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Effect of fractional orders in the velocity control of a servo system

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ABSTRACT

The application of fractional-order PID controllers is now an active field of research. This article investigates the effect of fractional (derivative and integral) orders upon system's performance in the velocity control of a servo system. The servo system consists of a digital servomechanism and an open-architecture software environment for real-time control experiments using MATLAB/Simulink tools. Experimental responses are presented and analyzed, showing the effectiveness of fractional controllers. Comparison with classical PID controllers is also investigated.

1. Introduction

Fractional calculus (FC) is the area of mathematics that extends derivatives and integrals to an arbitrary order (real or, even, complex order) which emerged at the same time as the classical differential calculus. FC generalizes the classical differential operator $D_t^n \equiv \mathrm{d}^n/\mathrm{d}t^n$ to a fractional operator D_t^α, where α can be a complex number [1–4]. However, its inherent complexity delayed the application of the associated concepts.

Nowadays, the fractional calculus is applied in science and engineering, being recognized its ability to yield a superior modeling and control in many dynamical systems. We may cite its adoption in areas such as viscoelasticity and damping, diffusion and wave propagation, electromagnetism, chaos and fractals, heat transfer, biology, electronics, signal processing, robotics, system identification, traffic systems, genetic algorithms, percolation, modeling and identification, telecommunications, chemistry, irreversibility, physics, control, economy and finance etc. [1–5].

In what concerns the area of control systems, the fractional controllers are now extensively investigated [4,6–8]. In [9], Ma and Hori used a PI$^\alpha$D-controller for the speed control of a two-inertia system. The superior robustness performance against input torque saturation and load inertia variation are shown by comparison with integer order PID control. In [10], Feliu-Batlle, Pérez and Rodríguez applied fractional algorithms in the control of main irrigation canals, which reveals to be robust to changes in the time delay and the gain. In [11], Valério and Sá da Costa introduced a fractional controller in a two degree of freedom flexible robot, achieving a stable response for the position of its tip. In [12], Sabatier, Poullain, Latteux and Oustaloup applied the CRONE control to a robust speed control of a low damped electromechanical system with backlash. The CRONE controller ensures robust speed control of the load in spite of plant parametric variations and speed observation errors.

In spite of the recent progresses, the truth is that simple and effective tuning rules, such as those for classical PID controllers, are still lacking. In this article, we use fractional PID controllers in the velocity control of an experimental servo system. Firstly, the gains of the fractional-order PID controller are obtained through the well-known Ziegler–Nichols (Z–N) heuristic rules [13] applied to the conventional PID controller. After that, we study the effect of the fractional orders upon the real-system control performance. The experiments show that the extra parameters (derivative and integral orders) provided by the fractional controllers can effectively be used to enhance the system performance and to adjust more carefully the dynamics of a control system.

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This paper is organized as follows. Section 2 presents the fundamentals of fractional-order control systems while Section 3 outlines the Oustaloup's frequency approximation method. Section 4 describes the experimental servo system set-up and Section 5 gives the open-loop Ziegler–Nichols tuning rules. Section 6 shows the experimental results obtained from the application of several types of fractional-order PID controllers. Finally, Section 7 draws the main conclusions.

2. Fractional-order control systems

In general, a fractional-order control system can be described by a Linear Time Invariant (LTI) fractional-order differential equation of the form:

$$a_n D_t^{\beta_n} y(t) + a_{n-1} D_t^{\beta_{n-1}} y(t) + \cdots + a_0 D_t^{\beta_0} y(t) = b_m D_t^{\alpha_m} u(t) + b_{m-1} D_t^{\alpha_{m-1}} u(t) + \cdots + b_0 D_t^{\alpha_0} u(t) \quad (1)$$

or by a continuous transfer function of the form:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\alpha_m} + b_{m-1} s^{\alpha_{m-1}} + \cdots + b_0 s^{\alpha_0}}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}} + \cdots + a_0 s^{\beta_0}} \quad (2)$$

where $\beta_k,\ \alpha_k\ (k = 0, 1, 2, \ldots)$ are real numbers, $\beta_k > \cdots > \beta_1 > \beta_0, \alpha_k > \cdots > \alpha_1 > \alpha_0$ and $a_k,\ b_k\ (k = 0, 1, 2, \ldots)$ are arbitrary constants.

