HOMEOMORPHISMS OF SUFFICIENTLY LARGE $P^2$-IRRREDUCIBLE 3-MANIFOLDS

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Let $V$ be a compact connected $PL$ 3-manifold which is irreducible, sufficiently large, and contains no embedded projective plane having a trivial normal bundle. Denote by $PL(V\text{ rel }\partial)$, $G(V\text{ rel }\partial)$ the simplicial spaces of $PL$ homeomorphisms, respectively, homotopy equivalences of $V$ which restrict to the identity on $\partial V$. Waldhausen[3] showed that the inclusion $PL(V\text{ rel }\partial)\to G(V\text{ rel }\partial)$ induces an isomorphism on $\pi_0$. Laudenbach[1] extended this to $\pi_1$. Pushing their techniques further, we prove in this paper:

**Theorem.** The inclusion $PL(V\text{ rel }\partial)\to G(V\text{ rel }\partial)$ is a homotopy equivalence.

Since $V$ is a $K(\pi_1,1)$, it is an easy application of obstruction theory to determine the homotopy type of $G(V\text{ rel }\partial)$ when $\partial V = \emptyset$. One finds that $\pi_0G(V)$ is isomorphic to the group of outer automorphisms of $\pi_1V$, $\pi_1G(V)$ is isomorphic to the center of $\pi_1V$, and all higher homotopy groups of $G(V)$ vanish. Furthermore, Waldhausen[4] showed that when the center of $\pi_1V$ is nontrivial and $V$ is orientable, $V$ is a Seifert manifold[2]. In this case either $V$ is the 3-torus, or the center of $\pi_1V$ is $\mathbb{Z}$, generated by an orbit of an $S^1$ action on $V$. Thus the identity component of $PL(V)$ contains a Lie subgroup ($\{1\}$, $S^1$, or $S^1\times S^1\times S^1$) as a deformation retract, when $V$ is closed and orientable. Similar statements probably hold also in the non orientable case. When $\partial V \neq \emptyset$, the components of $G(V\text{ rel }\partial)$ are contractible, though $G(V\text{ rel }\partial)$ itself need not be, e.g. for $V = S^1\times S^1\times I$.

It is a well-known consequence of the triangulation theorems of Bing and Moise that the simplicial spaces of $PL$ and topological homeomorphisms of any 3-manifold have the same homotopy type, so our theorem holds also in $TOP$. To pass to the differentiable category one would need to know that $Diff(D^3\text{ rel }\partial)$ is contractible (the "Smale conjecture").

The significance of the theorem is that it gives the first examples of manifolds $M^n$, $n > 2$, for which the full homotopy type of $PL(M^n\text{ rel }\partial)$ or $TOP(M^n\text{ rel }\partial)$ is known, excluding of course the trivial case $M = D^n$.

In some cases when $V \subset \mathbb{R}^4$ the theorem has also been obtained by T. Akiba.

§1. A SPECIAL CASE

For notational simplicity we will omit "rel $\partial$" from now on. Thus $PL(V)$ means $PL(V\text{ rel }\partial)$, etc.

Let $V$ be a 3-disc with a finite number of 1-handles attached. In this section we prove that $PL(V)$, like $G(V)$, is contractible. This special case of the theorem contains the essential ideas of the general case while avoiding most of the technical complexities. Also, it will be used in the proof of the general case.

Let $M \subset V$ be the co-core of one of the 1-handles of $V$. Thus $M$ is a 2-disc. Denote by $E(M, V)$ the simplicial space of $PL$ embeddings $M \to V$ which are proper ($M \cap \partial V = \partial M$) and agree with the given $M \subset V$ on $\partial M$. Restriction to $M \subset V$ gives a fibration:

$$PL(V')\to PL(V)\to E(M, V),$$

where $V'$ is $V$ cut open along $M$. If we can show that $E(M, V)$ is contractible, then $PL(V)$ deforms into $PL(V')$ which is contractible by induction on the number of handles. (The induction starts with $PL(D^3) = \ast$ by the Alexander trick.)

Let $M \times I \subset V$, $I = [0, 1]$, be a collar on the given $M = M \times 1$ and set $N = M \times 0 \subset V$.

**Proposition 1.** The subspace $E(M, V - N)$ of $E(M, V)$ consisting of embeddings disjoint from $N$ is a deformation retract of $E(M, V)$.

