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A Trivalent Graph of Girth Ten

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Among trivalent or cubic graphs (regular graphs of valency, or degree, equal to three), "cages" present a considerable interest owing to their symmetry.

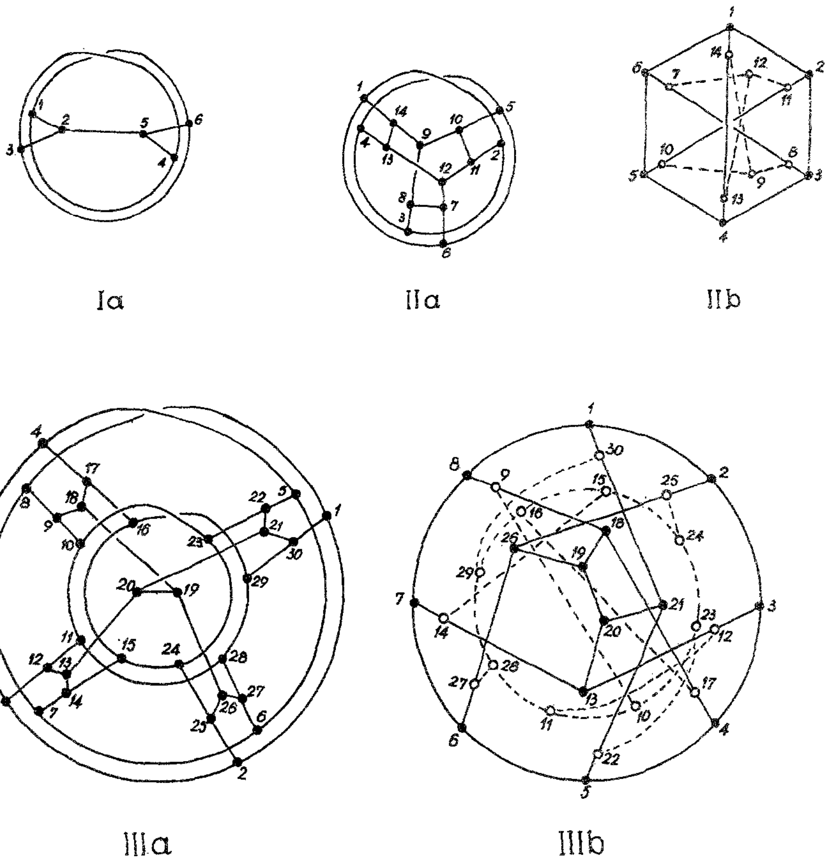

FIG. 1 (*Continued on next page*).

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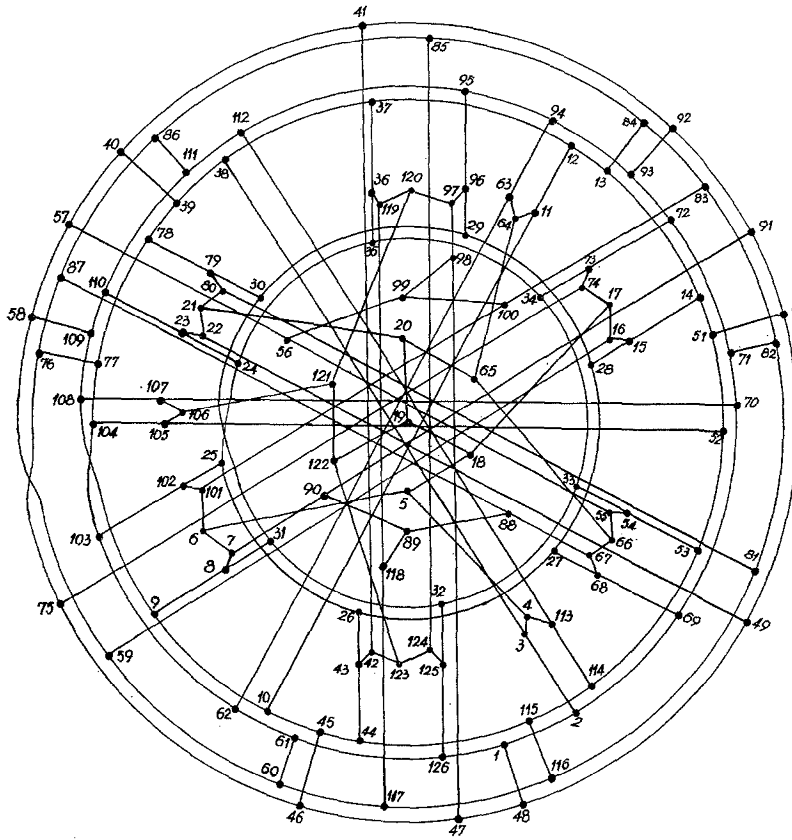