The generalized operator ${}_aD_t^{\alpha}$, where a and t are the limits and α the order of operation, is usually given by the Riemann–Liouville definition ($\alpha > 0$):

$${}_aD_t^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{x(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \quad n-1 < \alpha < n \quad (3)$$

where $\Gamma(z)$ represents the Gamma function of z. Another common definition is that given by the Grünwald–Letnikov approach ($\alpha \in \Re$):

$${}_aD_t^{\alpha} x(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} x(t - kh) \quad (4)$$

where h is the time increment and $[v]$ means the integer part of v.

As shown by (3) and (4), the fractional-order derivatives are *global* operators having a memory of all past events. This property is used to model hereditary and memory effects in most materials and systems.

The fractional-order derivatives can also be defined in the transform domain. It is shown that the Laplace transform (L) of a fractional derivative of a signal $x(t)$ is given by:

$$L\{D^{\alpha} x(t)\} = s^{\alpha} X(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} x(t)\Big|_{t=0} \quad (5)$$

where $X(s) = L\{x(t)\}$. Considering null initial conditions, expression (5) reduces to the suitable form ($\alpha \in \Re$):

$$L\{D^{\alpha} x(t)\} = s^{\alpha} X(s) \quad (6)$$

which is a direct generalization of the integer order scheme with the multiplication of the signal transform $X(s)$ by the Laplace s-variable raised to a real value α. The Laplace transform reveals to be a valuable tool for the analysis and design of fractional-order control systems.

The fractional-order controllers were introduced by Oustaloup [2], who developed the so-called *Commande Robuste d'Ordre Non Entier* (CRONE) controller. More recently, Podlubny [8] proposed a generalization of the PID controller, the $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$-controller, involving an integrator of order λ and a differentiator of order μ. The transfer function $G_c(s)$ of such a controller has the form:

$$G_c(s) = \frac{U(s)}{E(s)} = K_P + K_I s^{-\lambda} + K_D s^{\mu}, \quad \lambda, \mu > 0 \quad (7)$$

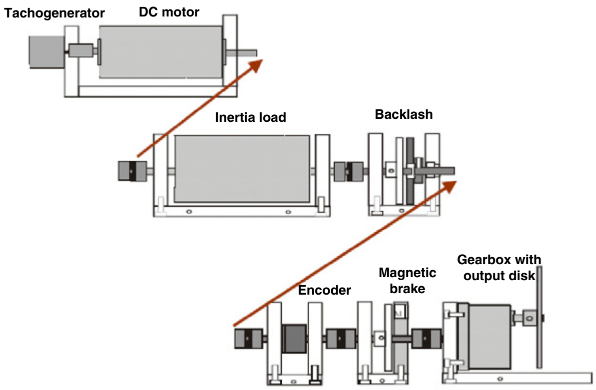
where $E(s)$ is the error signal and $U(s)$ the controller's output. The constants ($K_P,\ K_I,\ K_D$) are the proportional, integral, and derivative gains of the controller, respectively.

The $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$-controller is represented by a fractional integro-differential equation of type:

$$u(t) = K_P e(t) + K_I D^{-\lambda} e(t) + K_D D^{\mu} e(t). \quad (8)$$

Clearly, depending on the values of the orders λ and μ, we get an infinite number of choices for the controller's type (defined continuously on the $(\lambda,\ \mu)$-plane). For instance, taking $(\lambda,\ \mu) \equiv (1,\ 1)$ gives a classical PID controller, $(\lambda,\ \mu) \equiv (1,\ 0)$ gives a PI controller, $(\lambda,\ \mu) \equiv (0,\ 1)$ gives a PD controller and $(\lambda,\ \mu) \equiv (0,\ 0)$ gives a P-controller. All these classical types of PID controllers are the particular cases of the fractional $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$-controller. Thus, the $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$-controller is more flexible and gives the possibility of adjusting more carefully the dynamical properties of a control system [4].