Assuming this, it remains to show that $E(M, V - N)$ is contractible. Consider the fibration
Proof of Proposition 1. Let $M \to V$, $t \in D^k$, represent an element of $\pi_n(E(M, V), E(M, V - N))$. Let $(N \times [-1, 1], \partial N \times [-1, 1])$ be a bicollar neighborhood of $(N, \partial N)$ in $(V, \partial V)$, which we may assume to be disjoint from $\partial M$.

**Lemma 1.** There exist finite covers $\{B_i\}$ and $\{B'_i\}$ of $D^k$ by $k$-balls, with $B'_i \subset \text{int } B_i$, such that $M_i$ is transverse to a slice $N = N \times s_i \subset N \times [-1, 1]$ for $t \in B_i$. Also, we may assume that $N_i \neq N_j$ if $i \neq j$.

**Proof.** Triangulate so that the composition

$$\bigcup_i M_i \cap (N \times [-1, 1] \times D^k) \to N \times [-1, 1] \times D^k \to [-1, 1] \times D^k$$

is simplicial. The triangulation $T$ of $[-1, 1] \times D^k$ intersects each slice $[-1, 1] \times t$ in a triangulation $T_t$. For fixed $t$ and for $s$ in the interior of a one-simplex of $T_t$, $M_i \cap N \times s$ is independent of $s$, up to isotopy, and hence $M_i$ is transverse to $N \times s$. Thus in $[-1, 1] \times D^k$ we need only avoid a subpolyhedron $X$ which is zero-dimensional in each $t$-slice, namely, the simplices of $T$ which project to $D^k$ non-degenerately. So choose finite covers $\{B_i\}$ and distinct $s_i \in [-1, 1]$ such that $s_i \times B_i \cap N = \emptyset$ for each $i$. □

To prove the proposition we will construct a family $h_{u, 0 \leq u \leq 1}$ of isotopies of $M$, which for $t \in B_i$ eliminates all the circles of intersection of $M_i$ with $N_i$. This is sufficient to make $M$ disjoint from $N$ over $D^k$: "average" the $N_i$'s together via a partition of unity subordinate to $\{B_i\}$ to get a slice $N \times s(t)$ disjoint from the isotoped $M_i$, then ambient isotope $N \times s(t)$ to $N$. If $M_i$ is disjoint from $N_i$ and $N_i$, it is disjoint from the region between $N_i$ and $N_i$, since $M_i$ is connected and $\partial M_i \cap N \times [-1, 1] = \emptyset$.

Let $\mathcal{C}_i$ be the collection of circles in $M_i \cap N_i$ for all $i$ such that $t \in B_i$. Each circle $C \in \mathcal{C}_i$, with $C \subset N_i$, say, belongs to a family $C_i \subset N_i$, $t \in B_i$, which varies by a $k$-parameter isotopy. $C_i$ bounds unique discs $D^2_i(C_i) \subset M_i$ and $D^2_i(C_i) \subset N_i$. The inclusion relations among the $D^2_i$'s define a partial ordering $\prec_\mathcal{C}$ on $\mathcal{C}_i$: $C_i \prec_\mathcal{C} C'_i$ if $D^2_i(C_i) \subset D^2_i(C'_i)$. Similarly we have $\prec_\mathcal{N}$ on $\mathcal{N}_i$.

The basic construction is the following. Fix a $t \in D^k$. If $C_i \in \mathcal{C}_i$ is minimal in $\prec_\mathcal{C}$, then $D^2_i(C_i)$ or $D^2_i(C'_i)$ is an embedded 2-sphere in $V$, which bounds a (unique) 3-disc $D^3_i(C_i)$ in $V$ since $V$ is irreducible. Choose a homeomorphism of $(D^3_i(C_i), D^2_i(C_i), D^2_i(C'_i))$ with a standard lens-shaped model $(D^3; D^2, D^2, D^2)$, where $\partial D^3 = D^2 \cup D^2 \cup D^2 \cap D^2 = S^1$. There is then an evident isotopy of $D^2_i(C_i)$ across $D^3_i(C_i)$ to $D^2_i(C'_i)$ which, if continued slightly to the other side of $D^2_i(C_i)$, gives an isotopy of $M_i$ eliminating $C_i$.