IVa

Fig. 1. n-Cages: (I) $n = 4$; (II) $n = 6$; (III) $n = 8$; (IV) $n = 12$. Representations (a) evidence connection of opposite vertices of n-gon by an $n/2$ path. Representations (b) evidence an $(n - 2)$-cage (full edges, black circle vertices) within an n-cage (all edges and vertices).

The known n-cages (where n is positive and even) [3–6] with girths $n = 4$ (I, Thomsen graph[1]), 6 (II, Heawood graph), 8 (III, Tutte graph), and 12 (IV, Benson graph [1]), all contain as a subgraph an n-gon whose pairs of opposite vertices are connected by paths of length $n/2$. Applying this observation to graphs of girth 10, i.e., starting with a decagon whose pairs of opposite vertices are connected by paths of length 5, the graph V

[1] Roman numerals indicate graphs. Letters following a Roman numeral discriminate among isomorphic geometric realizations. The numbering of vertices, evidencing Hamiltonian circuits (spanning cycles) in all graphs under discussion, is invariant in isomorphic graphs presented in Figs. 1–4.

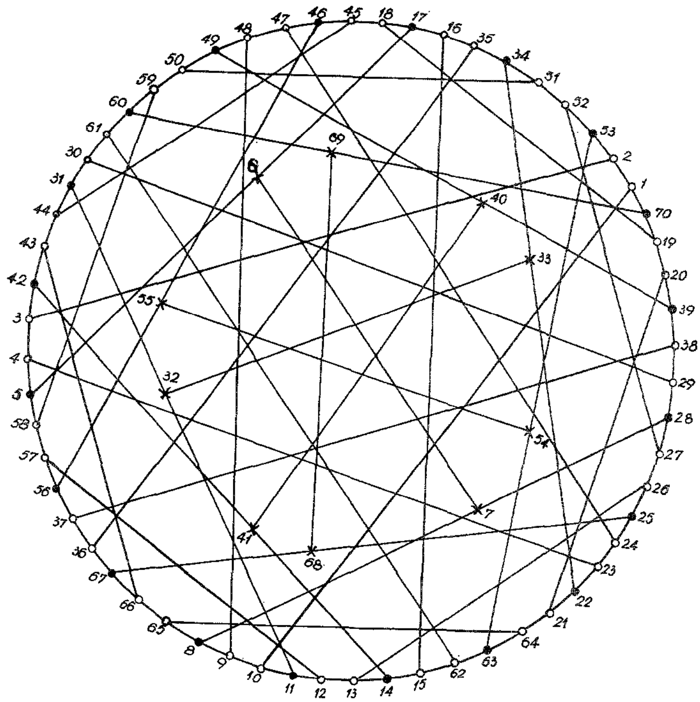

FIG. 2. The trivalent graph (V) of girth ten.

