in many applied problems. One such problem—the problem of inverting a perturbed operator—is considered in Section 2.

Representations of the inverse matrix in the form of bilinear series are obtained. The formulas allow one to calculate the inverse matrix without using numerical inversion. The main results of the paper are formulated in Theorems 1 and 2. The former specifies the bilinear formula for the inverse matrix in terms of some numbers and vectors related to the matrix itself. The latter gives another representation with more quickly converging series that allows one to diminish the truncation error in numerical calculations.

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