We must somehow piece together all these little isotopies for the various $C_i \in \mathcal{C}_i$ in such a way that the resulting isotopy $h_{u,0} \equiv h_{u}$ is continuous in $t$. To begin, we construct a (PL) family of functions $\varphi : \mathcal{C} \to (0, 2)$ which are to tell in what order the circles of $\mathcal{C}$ are to be eliminated. $\varphi$ is to satisfy:

1. $\varphi_i(C_i) < \varphi_i(C'_i)$ whenever $C_i \prec_\mathcal{C} C'_i$,
2. $\varphi_i(C_i) < 1$ if $C_i \subset N_i$ and $t \in B'_i$,
3. $\varphi_i(C_i) > 1$ if $C_i \subset N_i$ and $t \in \partial B_i$.

Such a family $\varphi$ clearly exists.

Let $C_i \in \mathcal{C}_i$, with $C_i \subset N_i$. In the graph $\Gamma \subset D^k \times [0, 2]$ of $\cup \varphi$, the points $\varphi_i(C_i)$ form a sheet lying over $B_i$. Thicken this sheet to a "plate" intersecting $t \times [0, 2]$ in an interval $I(C_i) = [\varphi_i(C_i), \varphi_i(C'_i) + \varepsilon]$. We can assume $\varepsilon$ is chosen small enough so that $I(C_i) \cap I(C'_i) = \emptyset$ only near the intersection of the sheets of $\Gamma$ corresponding to $C_i$ and $C'_i$, so that neither $C_i \prec_\mathcal{C} C'_i$ nor $C'_i \prec_\mathcal{C} C_i$.

$I(C_i)$ is intended to be the $u$-support of the little isotopy $h_{u,0}(C_i)$ of $M_i$ which eliminates $C_i$ as above, assuming we have already eliminated all other circles of $\mathcal{C}_i$ with smaller $\varphi$-value, thereby
making $C_i$ minimal in $<_\omega$. However it might happen that for some $C'_i \in \mathcal{C}_i$ with $\varphi_i(C_i') < \varphi_i(C_i)$, we also have $C_i <_\omega C'_i$.

In this case the isotopy which eliminates $C'_i$ would automatically eliminate $C_i$. So we reduce the graph $\Gamma$ by deleting all $\varphi_i(C_i)$ such that $\varphi_i(C_i) > \varphi_i(C'_i)$ for some $C'_i$ with $C_i <_\omega C'_i$. Also, we reduce the intervals $I(C_i)$ by deleting points $u \in I(C_i)$ such that $u > \varphi_i(C'_i)$ for some $C'_i$ with $C_i <_\omega C'_i$.

As a result, if two reduced intervals $I(C_i)$ and $I(C'_i)$ overlap, then $C_i$ and $C'_i$ are unrelated both in $<_\omega$ and in $<_\omega$. Hence $D^3(C_i) \cap D^3(C'_i) = \emptyset$ and the isotopies $h_{u_i}(C_i)$ and $h_{u_i}(C'_i)$ are completely independent of each other, having disjoint supports in $V$.

Now for fixed $t \in D^k$, the $h_{u_i}(C_i)$'s can be strung together in the obvious way: Proceeding upward in the reduced $\Gamma$ from $u = 0$ to $u = 1$, on the reduced interval $I(C_i)$, which is an initial segment of $[\varphi_i(C_i), \varphi_i(C'_i) + \epsilon]$, take the restriction of $h_{u_i}(C_i)$ to this initial segment. And on the overlaps of these reduced $I(C_i)$'s, do (the initial segments of) the corresponding $h_{u_i}(C_i)$'s concurrently.*

To get $h_{u_i}$ simultaneously for all $t \in D^k$, first triangulate $\Gamma$ and $D^k$ so that the projection $\Gamma \to D^k$ is simplicial. (The reduced graph is then a subcomplex of $\Gamma$.) Assume inductively that $h_{u_i}$ has already been defined over the boundary of an $l$-simplex $\Delta_l'$ in $D^k$. In particular, the $h_{u_i}(C_i)$'s have been defined over $\partial \Delta_l'$ as $PL(l - 1)$-parameter families. To extend $h_{u_i}(C_i)$ to an $l$-parameter family over the interior of $\Delta_l'$ it is only necessary to extend the homeomorphism of $(D^3(C_i); D^3_\partial(C_i), D^3_+)$ with the standard model $(D^2; D^2_+, D^2_\ast)$ from $t \in \partial \Delta_l'$ to $t \in \Delta_l'$. This is done as follows. A family of homeomorphisms $f_t: D^3_l \to D^3_l$ for $t \in B_l$ is chosen at the start by isotopy extension. Over $\partial \Delta_l'$ we have by induction an extension of $f_t$ to $F_t: D^3(C_i) \to D^3$, which we wish to extend over $\Delta_l'$. By isotopy extension $f_t$ can be extended to $F_t: D^3(C_i) \to D^3$ for $t \in \Delta_l'$. The obstruction to extending $F_t$ from $\partial \Delta_l'$ to $\Delta_l'$ is the homotopy class of $\partial \Delta_l' \to PL(D^3_\partial D^3_\ast)$, $t \mapsto F_t \ast F_t^{-1}$. But $PL(D^3_\partial D^3_\ast)$ is contractible by the Alexander trick. Thus each isotopy $h_{u_i}(C_i)$ for $t \in \partial \Delta_l'$ extends over $\Delta_l'$, and the same prescription for building $h_{u_i}$ from the $h_{u_i}(C_i)$'s for fixed $t$ now works for all $t \in \Delta_l'$ simultaneously. So $h_{u_i}$ for $t \in \partial \Delta_l'$ extends to $h_{u_i}$ for $t \in \Delta_l'$.