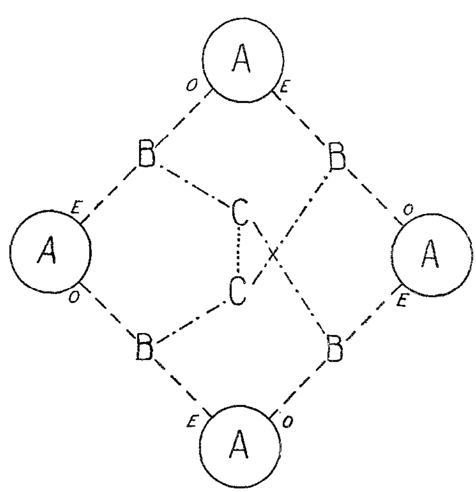

FIG. 3. Structural diagram of graph V. Each A is a circuit of 10 vertices. Each B and C represents 5 separated vertices. Four kinds of edge (—, ---, -.-, and ···) and three kinds of vertex (A, B, and C) are evident. In the A-circuits, E and O designate even and odd numbered vertices, respectively.

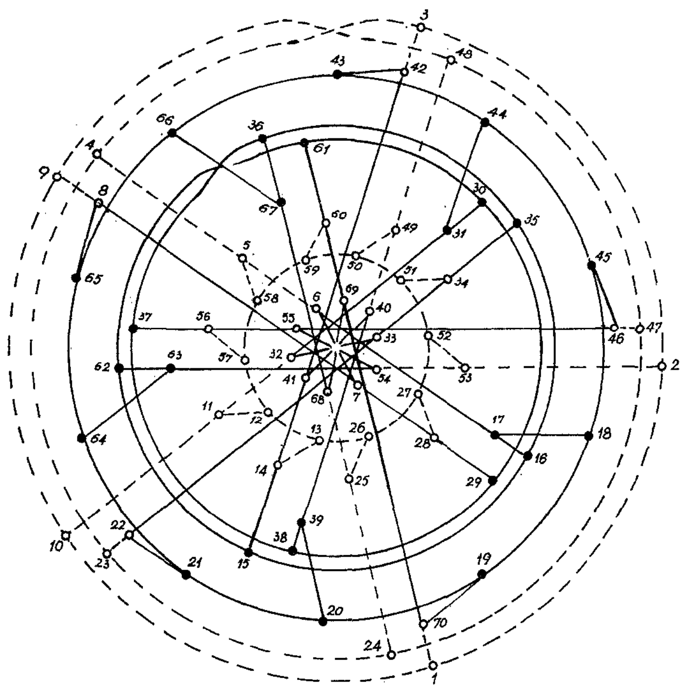

FIG. 4. Graph V in representation evidencing connection of opposite vertices of decagons by paths of length 5, as well as an 8-cage (full edges, black vertices) within the graph V.

was obtained. It has four such decagons and 70 vertices. This graph V has four kinds of edges and three kinds of vertices: 40 of these form the four decagons (drawn as open circles in Fig. 2), 20 (drawn as black circles) are adjacent to, and 10 vertices (drawn as ×) are not adjacent to, the first 40 vertices. A symbolic structural diagram evidencing the kinds of vertices and edges ($4E/3V$ in the notation of R. M. Foster) is shown in Fig. 3. The diameter of graph V is 6 (cf. vertices 1 and 29, 8 and 14, or 6 and 32 in Fig. 2).

An $(n-2)$-cage is made apparent in n-cage diagrams, presented in Fig. 1, by using full and dashed lines, as well as open and black circles. A similar relationship between the 8-cage and graph V is shown in Fig. 4.

The group of graph V is of order 80 since each of the four decagons can be mapped on itself or on the other three decagons in 2×10 ways. Being intransitive with respect to vertices and edges, graph V is not even 0-regular [5, 6]. Graphs I–III are edge- and vertex-transitive, graph IV only edge-transitive.

If a generalized (k, n)-cage is defined as a regular connected graph of valency k and girth n which posseses the minimum number of vertices among such graphs, then it is conjectured that graph V is a (3, 10)-cage.[2] Even if this conjecture holds, however, it does not seem appropriate to denominate graph V as a 10-cage because it is not s-regular[3]: the earlier definition of n-cages [5] emphasized their s-regularity (this emphasis is less apparent, however, in the more recent definition [6]).

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[2] Personal communication by R. M. Foster. Previously known trivalent graphs of girth 10 (cf. [2]) had at least 80 vertices.

[3] Or s-unitransitive [3].