(a) Set-up.



(b) Mechanical construction.

Fig. 1. The modular servo system (MSS) [15].

3. Oustaloup's approximation method

In order to implement the term s^α ($\alpha \in \Re$) of the fractional controller, a frequency-band limited approximation is used by cutting out both high and low frequencies of transfer $(s/\omega_u)^\alpha$ to a given frequency range $\omega \in [\omega_b, \ \omega_h]$, distributed geometrically around the unit gain frequency $\omega_u = (\omega_b\omega_h)^{1/2}$ [2,14]. The resulting continuous transfer function of such approximation is given by the formula:

$$D_N(s) = \left(\frac{\omega_u}{\omega_h}\right)^\alpha \prod_{k=-N}^{N} \frac{1+s/\omega'_k}{1+s/\omega_k} \tag{9}$$

where the zero and pole of rank k can be evaluated, respectively, as:

$$\omega'_k = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+\frac{1}{2}-\frac{\alpha}{2}}{2N+1}} \omega_b, \qquad \omega_k = \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+\frac{1}{2}+\frac{\alpha}{2}}{2N+1}} \omega_b. \tag{10}$$

Taking N, ω_b, ω_h, and α, permits the determination of the values of the set of zeros and poles of (10) and, consequently, the synthesis of the desired transfer function (9).

4. The experimental servo system

The Modular Servo System (MSS) consists of the Inteco (http://www.inteco.com.pl) digital servomechanism and an open-architecture software environment for real-time control experiments. The MSS supports the real-time design and implementation of advanced control methods using MATLAB/Simulink tools.

Fig. 1(a) illustrates the MSS set-up, which consists of several modules mounted in a metal rail and coupled with small clutches. The modules are arranged in the chain such that the DC motor with the generator module is at the front and the gearbox with the output disk is at the end of the chain (Fig. 1(b)).

The DC motor can be coupled with the modules of inertia, magnetic brake, backlash and gearbox with the output disk. The angle of rotation of the DC motor shaft is measured using an incremental encoder. The generator is connected directly to the DC motor and generates voltage proportional to the angular velocity.

The servomechanism is connected to a computer where a control algorithm is implemented based on the measurement of the angular position and/or velocity. The accuracy of measurement of the position is 0.1% while the accuracy of measured velocity is 5%. The armature voltage of the DC motor is controlled by a PWM signal $v(t)$ excited by a dimensionless control signal in the form $u(t) = v(t)/v_{\max}$. The admissible controls satisfy $|u(t)| \le 1$ and $v_{\max} = 12$ V [15].

5. Ziegler–Nichols tuning rules

In 1942, Ziegler and Nichols [13] recognized that the step responses of a large number of process control systems exhibits an S-shaped curve.

The S-shaped curve is characteristic of many higher-order systems, and such plant transfer function may be approximated by a first-order system plus a time delay of t_d seconds:

$$\frac{Y(s)}{U(s)} = \frac{Ae^{-t_d s}}{\tau s + 1} \tag{11}$$

Table 1
Ziegler–Nichols tuning for the controller $G_c(s) = K_P\left(1 + \frac{1}{T_I s} + T_D s\right)$, for a decay ratio of 0.25.

Type of controller	K_p	T_I	T_D
P	$\frac{1}{RL}$	∞	0
PI	$\frac{0.9}{RL}$	$\frac{L}{0.3}$	0
PID	$\frac{1.2}{RL}$	$2L$	$0.5L$