One final remark: If $C_i \in \mathcal{C}_i$, with $C_i \subseteq N_{u_i}$ then as $t$ crosses $\partial B_{u_i}$, $C_i$ is suddenly dropped from $\mathcal{C}_i$. This could cause a discontinuity in the family $h_{u_i}$. However at $\partial B_{u_i}, \varphi_i(C_i) > 1$. So since we restrict $u$ to $[0, 1]$ this difficulty does not actually occur. (The requirement that $\varphi_i(C_i) < 1$ for

*Note that if $\varphi_i(C_i)$ is not deleted from $\Gamma$, then $C_i$ is not moved by $h_{u_i}$ for $u < \varphi_i(C_i)$.
Theorem 1. If $t \in B'$, guarantees that $h_{u}$ for $u \in [0,1]$ will eliminate $C_{t}$ over $B'$, as desired, provided that $\epsilon$ is small enough so that $I(C_{t}) = [\varphi(C_{t}), \varphi(C_{t}) + \epsilon] \subset [0,1]$ for $t \in B'$.

2. THE GENERAL CASE

Let $V$ be $P^{2}$-irreducible and sufficiently large. We distinguish two cases: (1) $\partial V \neq \emptyset$, (2) $\partial V = \emptyset$. In (1), $\pi_{1}G(V) = 0$ for $k \geq 1$ and in (2), $\pi_{1}G(V) = 0$ for $k \geq 2$. We will prove the corresponding statements with $G$ replaced by $PL$. This suffices to prove the theorem since Waldhausen and Laudenbach have already shown that $PL(V) \rightarrow G(V)$ induces an isomorphism on $\pi_{0}$ and $\pi_{1}$. From now on we assume $k \geq 1$ in case (1) and $k \geq 2$ in case (2).

As in §1, consider the fibrations

$$PL(V') \rightarrow PL(V) \rightarrow E(M, V),$$

$$PL(M \times I) \rightarrow E \rightarrow E(M, V - N),$$

where now $M$ is an incompressible surface in $V$ (and $N$ is a parallel copy of $M$). If $\pi_{0}E(M, V) = 0$, then $\pi_{1}PL(V) = \pi_{1}PL(V')$. By hypothesis, $V$ possesses a hierarchy, so that $V$ can be reduced to a disjoint union of discs by a finite number of such cutting operations $V \rightarrow V'$. Hence we would have $\pi_{1}PL(V) = \pi_{1}PL(D^{2}) = 0$.

**Proposition 2.** $\pi_{1}(E(M, V), E(M, V - N)) = 0$ for $k \geq 1$ if $\partial M \neq \emptyset$ and for $k \geq 2$ if $\partial M = \emptyset$.

Assuming this, we have $\pi_{1}E(M, V) = \pi_{1}E(M, V - N) = \pi_{1-1}PL(M \times I)$. In case (1), $\partial M \neq \emptyset$, and so $\pi_{1-1}PL(M \times I) = 0$ by §1. In case (2), where $\partial M = \emptyset$ for the first cut, $\pi_{1-1}PL(M \times I) = 0$ ($k \geq 2$) by case (1).

It remains only to prove Proposition 2, which is a simple matter of combining Laudenbach’s methods with the machinery of §1. Let $M \rightarrow V$, $t \in D^{k}$, representing an element of $\pi_{1}(E(M, V), E(M, V - N))$, be transverse to $N_{t}$ over $B_{t} \subset D^{k}$ as before. The proof of Proposition 1 applies in this setting to eliminate over $B'_{t} \subset B_{t}$ all the circles in $M_{t} \cap N_{t}$ which are contractible in $V$ (hence also in $M_{t}$ and $N_{t}$ since $\pi_{1}M_{t}$ and $\pi_{1}N_{t}$ inject into $\pi_{1}V$). So suppose only non-contractible circles remain in $M_{t} \cap N_{t}$.

Let $p: \tilde{V} \rightarrow V$ be the covering with $p \circ \pi = \pi_{1}M_{t} \subset \pi_{1}V$. Let $\tilde{M}_{t} \rightarrow \tilde{V}$ be a lift of $M_{t} \rightarrow V$ (so $p$ restricts to a homeomorphism of $\tilde{M}_{t}$ onto $\pi_{1}V$). Let $\tilde{N}_{t}$ separate $\tilde{V}$ into two components. Let $Y_{\alpha}$ be the closure of the component not containing $\partial M_{t}$ if $\partial M_{t} \neq \emptyset$, or if $\partial M_{t} = \emptyset$, the closure of the component not containing $\tilde{M}_{t}$ for $t \in \partial D^{k}$. $Y_{\alpha}$ is well-defined in the latter case since $\partial D^{k}$ is connected if $k \geq 2$. This is the only place where $k \geq 2$ in case (2) is required. It is not hard to extend this case to $k = 1$ by including some extra data about paths to a basepoint. (See [1], II.7.3.)

Now fix a $t \in B$. Let $Y_{\alpha}$ be minimal, with respect to inclusion, among the (finite number of) $Y_{\alpha}$‘s intersecting $\tilde{M}_{t}$, and let $C_{i}$ be a component of $\tilde{M}_{t} \cap Y_{\alpha}$. Assuming only that there exists a homotopy of $M_{t}$ (rel $\partial$) into $V - N$, Laudenbach shows ([1], Corollary II.4.2 and Lemmas II.5.4(3) and (5)) that $C_{i}$ is one end of a (unique) h-cobordism $W(C_{i}) \subset Y_{\alpha}$, the other end $C'$ lying in $\tilde{N}_{\alpha}$, and moreover that $p|W(C_{i})$ is a homeomorphism. Since $V$ is irreducible, $p(W(C_{i}))$ is a product h-cobordism. Choosing a product structure determines an isotopy of $M_{t}$ carrying $p(C_{i})$ across $p(W(C_{i}))$, eliminating $C_{i} \subset \tilde{M}_{t} \cap Y_{\alpha}$, and thus $\tilde{M}_{t} \cap Y_{\alpha}$ is disjoint from $N_{\alpha}$.

Adding parameters presents no new difficulties. Let $Y_{\alpha}$ be minimal among the $Y_{\alpha}$‘s intersecting $\tilde{M}_{t}$, for some $t \in D^{k}$. The inductive step is to produce a $k$-parameter family of isotopies $h_{u}$ of $M_{t}$, $t \in D^{k}$, which eliminates $M_{t} \cap Y_{\alpha}$ over $B'$.

**Lemma 2.** Let $C$ be a compact connected surface other than $S^{2}$ or $P^{2}$. Then $PL(C \times I \text{ rel } C \times 0)$ is contractible.

**Proof.** Let $F$ be the homotopy fiber of $PL(C) \rightarrow G(C)$ and consider the map of fibrations
The vertical map to \( F \) comes from interpreting \( F \) as the space of pairs \((f, \gamma)\), where \( f \in PL(C) \) and \( \gamma \) is a path in \( G(C) \) from \( f \) to \( 1 \). An element of \( PL(C \times I \text{ rel } C \times 0 \cup \partial C \times I) \) determines such a pair by projection of \( C \times I \) to \( C \). The map \( PL(C \times I) \to \Omega G(C) \cong G(C \times I) \) is a homotopy equivalence by \( \S 1 \) if \( \partial C \neq \emptyset \) and by case (1) if \( \partial C = \emptyset \). Hence \( PL(C \times I \text{ rel } C \times 0) \), which can be identified with \( PL(C \times I \text{ rel } C \times 0 \cup \partial C \times I) \), is homotopy equivalent to \( F \). By surface theory \( PL(C) \to G(C) \) is a homotopy equivalence if \( C \) is not \( S^2 \) or \( P^2 \), so \( F \) is contractible. \( \Box \)

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