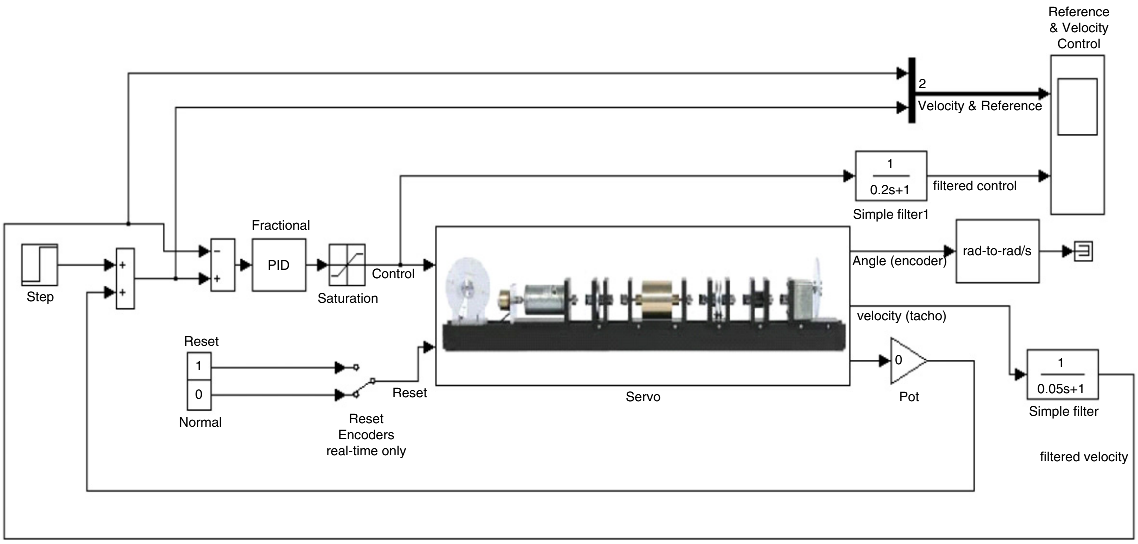


Fig. 2. Real-time model with the fractional PID controller (adapted from [15]).

where τ is the system time constant and A is the gain. The constants (A, t_d, τ) are determined from the unit step response of the process. If a tangent is drawn at the inflection point of the S-shaped curve, then the slope of the line is $R = A/\tau$ and the intersections of the tangent line with the time axis and line $y(t) = A$ identifies the time delay $L = t_d$ and time constant τ [16].

The choice of controller parameters is designed to result in a closed-loop step response transient with a decay ratio of approximately 0.25 in one period of oscillation. This corresponds to a damping factor $\zeta = 0.21$ and is a good compromise between quick response and adequate stability margins. Table 1 lists the controller parameters suggested by Ziegler and Nichols (Z–N) to tune the proportional gain K_p, integral time T_I, and derivative time T_D.

Once the values of T_I and T_D have been obtained, the gains K_I and K_D, are computed as:

$$K_I = \frac{K_P}{T_I}, \qquad K_D = K_P T_D. \tag{12}$$

In general, the controller settings according to Z—N rules provide a good closed-loop response for many systems.

6. Experimental results

This section investigates the application of several types of fractional-order PID controllers in the control of the angular velocity of the servo system.

The MSS set-up for the experiments includes the modules of DC motor with tacho-generator, inertia load, encoder and gearbox with output disk (see Fig. 1).

The real-time control experiments are performed using the MATLAB/Simulink real-time model given in Fig. 2. A fixed-step solver (Euler's integration method) of a fixed-step size set to 0.01 (sampling period of $T = 0.01$ s) is chosen.

For the identification experiment, a unit step input is applied to the servo system and the process reaction curve is acquired, as shown in Fig. 3. Following the method of Ziegler–Nichols, as described in Section 5, we get the system parameters $A = 187.2106$, $\tau = 1.1841$ and $L = 0.1753$. The controller parameters are then calculated according to the formulae given in Table 1.

The fractional term s^α ($\alpha \in \Re$) in the fractional PID controller transfer function (7) is implemented by using the Oustaloup's frequency approximation method described in Section 3. The values used were $N = 5$, $\omega_b = 1$ rad/s and $\omega_h = 1000$ rad/s.

The fractional-order controllers are implemented in digital form by discretization of the continuous controller transfer functions. The discretization technique used consists in the bilinear (or Tustin's) approximation with a sampling period of $T = 0.01$ s.

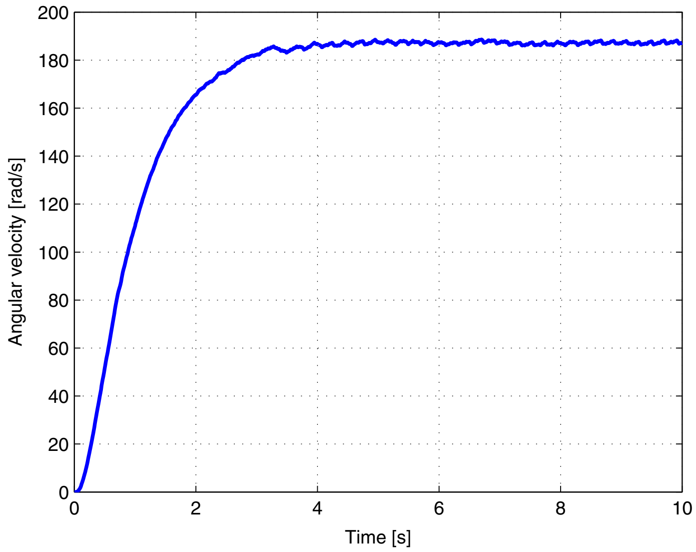


Fig. 3. Unit step response of the servo system.

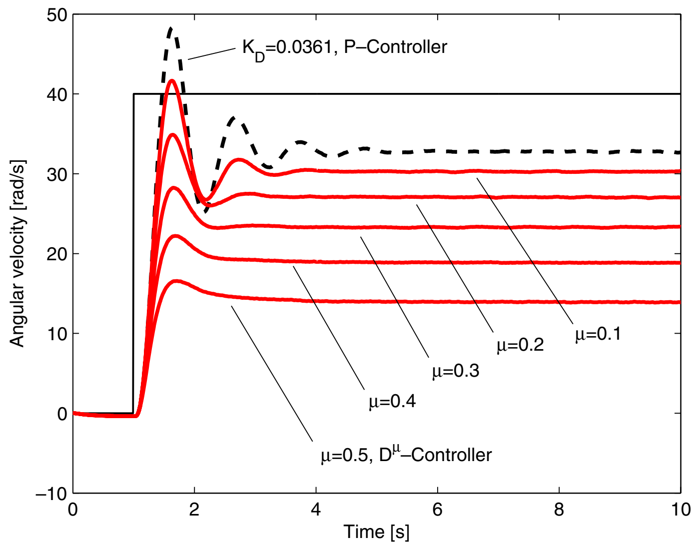


Fig. 4. Response of the real system with the D^μ-controller and $\mu = \{0.1, 0.2, 0.3, 0.4, 0.5, 1\}$.

In the following experiments, a step input of amplitude 40 rad/s is applied to the closed-loop servo system (Fig. 2) and the angular velocity *versus* time is acquired for different types of fractional-order PID controllers. The experimental results are presented and analyzed.

6.1. The D^μ*-controller*

The transfer function of a fractional D^μ-controller is given by:

$$G_c(s) = K_D s^\mu, \quad \mu > 0 \tag{13}$$

where the gain K_D and the derivative order μ are the parameters to be tuned.

The fractional controller is designed by adopting the proportional gain of the P-controller obtained from the Ziegler–Nichols rules, that is, $K_D = 1/RL = 0.0361$.

Fig. 4 depicts the experimental step responses of the angular velocity for several values of derivative order $\mu = \{0.1, 0.2, 0.3, 0.4, 0.5, 1\}$ while maintaining the derivative gain $K_D = 0.0361$.

The responses reveal that the steady-sate error increases as the order μ increases. The variation of the gain K_D was also tested (with a fixed value of the derivative order) and the system showed a diminishing steady-state error as K_D increases. However, the overshoot and settling time are more acceptable for the case where the order μ is changed. We verify that

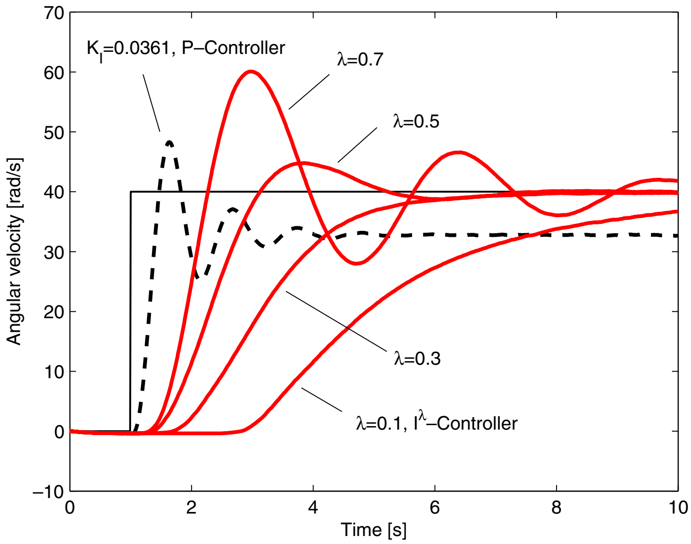


Fig. 5. Response of the real system with the I^{λ}-controller and $\lambda = \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 1\}$.

the extra degree of tuning provided by the fractional controller, in comparison to the classical P-controller, may be useful to yield a satisfactory control.

6.2. The I^{λ}*-controller*

The transfer function of a fractional I^{λ}-controller is given by:

$$G_c(s) = \frac{K_I}{s^{\lambda}}, \quad \lambda > 0 \tag{14}$$

where the gain K_I and the integrative order λ are the parameters to be tuned.

In order to assure a good steady-state error, the term $1/s^{\lambda}$ must be implemented by means of an integer integrator [16, 17]. The modified I^{λ}-controller is then given in the form:

$$G_c(s) = K_I \frac{s^{1-\lambda}}{s}, \quad 0 < \lambda < 1. \tag{15}$$

The I^{λ}-controller is designed by adopting the proportional gain of the P-controller obtained from the Ziegler–Nichols rules, that is, $K_I = 1/RL = 0.0361$.

Fig. 5 shows the experimental step responses of the angular velocity for several values of integrative order $\lambda = \{0.1, 0.3, 0.5, 0.7, 1\}$ while maintaining the integral gain $K_I = 0.0361$. The variation of gain K_I (with integrative order fixed) was also tested. We observed that the steady-state error is very small. Note that the real system is nonlinear and, therefore, the oscillations are damped very quickly. Also, we verify that the fractional order λ is a very useful parameter for adjusting the dynamics of the control system. In fact, the order λ has a large influence upon the system dynamics, as illustrated in Fig. 5.

6.3. The PI^{λ}*-controller*

The transfer function of a fractional PI^{λ}-controller is given by:

$$G_c(s) = K_P + \frac{K_I}{s^{\lambda}}, \quad \lambda > 0 \tag{16}$$

where the proportional gain K_P, the integral gain K_I and the integration order λ are the parameters to be tuned. The term K_I/s^{λ} is implemented as in (15).

The fractional controller is designed by adopting the controller parameters of the PI controller obtained from the Ziegler–Nichols rules, that is, $K_P = 0.9/RL = 0.0325$ and $K_I = 0.3K_P/L = 0.0556$.

Fig. 6 shows the experimental step responses of the angular velocity for several values of integrative order $\lambda = \{0.3, 0.5, 0.7, 0.9, 1\}$ while maintaining the gains $K_P = 0.0325$ and $K_I = 0.0556$. The variation of integral gain K_I (with integrative order fixed) was also tested. As in previous case, the steady-state error is very small. The steady-state behavior could be also improved by multiplying the fractional controller by a term of the form $(s + \eta)/s$, with η being a small value [10]. Note the influence of the order λ in the system overshoot and settling time. An adequate phase margin can be easily established by a proper choice of fractional order λ. However, the output converges to its final value more slowly, as should be expected by a weak fractional integral term.

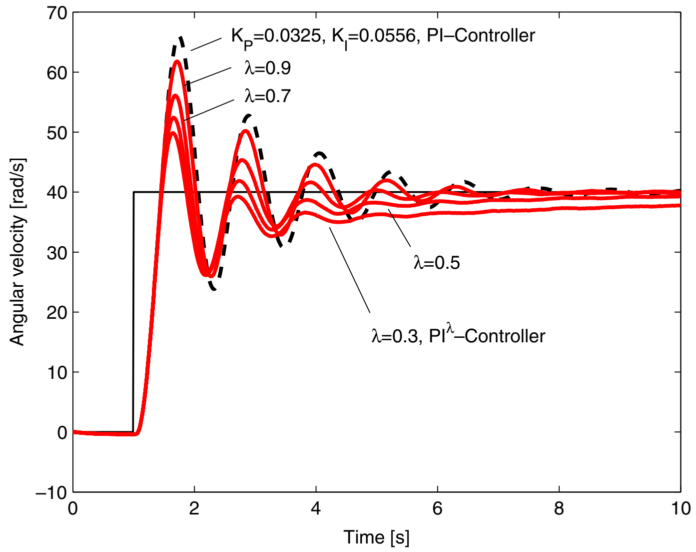


Fig. 6. Response of the real system with the PI^{λ}-controller and $\lambda = \{0.3, 0.5, 0.7, 0.9, 1\}$.

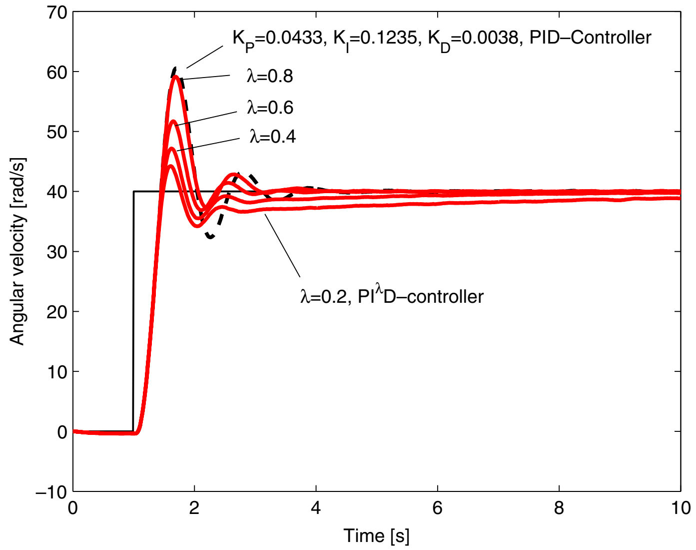


Fig. 7. Response of the real system with the $PI^{\lambda}D$-controller and $\lambda = \{0.2, 0.4, 0.6, 0.8, 1\}$.

6.4. The $PI^{\lambda}D$*-controller*

The transfer function of a fractional $PI^{\lambda}D$-controller is:

$$G_c(s) = K_P + \frac{K_I}{s^{\lambda}} + K_D s, \quad \lambda > 0 \tag{17}$$

where the proportional gain K_P, the integral gain K_I, the derivative gain K_D and the integrative order λ are the parameters to be tuned. The term K_I/s^{λ} is implemented as in (15).

The fractional controller is designed by adopting the controller parameters of the PID controller obtained from the Ziegler–Nichols rules, that is, $K_P = 1.2/RL = 0.0433$, $K_I = K_P/(2L) = 0.1235$ and $K_D = 0.5LK_P = 0.0038$.

Fig. 7 shows the experimental step responses of the angular velocity for several values of integrative order $\lambda = \{0.2, 0.4, 0.6, 0.8, 1\}$ while maintaining the gains $K_P = 0.0433$, $K_I = 0.1235$ and $K_D = 0.0038$. Once more, we note the influence of the order λ upon the system performance, particularly in the overshoot and settling time. The slow convergence of the response to its final value, due to a weak fractional integral, is also evident.

7. Conclusions

In this article we investigated the velocity control of a servo system by using several fractional-order PID controllers. It was shown that the fractional controllers can effectively enhance the control system performance providing extra tuning parameters useful for the adjustment of the control system dynamics.

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